On the adjacent vertex distinguishing edge chromatic number of graphs*

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Abstract

An adjacent vertex distinguishing edge coloring or an avd-coloring of a simple graph G is a proper edge coloring of G such that for any two adjacent and distinct vertices u and v in G, the set of colors assigned to the edges incident to u differs from the set of colors assigned to the edges incident to v. In this paper, we prove that graphs with maximum 3 and with no any isolated edges partly satisfy the adjacent vertex distinguishing edge coloring conjecture.

Keywords: Adjacent vertex distinguishing edge coloring, Adjacent vertex distinguishing edge chromatic number .
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1. Introduction

In this paper we consider only simple, finite and undirected graphs. Let $V(G)$, $E(G)$, $\Delta(G)$, $\delta(G)$, $d(u)$ and $N(u)$ denote the vertex set, the edge set, the maximum degree, the minimum degree, the degree of vertex v and the set of vertices adjacent to u of graph G respectively. A proper edge coloring of a simple graph G is called adjacent vertex distinguishing if for any two adjacent and distinct vertices u and v in G, the set of colors assigned to the edges incident to u differs from the set of colors assigned to the edges incident to v, where the set of colors assigned to the edges incident to u is denoted by $f[u] = \{f(uv)|uv \in E(G)\}$. And the minimal number of colors required for a adjacent vertex distinguishing edge coloring of G is denoted by $\chi'_{as}(G)$. It is clear that every graph with isolated edges does not have any avd-coloring.

In 2002, Zhang Zhongfu [1] first introduced the concept of adjacent vertex distinguishing edge coloring of G. Adjacent vertex distinguishing edge colorings are studied in [1 ∼ 6], where different names such as adjacent strong edge coloring[1] and 1-strong edge coloring [3] are used to refer to an avd-coloring. Adjacent vertex distinguishing edge colorings are related to vertex distinguishing edge colorings in which the condition $f[u] \neq f[v]$ holds for every pair of vertices u and v, not necessarily adjacent. This concept has been studied in many papers. [9 ∼ 12]

Another interesting problem arises when we drop the condition that the edge coloring is proper and allow the incident edges to have the same color. Such as, 1. There exists a finite set of real numbers which can be used to weight the edges of any graph with no isolated edges so that adjacent vertices have different sums of incident edge weights.

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2. There is a finite set which can be used to color the edges of any graph with no isolated edges so that adjacent vertices meet different multisets (i.e. duplicate elements are allowed) of colors. (These results are proved by Karonski in [13])

This shows that by dropping the condition of being proper from the definition of avd-coloring. a constant number of colors would be sufficient. Obviously, when the edge colorings are required to be proper this not the case. And adjacent vertex distinguishing edge coloring conjecture given by Zhang Zhongfu [1] is as follows.

Conjecture. The avd-chromatic number of every simple connected graph G such that $G \neq C_5$ (the cycle with order 5) and $G \neq K_2$ (without any isolated edge) is at most $\Delta(G) + 2$.

The conjecture appears to be difficult even when the graph G are some special graphs. It is clear, for any simple graph G, $\chi'_{as}(G) \geq \Delta(G)$. If graph G has two adjacent vertices of maximal degree, then $\chi'_{as}(G) \geq \Delta(G) + 1$.

Balister etc. proved that if G is a graph with no isolated edges, then the avd-chromatic number of G is at most $\Delta + O(log\chi(G))$. In [2], Hamed H. proved that if G is a graph with no isolated edges and maximum degree $\Delta > 10^{20}$, then the avd-chromatic number of G is at most $\Delta + 300$. In [7], M.Ghandehari, etc. prove that 1. if $\Delta(G) \geq 10^6$ then $\chi_{at}(G, 1) \geq \Delta + 27\Delta\sqrt{\ln \Delta}$. 2. for the k-strong edge chromatic number of every r-regular graph , where $r > 100$ is less than or equal to $r + (k + 2)\log_2 r$

Notwithstanding they obtain some bounds of general graph, but the bound is so far away $\Delta + 2$. In this paper we shall prove the graphs with maximum degree 3 and with no any isolated edges partly s atisfy the adjacent vertex distinguishing edge coloring conjecture .

Definitions not given here may be found in [8,14,15].

2. Main result

Lemma 2.1.[1] If G is a cycle or a path, then G satisfy the conjecture of adjacent vertex distinguishing edge coloring. ■

Lemma 2.2. Let G be a graph consisting of two components G_1 and G_2. If G_1 and G_2 are k-adjacent vertex distinguishing edge coloring, then so is G. ■

Lemma 2.2 is obvious, by Lemma 2.1 and Lemma 2.2, we may get lemma 2.3.

Lemma 2.3. If G ($G \neq C_5$) is a graph with no isolated edges and maximum degree $\Delta(G) = 2$, then G satisfies adjacent vertex distinguishing edge coloring conjecture. ■

Theorem 2.4 Let G ($G \neq C_5$) be a graph with no isolated edges and maximum degree $\Delta(G) = 3$ and minimum degree $\delta(G) = 1$, suppose u is vertex of degree 1 , if $G^* = G - u$ has a 5-adjacent vertex distinguishing edge coloring f^*, then G also has a 5-adjacent vertex distinguishing edge coloring f.

Proof. By Lemma 2.2 and Lemma 2.3, we may suppose that G is connected. Let $C = \{1, 2, 3, 4, 5\}$ denote a color set, suppose N(u)={v}. Now we extend f^* of G^* to a 5-adjacent vertex distinguishing edge coloring f of G.

Case 1. If $d(v) = 2$, let $N(v) = \{u, w\}$, obviously, $d(w) \geq 2$. then, let $f(uv) \in C - f^*[w]$. The coloring of other edges is the same to f^*. Then $f[u] \neq f[v], f[v] \neq f[w]$.

Case 2. If $d(v) = 3$, let $N(v) = \{u, w, x\}$.

Subcase 2.1. If $d(w) + d(x) \leq 4$, then let $f(uv) \in C - f^*[w] - f^*[x]$.

Subcase 2.2. If $d(w) + d(x) = 5$, then If $f^*(wv) \in f^*[x]$, let $f(uv) \in C - f^*[x]$. If $f^*(wv) \notin f^*[x]$, let $f(uv) \in C - f^*(wv) - f^*(vx)$. The coloring of other edges is the same to f^*. Then $f[u] \neq f[v], f[v] \neq f[w], f[v] \neq f[x]$.

Subcase 2.3. If $d(w) + d(x) = 6$, then,

2.3.1. If $f^*(vw), f^*(vx) \in f^*[w] \cap f^*[x]$, then $| f^*[w] \cup f^*[x] | \leq 4$. Let $f(uv) \in C - f^*[w] - f^*[x]$. The coloring of other edges is the same to f^*. Then $f[u] \neq f[v], f[v] \neq f[w], f[v] \neq f[x]$.

2.3.2. If $f^*(vx) \in f^*[w]$, but $f^*(vw) \notin f^*[x]$. Let $f(uv) \in C - f^*[w]$. The coloring of other edges is the same to f^*. Then $f[u] \neq f[v], f[v] \neq f[w], f[v] \neq f[x]$.

2.3.3. If $f^*(vx) \notin f^*[w]$, but $f^*(vw) \in f^*[x]$. Let $f(uv) \in C - f^*[x]$. The coloring of other edges is the same to f^*. Then $f[u] \neq f[v], f[v] \neq f[w], f[v] \neq f[x]$.

2.3.4. If $f^*(vx) \notin f^*[w]$, $f^*(vw) \notin f^*[x]$. Let $f(uv) \in C - f^*(vw) - f^*(vx)$. The coloring of other edges is the same to f^*. Then $f[u] \neq f[v], f[v] \neq f[w], f[v] \neq f[x]$.

With all cases considered , f is a 5-adjacent vertex distinguishing edge coloring of G, so the theorem holds. ■

Lemma 2.5 Let G ($G \neq C_5$) be a graph with no isolated edges and maximum degree $\Delta(G) = 3$ and minimum degree $\delta(G) = 1$, suppose u is vertex of degree 1 , if $G^* = G - u$ has a 6-adjacent vertex distinguishing edge coloring f^*, then G also has a 6-adjacent vertex distinguishing edge coloring f.

The proof of the lemma is similar to theorem 2.4. ■

Theorem 2.6 Let G ($G \neq C_5$) be a graph with maximum degree $\Delta(G) = 3$ and minimum degree $\delta(G) = 2$, H be a graph with maximum degree $\Delta(G) = 3$ and minimum degree $\delta(G) = 1$, if H has a 5-adjacent vertex distinguishing edge coloring, then G has also a 5-adjacent vertex distinguishing edge coloring.

Proof. We will prove the theorem by induction on the number of vertices ($| V(G) |= n$) of G. By Lemma 2.2 and Lemma 2.3, we may suppose that G is connected. Let $C = \{1,2,3,4,5\}$ denote a color set. If $| V(G) | \leq 4$, the result is obvious. Suppose G has a 5-adjacent vertex distinguishing edge coloring f for $| V(G) | < k$. Suppose d(u)=2, let w, v be the neighbor of u, namely, $N(u) = \{w, v\}$.

———- If vertices w and v are not adjacent, obviously, $4 \leq d(v) + d(w) \leq 6$, then,

Case 1. If $4 \leq d(v) + d(w) \leq 5$, we denote a new graph $G^* = G - u$, obviously, $\Delta(G^*) = 3$ and $\delta(G^*) = 1$ or $\Delta(G^*) = 2$. By the condition (i.e. H has a 5-adjacent vertex distinguishing edge coloring,) of the theorem 2.6 and lemma 2.3, G^* has a 5-adjacent vertex distinguishing edge coloring f^*. Now we extend f^* of G^* to a 5-adjacent vertex distinguishing edge coloring f of G.

Subcase 1.1. If $d(v) = d(w) = 2$, let $N(w) = \{u, x\}, N(v) = \{u, y\}$.

1.1.1. If $d(x) = d(y) = 2$, let $f(uw) \in C - f^*[x] - f^*(vy), f(uv) \in C - f^*[y] - f(uw) - f^*(wx)$. The coloring of other edges is the same to f^*. Then $f[u] \neq f[v], f[u] \neq f[w], f[w] \neq f[x].f[v] \neq f[y]$.

1.1.2. If $d(x) = 3, d(y) = 2$, let $f(uw) \in C - f^*(wx) - f^*(vy), f(uv) \in C - f^*[y] - f^*(wx) - f(uw)$. The coloring of other edges is the same to f^*. Then $f[u] \neq f[v], f[u] \neq f[w], f[w] \neq f[x].f[v] \neq f[y]$.

1.1.3. If $d(x) = 2, d(y) = 3$, the proof is similar to 1.1.2.

1.1.4. If $d(x) = d(y) = 3$, let $f(uw) \in C - f^*(wx) - f^*(vy), f(uv) \in C - f^*(vy) - f(uw) - f^*(wx)$. The coloring of other edges is the same to f^*. Then $f[u] \neq f[v], f[u] \neq f[w], f[w] \neq f[x].f[v] \neq f[y]$.

Subcase 1.2. If $d(v) = 2, d(w) = 3$, let $N(v) = \{u, z\}, N(w) = \{u, x, y\}$.

1.2.1. If $d(x) = d(y) = d(z) = 2$, let $f(uw) \in C - f^*(wx) - f^*(wy) - f^*(vz), f(uv) \in C - f^*[z] - f(uw)$. The coloring of other edges is the same to f^*. Then $f[u] \neq f[v], f[u] \neq f[w], f[w] \neq f[x], f[w] \neq f[y], f[v] \neq f[z]$.

1.2.2. If $d(x) = d(y) = 2, d(z) = 3$, let $f(uw) \in C - f^*(wx) - f^*(wy) - f^*(vz), f(uv) \in C - f^*(vz) - f(uw)$. The coloring of other edges is the same to f^*. Then $f[u] \neq f[v], f[u] \neq f[w], f[w] \neq f[x], f[w] \neq f[y], f[v] \neq f[z]$.

1.2.3. If $d(x) = d(z) = 2, d(y) = 3$, then,

(1). If $f^*(vz) \in \{f^*(wx)\} \cup f^*[y]$, let $f(uw) \in C - f^*(wx) - f^*[y]$.

(2). If $f^*(vz) \notin \{f^*(wx)\} \cup f^*[y]$, then,

(2).1. If $f^*(wx) \in f^*[y]$, let $f(uw) \in C - f^*[y] - f^*(vz)$.

(2).2. If $f^*(wx) \notin f^*[y]$, let $f(uw) \in C - f^*(wx) - f^*(wy) - f^*(vz)$.

Let $f(uv) \in C - f^*[z] - f(uw)$. The coloring of other edges is the same to f^*. Then $f[u] \neq f[v], f[u] \neq f[w], f[w] \neq f[x], f[w] \neq f[y], f[v] \neq f[z]$.

1.2.4. If $d(y) = d(z) = 2, d(x) = 3$, the proof is similar to 1.2.3.

1.2.5. If $d(x) = d(y) = d(z) = 3$, firstly, we denote a new graph $G' = G^* + wv = G - u + wv$, obviously, $\Delta(G') = 3, \delta(G') = 2$. By the induction assumption, G' has a 5-adjacent vertex distinguishing edge coloring f'. Now we extend f' of G' to a 5-adjacent vertex distinguishing edge coloring f of G.

Let $f(uw) = f'(wv), f(uv) = C - f(uw) - f'(vz)$. The coloring of other edges is the same to f^*. Then $f[u] \neq f[v], f[u] \neq f[w], f[w] \neq f[x], f[w] \neq f[y], f[v] \neq f[z]$.

1.2.6. If $d(y) = d(z) = 3, d(x) = 2$ or $d(x) = d(z) = 3, d(y) = 2$ or $d(x) = d(y) = 3, d(z) = 2$, the proof of this case is similar to 1.2.5.

Subcase 1.3. If $d(v) = 3, d(w) = 2$, the proof is similar to subcase 1.2.

Case 2. If $d(v) + d(w) = 6$, i.e. $d(v) = d(w) = 3$, we denote a new graph $G' = G + a + ua$, where $a \notin V(G)$, obviously, $\Delta(G') = 3$ and $\delta(G') = 1$. By the condition (i.e. H has a 5-adjacent vertex distinguishing edge coloring,) of the theorem 2.6, then G' has a 5- adjacent vertex distinguishing edge coloring, then obviously, G has 5- adjacent vertex distinguishing edge coloring.

——— If vertices w and v are adjacent, obviously, $4 \leq d(v) + d(w) \leq 6$, then,

We denote a new graph $G^* = G - u$, obviously, $\Delta(G^*) = 3$ and $\delta(G^*) = 1$ or $\Delta(G^*) = 3$ and $\delta(G^*) = 2$ or $\Delta(G^*) = 2$. By the condition (i.e. H has a 5-adjacent vertex distinguishing edge coloring,) of the theorem 2.6, the induction assumption and lemma 2.3, G^* has a 5-adjacent vertex distinguishing edge coloring f^*. Now we extend f^* of G^* to a 5-adjacent vertex distinguishing edge coloring f of G.

Case 1. If $d(w) = 2, d(v) = 3$, let $N(v) = \{u, w, x\}, N(w) = \{u, v\}$.

Subcase 1.1. If x is a vertex with degree 2, let $f(uw) \in C - f^*(wv), f(uv) \in C - f^*(wv) - f(uw) - f^*(vx)$. The coloring of other edges is the same to f^*. Then $f[u] \neq f[v], f[u] \neq f[w], f[w] \neq f[v].f[v] \neq f[x]$.

Subcase 1.2. If x is a vertex with degree 3, let $f(uv) \in C - f^*[x] - f^*(vw), f(uw) \in C - f^*(vw) - f(uv)$. The coloring of other edges is the same to f^*. Then $f[u] \neq f[v], f[u] \neq f[w], f[w] \neq f[v].f[v] \neq f[x]$.

Case 2. If $d(w) = 3, d(v) = 2$, the proof is similar to case 1

Case 3. If $d(w) = d(v) = 3$, let $N(w) = \{u, v, x\}, N(v) = \{u, w, y\}$.

Subcase 3.1 If $d(x) = d(y) = 2$, let $f(uw) = C - f^*(wv) - f^*(wx) - f^*(vy), f(uv) \in C - f(uw) - f^*(wv) - f^*(vy)$. The coloring of other edges is the same to f^*. Then $f[u] \neq f[v], f[u] \neq f[(w], f[w] \neq f[v].f[w] \neq f[x], f[v] \neq f[y]$.

Subcase 3.2 If $d(x) = 2, d(y) = 3$,

3.2.1. If $f^*(wv) \in f^*[y]$, let $f(uv) \in C - f^*[y] - f^*(wx), f(uw) \in C - f(uv) - f^*(wv) - f^*(wx)$.

3.2.2. If $f^*(wv) \notin f^*[y]$, let $f(uv) \in C - f^*(wv) - f^*(vy) - f^*(wx), f(uw) \in C - f(uv) - f^*(wv) - f^*(wx)$.

The coloring of other edges is the same to f^*. Then $f[u] \neq f[v], f[u] \neq f[(w], f[w] \neq f[v].f[w] \neq f[x], f[v] \neq f[y]$.

Subcase 3.3 If $d(x) = 3, d(y) = 2$, the proof is similar to subcase 3.2.

Subcase 3.4 If $d(x) = d(y) = 3$,

3.4.1. If $f^*(wv) \in f^*[x]$ and $f^*(wv) \in f^*[y]$, let $f(uw) \in C - f^*[x] - f^*(vy), f(uv) \in C - f^*[y] - f(uw)$.

3.4.2. If $f^*(wv) \in f^*[x]$, but $f^*(wv) \notin f^*[y]$, let $f(uw) \in C - f^*[x] - f^*(vy), f(uv) \in C - f^*(wv) - f^*(vy) - f(uw)$.

3.4.3. If $f^*(wv) \notin f^*[x]$, but $f^*(wv) \in f^*[y]$, the proof is similar to 3.4.2.

3.4.4. If $f^*(wv) \notin f^*[x]$ and $f^*(wv) \notin f^*[y]$, let $f(uw) \in C - f^*(wv) - f^*(wx) - f^*(vy), f(uv) \in C - f(uw) - f^*(wv) - f^*(vy)$.

The coloring of other edges is the same to f^*. Then $f[u] \neq f[v], f[u] \neq f[(w], f[w] \neq f[v].f[w] \neq f[x], f[v] \neq f[y]$.

With all cases considered , the theorem holds. ■

Theorem 2.7 If G ($G \neq C_5$) is a graph with maximum degree $\Delta(G) = 3$ and minimum degree $\delta(G) \leq 2$, then G has a 6-adjacent vertex distinguishing edge coloring.

Proof. We will prove the theorem by induction on the number of vertices ($|V(G)| = n$) of G. By Lemma 2.2 and Lemma 2.3, we may suppose that G is connected. Let $C = \{1,2,3,4,5,6\}$ denote a color set. If $|V(G)| \leq 4$, the result is obvious. Suppose G has a 6-adjacent vertex distinguishing edge coloring f for $|V(G)| < k$.

Case 1. If $\delta(G) = 1$, suppose d(u)=1, We denote a new graph $G^* = G - u$, obviously, $\Delta(G^*) = 3$ and $\delta(G^*) \leq 2$ or $\Delta(G^*) = 2$. By the induction assumption and lemma 2.3, G^* has a 6-adjacent vertex distinguishing edge coloring f^*. then by lemma 2.5, G has a 6-adjacent vertex distinguishing edge coloring.

Case 2. If $\delta(G) = 2$, suppose d(u)=2, suppose w, v be the neighbor of u, namely, $N(u) = \{w, v\}$.

Subcase 2.1. If vertices w and v are not adjacent and d(v)+d(w)=6. i.e. $d(v) = d(w) = 3$, $uv \notin E(G)$. We denote a new graph $G^* = G - u$, obviously, $\Delta(G^*) = 3$ and $\delta(G^*) = 2$ or $\Delta(G^*) = 2$. By the induction assumption and lemma 2.3, G^* has a 6-adjacent vertex distinguishing edge coloring f^*. Now we extend f^* of G^* to a 6-adjacent vertex distinguishing edge coloring f of G.

let $N(w) = \{a, b, u\}, N(v) = \{u, g, h\}$. We first color edge uw,

If $d(a) = d(b) = 3$, then,

(1). When $|f^*[a] \cap f^*[b]| = 0$, then let $f(uw) \in C - f^*(wa) - f^*(wb)$.

(2). When $1 \leq |f^*[a] \cap f^*[b]| \leq 3$, then $\exists\ \alpha \in C$ such that $\alpha \notin f^*[a]$ *and* $\alpha \notin f^*[b]$, let $f(uw) = \alpha$.

If $d(a) = 2, d(b) = 3$, then let $f(uw) \in C - f^*[a] - f^*(wa)$.

If $d(a) = d(b) = 2$, then let $f(uw) \in C - f^*(wa) - f^*(wb)$.

Edge uv is colored by as follows,

(1). If $d(g) = d(h) = 3$, then,

When $|f^*[g] \cap f^*[h]| = 0$, let $f(uv) \in C - f(uw) - f^*(vg) - f^*(vh)$.

When $|f^*[g] \cap f^*[h]| = 1$, then $\exists\ \alpha \in C$ such that $\alpha \notin f^*[g]$ *and* $\alpha \notin f^*[h]$,

If $\alpha \neq uw$, let $f(uv) = \alpha$.

If $\alpha = uw$, because $|f^*[g] \cap f^*[h]| = 1 \Rightarrow \exists\ \beta \in C$ such that $\beta \in f^*[g]$ *and* $\beta \in f^*[h]$, If $\beta \neq f^*(vg)$ *and* $\beta \neq f^*(vh)$, then let $f(uv) \in C - f^*(vg) - f^*(vh) - \alpha$. If $\beta = f^*(vg)$ then $\beta \neq f^*(vh)$, let $f(uv) \in C - f^*[h] - f(uw)$.

When $2 \leq |f^*[g] \cap f^*[h]| \leq 3$, then $\exists\ \alpha, \beta \in C$ $(\alpha \neq \beta)$, such that $\alpha, \beta \notin f^*[g]$ *and* $\alpha, \beta \notin f^*[h]$, let $f(uv) \in \{\alpha, \beta\} - f(uw)$.

(2). If $d(g) = 2, d(h) = 3$, then let $f(uv) \in C - f^*[h] - f^*(vg) - f(uw)$.

(3). If $d(g) = d(h) = 2$, then let $f(uv) \in C - f(uw) - f^*(vg) - f^*(vh)$.

The coloring of other edges is the same to f^*. Then $f[u] \neq f[v], f[u] \neq f[w], f[w] \neq f[x], f[w] \neq f[y], f[v] \neq f[z]$.

Subcase 2.2. The proofs of case 2 in theorem 2.7 are the same as that of theorem 2.6 except for when vertices w and v are not adjacent and d(v)+d(w)=6.

The coloring of other edges is the same to f^*. Then $f[u] \neq f[v], f[u] \neq f[(w], f[w] \neq f[v].f[w] \neq f[x], f[v] \neq f[y]$. With all cases considered , the theorem holds. ■

Theorem 2.8. If G is a 3-regular graph containing a cut edge at least, suppose uv is a cut edge of G, if $G - uv$ has a 5- adjacent vertex distinguishing edge coloring , then G has a 5- adjacent vertex distinguishing edge coloring .

Proof. Because uv is a cut edge of G, then $G - uv$ has two components G_1 and G_2, where G_1 contains u, G_2 contains v, moreover, $G - uv$ has a 5- adjacent vertex distinguishing edge coloring, then G_1 has a 5- adjacent vertex distinguishing edge coloring f_1; G_2 has a 5- adjacent vertex distinguishing edge coloring f_2. Now we will establish a 5- adjacent vertex distinguishing edge coloring f according to f_1 and f_2. Let C={1,2,3,4,5} be a color set, $N(u) = \{u_1, u_2, v\}$, $N(v) = \{v_1, v_2, u\}$.

Step 1. First, we color edge uv by f_1 (without considering f_2), obviously, $1 \leq |f_1[u_1] \cap f_1[u_2]| \leq 3$.

— If $f_1[u_1] \cap f_1[u_2] = 1$, then $| f_1[u_1] \cup f_1[u_2] |= 5$. If $f_1(u_1u) \notin f_1[u_2]$ and $f_1(u_2u) \notin f_1[u_1]$, then let $f(uv) \in C - f_1(u_1u) - f_1(u_2u)$; If $f_1(u_1u) \in f_1[u_2]$, then let $f(uv) \in C - f_1(u_2)$; If $f_1(u_2u) \in f_1[u_1]$, then let $f(uv) \in C - f_1(u_1)$.

— If $2 \leq| f_1[u_1] \cap f_1[u_2] |\leq 3$, then existing a color $\alpha \in C$ such that $\alpha \notin f_1[u_1]$ and $\alpha \notin f_1[u_2]$, let $f(uv) = \alpha$.

Without loss of the generality, we assume $f(uv) = \theta$.

The coloring of G_1 remain invariability.

Step 2. Next, we will recolor some elements (vertices and edges) of G_2 according to f_1 and $f(uv)$,

Whether or no, by permutation of colors in f_2, we can obtain a new coloring f' of G_2, we can make $f'(vv_1) \neq f(uv)$, $f'(vv_2) \neq f(uv)$ and $\{f_1(uu_1), f_1(uu_2)\} \neq \{f'(vv_1), f'(vv_2)\}$.

— If $\theta \notin f'[v_1]$ and $\theta \notin f'[v_2]$, obviously, $f_1, f(uv)$ and f' form a 5- adjacent vertex distinguishing edge coloring of G.

— If $\theta \in f'[v_1]$ and $\theta \in f'[v_2]$, suppose $N(v_1) = \{v, v_1', v_1''\}$, $N(v_2) = \{v, v_2', v_2''\}$ without loss of the generality, we can assume $f'(v_1v_1') = f'(v_2v_2') = \theta$ then by permutation of colors of in f', (i.e. the permutation two colors θ *and* β in f', $\theta \Leftrightarrow \beta \in C - f'(vv_1) - f'(vv_2) - f'(v_1v_1'') - f'(v_2v_2'')$). We can obtain a new coloring f'' of G_2, then $f_1, f(uv)$ and f'' form a 5- adjacent vertex distinguishing edge coloring of G.

— If $\theta \in f'[v_1]$ and $\theta \notin f'[v_2]$ or $\theta \in f'[v_2]$ and $\theta \notin f'[v_1]$, without loss of the generality, we may assume $\theta \in f'[v_1]$ and $\theta \notin f'[v_2]$, we may assume $f'(v_1v_1') = \theta$. If $f'(vv_2) \notin f'[v_1]$, then, the conclusion is true. If $f'(vv_2) \in f'[v_1]$, then by permutation of colors of in f', (i.e. the permutation two colors θ *and* β in f', $\theta \Leftrightarrow \beta \in C - f'[v_2] - f'(vv_1)$). We can obtain a new coloring f'' of G_2, then $f_1, f(uv)$ and f'' form a 5- adjacent vertex distinguishing edge coloring of G.

Up to now , we establish a 6- adjacent vertex distinguishing edge coloring of G. ■

Lemma 2.9. If G is a 3-regular graph containing a cut edge at least, suppose uv is a cut edge of G, if $G - uv$ has a 6- adjacent vertex distinguishing edge coloring , then G has a 6- adjacent vertex distinguishing edge coloring .

The proof of the lemma is similar to theorem 2.8. ■

Theorem 2.10. If G is a 3-regular graph containing a cut edge at least, then G has a 6- adjacent vertex distinguishing edge coloring.

By theorem 2.7 and lemma 2.9, the theorem 2.10 is true. ■

We firstly introduced a lemma used in the proof of the following theorem before the proof as follows.

Lemma 2.11. Every 3-regular graph without cut edges has a perfect matching. ■

This lemma have been proved in [14]. Lemma 2.11 implies that the edge set of Every 3-regular graph G without cut edges can be partitioned into the union of a perfect matching M and certain vertex disjoint cycles. Suppose that $C_n = u_1u_2\cdots u_n$ is a cycle of $G - M$. In the sequel, we always use v_i to denote the neighbor of u_i in G which is different from u_{i-1} and u_{i+1}, namely, $N(u_i) = \{v_i, u_{i-1}, u_{i+1}\}, (i = 2, 3, \cdots, n-1)$, $N(u_1) = \{v_1, u_n, u_2\}$, $N(u_n) = \{v_n, u_{n-1}, u_1\}$. If $v_i \notin V(C_n)$, we call v_i a pendent vertex of C_n at the vertex u_i. Otherwise , u_iv_i forms a chord of C_n. Let $Q(C)$ denote the set of aa pendent vertices of C_n. Thus the edges between $V(C_n)$ and $Q(C)$ belong to the matching M. It is clear, edges $u_iv_i \in M, (i = 1, 2, \cdots, n)$.

If G is a 3-regular graph without cut edges, thus $G - M$ is the union of vertex-disjoint cycles, let H_1 denote the union of cycles of $G - M$ each of which has no chord in G. let H_2 denote the union of cycles of $G - M - V(H_1)$ each of which has at least one chord in G. Thus $G - M$ is partitioned into the vertex-disjoint union of H_1 and H_2.

Theorem 2.12. If G is a 3-regular graph without cut edges and G has a cycle $C_n \in H_1$, then $\chi'_{as}(G) \leq 7$.

Proof. By lemma 2.11, if G is a 3-regular graph without cut edges , then we know G consists of a perfect matching M and certain vertex disjoint cycles. Suppose that

$C_n = u_1u_2\cdots u_n$ is a cycle of $G - M$ which has no chord in G , then $G - V(C_n)$ is a cycle or a graph with $\Delta(G) = 3$ and with $\delta(G) = 2$. Moreover, when G is a cycle C_n, then $\chi'_{as}(C_n) \leq 6$; When G is a graph with $\Delta(G) = 3$ and with $\delta(G) = 2$, then $\chi'_{as}(G) \leq 6$ by theorem 2.7; Then $\chi'_{as}(G - V(C_n)) \leq 6$.

Suppose f^* is a 6-adjacent vertex distinguishing edge coloring of $G - V(C_n)$ with the color set $B = \{1, 2, 3, 4, 5, 6\}$. Now we establish a 7-adjacent vertex distinguishing edge coloring f of G with the color set $C = \{1, 2, 3, 4, 5, 6, 7\}$ as follows.

Step 1. Let $N(v_i) = \{u_i, v'_i, v''_i\}$, $i = 1, 2, \cdots, n$. $F^*[v_i] = \{f^*(v_i v'_i), f^*(v_i v''_i)\}$. Firstly we color edges u_iv_i, $(i = 1, 2, \cdots, n)$ of G with 7 ($\Rightarrow f[v_i] \neq f[v'_i], f[v_i] \neq f[v''_i]$).

Step 2. Next we color edge u_1u_2. Let $f(u_1u_2) \in B - F^*[v_1]$.

Step 3. Now we color edge u_2u_3. Let $f(u_2u_3) \in B - F^*[v_2] - f(u_1u_2)$.

Step 4. Finally, we color others edges u_iu_{i+1}, $(i = 3, 4, \cdots n - 1)$ and edge u_nu_1. Let $f(u_iu_{i+1}) = B - F^*[v_i] - f(u_{i-1}u_i) - f(u_{i-2}u_{i-1})$, $f(u_nu_1) \in B - F^*[v_n] - f(u_{n-1}u_n) - f(u_{n-2}u_{n-1})$. The coloring of other edges is the same to f^*. Then, we know for any u_i, $(i = 3, 4, \cdots, n - 1)$., $f[u_i] \neq f[v_i], f[u_i] \neq f[u_{i-1}], f[u_i] \neq f[u_{i+1}]$.

For vertices u_n, u_1, u_2, first, we will color afresh edges u_1u_2 and u_2u_3, let $f(u_1u_2) \in B - F^*[v_1] - f(u_{n-1}u_n) - f(u_nu_1)$; $f(u_2u_3) \in B - F^*[v_2] - f(u_1u_2) - f(u_{n-1}u_n)$. Then we may get vertices u_n, u_1, u_2 is also adjacent vertex distinguishing.

Example: For vertex u_3, $f(u_iv_i) = 7$; $f(u_2u_3) \in B - F^*[v_2] - f(u_1u_2)$: $f(u_3u_4) \in B - F^*[v_3] - f(u_2u_3) - f(u_1u_2) \Rightarrow f[u_3] \neq f[v_3], f[u_3] \neq f[u_2]$; $f(u_4u_5) \in B - F^*[v_4] - f(u_3u_4) - f(u_2u_3) \Rightarrow f[u_3] \neq f[u_4]$;

With all cases considered, the theorem is true. ■

In fact, it is easy to prove that for any 3-regular graphs $\chi'_{as}(G) \leq 7$. The reason of theorem 2.12 given here is that we want to only enucleate the structure of 3-regular graphs.

Theorem 2.13. If G is a 3-regular graph, then $\chi'_{as}(G) \leq 7$.

Proof. Suppose uv is a edge of G, then $G - uv$ is a a graph with $\Delta(G) = 3$ and with $\delta(G) = 2$, then $\chi'_{as}(G - uv) \leq 6$ by theorem 2.7; Let $N(u) = \{v, x, y\}, N(v) = \{u, w, z\}$. Suppose f^* is a 6-adjacent vertex distinguishing edge coloring of $G - uv$ with the color set $B = \{1, 2, 3, 4, 5, 6\}$. Now we establish a 7-adjacent vertex distinguishing edge coloring f of G with the color set $C = \{1, 2, 3, 4, 5, 6, 7\}$ as follows.

Case 1. If $0 \leq | \{f^*(ux), f^*(uy)\} \cap \{f^*(vw), f^*(vz)\} | < 2$, then let $f(uv) = 7$. The coloring of other edges is the same to f^*. Thus f is a 6-adjacent vertex distinguishing edge coloring.

Case 2. If $| \{f^*(ux), f^*(uy)\} \cap \{f^*(vw), f^*(vz)\} | = 2$, namely the two sets are equal.

(1). First we will color edge uv, let $f(uv) = 7$.

(2). Let $N(w) = \{w_1, w_2, v\}$. Next we will color anew edge vw.

——If $| f^*[w_1] \cap f^*[w_2] | = 0$, then let $f(vw) \in B - f^*(ww_1) - f^*(ww_2) - f^*(vw) - f^*(vz)$.

——If $| f^*[w_1] \cap f^*[w_2] | = 1$, then existing a color $\alpha \in B$ such that $\alpha \notin f^*[w_1]$ and $f^*[w_2]$. If $\alpha \notin \{f^*(vw), f^*(vz)\}$, let $f(vw) = \alpha$. If $\alpha \in \{f^*(vw), f^*(vz)\}$, because $| f^*[w_1] \cap f^*[w_2] | = 1$, then, 1). if $f^*(ww_1) \in f^*[w_2]$ then $f^*(ww_2) \notin f^*[w_1]$, then let $f(vw) \in B - f^*[w_2] - f^*(vw) - f^*(vz)$. 2). if $f^*(ww_1) \notin f^*[w_2], f^*(ww_2) \notin f^*[w_1]$, then let $f(vw) \in B - f^*(ww_1) - f^*(ww_2) - f^*(vw) - f^*(vz)$.

——If $| f^*[w_1] \cap f^*[w_2] | = 2$, then existing $\alpha, \beta \in B, (\alpha \neq \beta)$ such that $\alpha, \beta \notin f^*[w_1]$ and $f^*[w_2]$. If $\{\alpha, \beta\} \neq \{f^*(vw), f^*(vz)\}$, then let $f(vw) \in \{\alpha, \beta\} - f^*(vz)$. If $\{\alpha, \beta\} = \{f^*(vw), f^*(vz)\}$, then we will color anew edge ww_1, wv, suppose $N(w_1) = \{w, w'_1, w''_1\}$. Let $f(ww_1) = 7$, $f(vw) \in B - f^*(w_1w'_1) - f^*(w_1w''_1) - \alpha - \beta$.

——If $| f^*[w_1] \cap f^*[w_2] | = 3$, then existing $\alpha, \beta, \gamma \in B, (\alpha \neq \beta, \alpha \neq \gamma, \gamma \neq \beta)$ such that $\alpha, \beta, \gamma \notin f^*[w_1]$ and $f^*[w_2]$. Let $f(vw) \in \{\alpha, \beta, \gamma\} - \{ f^*(vw), f^*(vz)\}$. The coloring of other edges is the same to f^*. It is clear f is a 7-adjacent vertex distinguishing edge coloring. ■

Theorem 2.14. Let G is a 3-regular graph containing a triangle at least, vertices u, v, w form a triangle, where $d(u) = d(v) = d(w) = 3$, $N(u) = \{x, v, w\}$, $N(v) = \{v_1, u, w\}$, $N(w) = \{w_1, u, v\}$. Let $G^* = G - ux$, If G^* has a 5-adjacent vertex distinguishing edge coloring f^* , then G^* must exist a 5-adjacent vertex distinguishing edge coloring f such that $\{f(uv), f(uw)\} \neq \{f(xy), f(xz)\}$.

Proof. Because G^* have a 5-adjacent vertex distinguishing edge coloring f^*, let C={1,2,3,4,5} be a color set. Now by f^*, we establish a 5-adjacent vertex distinguishing edge coloring f such that $\{f(uv), f(uw)\} \neq \{f(xy), f(xz)\}$.

Case 1. If $\{f^*(uv), f^*(uw)\} \neq \{f^*(xy), f^*(xz)\}$, then f is f^*, the conclusion is true..

Case 2. If $\{f^*(uv), f^*(uw)\} = \{f^*(xy), f^*(xz)\}$, then,

Subcase 2.1. If $f^*(uw) \notin f^*[v_1]$, then let $f(vw) = f^*(uw), f(uw) = f^*(vw)$.

Subcase 2.2. If $f^*(uw) \in f^*[v_1]$ and $f^*(uv) \notin f^*[w_1]$, then let $f(vw) = f^*(uv)$, $f(uv) = f^*(vw)$.

Subcase 2.3. If $f^*(uw) \in f^*[v_1]$ and $f^*(uv) \in f^*[w_1]$, obviously, $| f^*[v_1] \cup \{f^*(xy), f^*(xz)\} | \leq 4$, $| f^*[w_1] \cup \{f^*(xy), f^*(xz)\} | \leq 4$ and $1 \leq | f^*[v] \cap f^*[w] | \leq 2$.

if $| f^*[v] \cap f^*[w] | = 1$, then,

—— if $f^*(wv) \in f^*[w_1]$), let $f(uw) \in C - f^*[w_1] - f^*(xy) - f^*(xz)$.

—— if $f^*(wv) \notin f^*[w_1]$, then let $f(uw) \in C - f^*(ww_1) - f^*(vw) - f^*(xy) - f^*(xz)$.

if $| f^*[v] \cap f^*[w] | = 2$, then,

—— if $f^*(wv) \in f^*[w_1]$ and $f^*(uw) \in f^*[v)$, then $f^*(uv) \notin f^*[w]$, then let $f(uw) \in C - f^*[w_1] - f^*(xy) - f^*(xz)$.

—— if $f^*(wv) \in f^*[w_1]$, $f^*(uw) \notin f^*[v)$ and $f^*(uv) \notin f^*[w)$, then let $f(uw) \in C - f^*[w_1] - f^*(xy) - f^*(xz)$.

—— if $f^*(wv) \in f^*[w_1]$ and $f^*(uv) \in f^*[w)$, then $f^*(uw) \notin f^*[v]$. If $f^*(wv) \in f^*[v_1]$, then let $f(uv) \in C - f^*[v_1] - f^*(xy) - f^*(xz)$. If $f^*(wv) \notin f^*[v_1]$, then let $f(uv) \in C - f^*(vv_1) - f^*(vw) - f^*(xy) - f^*(xz)$.

Up to now , we have established a 5-adjacent vertex distinguishing edge coloring f such that $\{f(uv), f(uw)\} \neq \{f(xy), f(xz)\}$. So the theorem holds. ■

Lemma 2.15. Let G is a 3-regular graph containing a triangle at least, vertices u, v, w form a triangle, where $d(u) = d(v) = d(w) = 3$, $N(u) = \{x, v, w\}$, $N(v) = \{v_1, u, w\}$, $N(w) = \{w_1, u, v\}$. Let $G^* = G - ux$, If G^* has a 6-adjacent vertex distinguishing edge coloring f^* , then G^* must has a 6-adjacent vertex distinguishing edge coloring f such that $\{f(uv), f(uw)\} \neq \{f(xy), f(xz)\}$.

The proof of the lemma is similar to theorem 2.14. ■

Theorem 2.16. Let G is a 3-regular graph containing a triangle at least, then G has a 6-adjacent vertex distinguishing edge coloring

Proof. Suppose vertices u, v, w form a triangle, where $d(u) = d(v) = d(w) = 3$, $N(u) = \{x, v, w\}$, $N(v) = \{v_1, u, w\}$, $N(w) = \{w_1, u, v\}$. We denote a new graph $G^* = G - ux$, then G^* is a graph with $\Delta(G^*) = 3$ and $\delta(G^* = 2$, by theorem 2.7, we know G^* has a 6-adjacent vertex distinguishing edge coloring f^*. Let C={1,2,3,4,5,6} be a color set.

Step 1. First we color edge ux, obviously, $0 \leq | f^*[y] \cap f^*[z] | \leq 3$.

Case 1. If $| f^*[y] \cap f^*[z] | = 0$, then let $f(ux) \in C - f^*(xy) - f^*(xz)$.

Case 2. $1 \leq | f^*[y] \cap f^*[z] | \leq 3$, then $3 \leq | f^*[y] \cup f^*[z] | \leq 5$, then let $f(ux) \in C - f^*[y] - f^*[z]$.

Step 2. Newt, we will recolor some edges of G for remain adjacent distinguishing, without loss of the generality, we can assume $f(ux) = \alpha$. By lemma 2.15 we know $| \{f(uv), f(uw)\} \cap \{ f(xy), f(xz)\} | \neq 2$, then,

Case 1. If $| f^*[w] \cap f^*[v] | = 1$, then,

Subcase 1.1. If $f^*(uw) \neq \alpha$ and $f^*(uv) \neq \alpha$, obviously, the conclusion is true.

Subcase 1.2. If $f^*(uv) = \alpha$, then recolor edge uv as follows,

1, If $f^*(wv) \in f^*[v_1]$ and $f^*(wu) \in \{f^*(xy), f^*(xz)\}$, then,

—— If $| f^*[v_1] \cup \{f^*(xy), f^*(xz), \alpha\} |= 6$, then, $f^*(wv) \notin \{f^*(xy), f^*(xz), \alpha\}$ and $f^*(wu) \in f^*[v_1]$, hence, let $f(wv) = f^*(wu), f(wu) = f^*(wv)$, $f(uv) \in C - f^*(vv_1) - f(wv) - f(wu) - \alpha$.

2, If $f^*(wv) \in f^*[v_1]$ but $f^*(wu) \notin \{f^*(xy), f^*(xz)\}$, then, let $f(uv) \in C - f^*[v_1] - f^*(wu) - \alpha$.

2, If $f^*(wv) \notin f^*[v_1]$ and $f^*(wu) \notin \{f^*(xy), f^*(xz)\}$, then, let $f(uv) \in C - f^*(vv_1) - f^*(wv) - f^*(wu) - \alpha$.

Case 2. If $| f^*[w] \cap f^*[v] |= 2$, it is easy to prove that $f(ux) = \alpha = f^*(wu)$ and $f^*(uv) \notin \{f^*(xy), f^*(xz)\}$, then, we recolor edge wu as follows,

Subcase 2.1. If $f^*(wv) \notin f^*[w_1]$, then let $f(wu) \in C - f^*[v] - f^*(ww_1)$.

Subcase 2.2. If $f^*(wv) \in f^*[w_1]$, then $| f^*[v] \cup f^*[w_1] | \leq 5$, hence let $f(wu) \in C - f^*[v] - f^*[w_1]$.

The coloring of other edges is the same to f^*. It is clear f is a 6-adjacent vertex distinguishing edge coloring. ■

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Reference

[1] Zhang Zhongfu, Liu Linzhong and Wang jianfang, On the adjacent strong edge coloring of Graph, Appl. Math. Letter, 15 (2002), 623 ~ 626.

[2] Hamed Hatami, Δ+ 300 is a bound on the adjacent vertex distinguishing edge chromatic number, Journal of Combinatorial Theory, Series B, Volume 95, Issue 2, November 2005, Pages 246 ~ 256.

[3] S. Akbari, H. Bidkhori and N. Nosrati, r-Strong edge colorings of graphs, Discrete Mathematics, Volume 306, Issue 23, 6 December 2006, Pages 3005 ~ 3010.

[4] Mohammad R. Salavatipour A polynomial time algorithm for strong edge coloring of partial k-trees, Discrete Applied Mathematics, Volume 143, Issues 1-3, 30 September 2004, Pages 285 ~ 291.

[5] Andrzej Czygrinow and Brendan Nagle Strong edge colorings of uniform graphs Discrete Mathematics, Volume 286, Issue 3, 28 September 2004, Pages 219 ~ 223.

[6] Daniel W. Cranston Strong edge-coloring of graphs with maximum degree 4 using 22 colors Discrete Mathematics, Volume 306, Issue 21, 6 November 2006, Pages 2772 ~ 2778.

[7] M.Ghandehari, etc. Two upper bounds for the strong edge chromatic number. to appear.

[8] A note on the adjacent vertex distinguishing total chromatic number of series-paralled graph, Theoretical Computer Science, to appear.

[9] A.C.Burris and R.H.Schelp, Vertex distinguishing proper edge colorings, J. Graph Theory 26, No.2(1997), 73 ~ 83.

[10] Cristina Bazgan, etc., On the vertex distinguishing proper edge colorings of graphs, Journal of Combinatorial Theory, Series B 75(1999), 288 ~ 301.

[11] P.N.Balister, etc., Vertex distinguishing colorings of graphs with $\Delta(G) = 2$, Discrete Mathematics 252(2002) 17 ~ 29.

[12] M.Aigner, E.Triesch etc., Irregular assignments and vertex-distinguishing edge colorings of graphs, Ann. Discrete Math. 52 (1990), 1 ~ 9.

[13] M.Karonski, etc., Edge weights and vertex colors, J.Combin.Theory Ser. B 91(2004) 151 $\sim$ 157.

[14] Bondy J A, Graph Theory with Application, the Macmillan, Press Ltd, 1976.

[15] Wang Zhiwen, Yan Li-hong, Zhang Zhongfu, vertex distinguishing equitable total chromatic number of join graph, Acta Mathematicae Applicatae Sinica, English Series, Vol.23.No.3(2007) 445 $\sim$ 450.