The Super Vertex-Graceful Labeling Of Graphs

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Abstract. Lee and Wei defined super vertex-graceful labeling in 2006. In this paper, the generalized Butterfly Graph B_n^t and $C_n^{(t)}$ graph are discussed, the generalized butterfly Graph B_n^t is super vertex-graceful when $t(t > 0)$ is even, B_n^0 is super vertex-graceful when $n \equiv 0, 3(mod4)$; For $C_3^{(t)}$, there are: $C_3^{(t)}$ is super vertex-graceful if and only if $t = 1, 2, 3, 5, 7$. Moreover, we propose two conjectures on super vertex-graceful labeling.

Keywords: Generalized Butterfly Graph B_n^t; $C_3^{(t)}$ Graph; Super Vertex-Graceful Graph.

1. Introdution

In 1967, Rosa [16] first introduced the concept of graph labeling and proved some interesting results. In 1980, Graham and Sloane [11] further developed the methods and new notations on graph labeling. Up to now, it has been discovered that theory of labeling graphs can be applied to coding theory, X-ray crystallography, radar, astronomy, circuit design, communication network addressing, data base management, etc [4, 5, 9, 15, 18].

In 2005, Lee, Pan and Tsai [13] called a graph G with p vertices and q edges is vertex-graceful if there is a bijection $f : V(G) \to \{1, 2, \cdots, p\}$ such that the induced mapping g from E to Z_q defined by $g(uv) = (f(u) + f(v))(modq)$ is a bijection. In 2006, Lee and Wei [14] defined a graph $G(V, E)$ to be super vertex-graceful if there is a bijection f from V to $\{0, \pm 1, \pm 2, \cdots, \pm\frac{|V|-1}{2}\}$ when $|V|$ is odd and from V to $\{\pm 1, \pm 2, \cdots, \pm\frac{|V|}{2}\}$ when $|V|$ is even such that the induced edge labeling g defined by $g(uv) =$

$f(u)+f(v)$ over all edges uv is a bijection from E to $\{0, \pm 1, \pm 2, \cdots, \pm \frac{|E|-1}{2}\}$ when $|E|$ is odd and from E to $\{\pm 1, \pm 2, \cdots, \pm \frac{|E|}{2}\}$ when $|E|$ is even. They showed that $K_{1,n} \times P_2$ are not super vertex-graceful for n odd; for $n \geq 3$, $P_n^2 \times P_2$ is super vertex-graceful if and only if $n = 3, 4, 5$; $P_{n_1} \times P_{n_2} \times \cdots \times P_{n_m}$ is not super vertex-graceful for each of $m, n_1, n_2, \cdots .n_m$ at least 3; and $C_n \times C_m$ is not super vertex-graceful. They conjecture that $P_n \times P_n$ is super vertex-graceful for $n \geq 3$.

Lee et al discussed the edge-graceful spectrum of butterfly graph. For $C_3^{(t)}$, there are the following: $C_3^{(t)}$ is graceful if and only if $t \equiv 0, 1(mod4)$ [3]; $C_3^{(t)}$ is harmonious if and only if $t \neq 4k+2$ [11]; $C_3^{(t)}$ has sum number [8]; $C_3^{(t)}$ is even edge-graceful[10]; $C_3^{(t)}$ is strongly multiplicative[1]; $C_3^{(t)}$ is 3-equitable if and only if $t \neq 6k+3$ [7]; $em(C_3^{(t)} = 1$ if $t = 1$ and $em(C_3^{(t)} = 2t-1$ if $t > 1$ [12]; $C_3^{(t)}$ is super edge-magic if and only if $t = 3, 4, 5, 7$ [17]; $C_3^{(t)}$ has total magic cordial labeling [6]; $C_3^{(t)}$) is super (a, d)-antimagic total [2].

In the paper, we define the generalized butterfly graph B_n^t and prove that is super vertex-graceful when $t(t > 0)$ is even, B_n^0 is super vertex-graceful when $n \equiv 0, 3(mod4)$, $C_3^{(t)}$ is super vertex-graceful if and only if $t = 1, 2, 3, 5, 7$. Moreover, we conjectures: 1. the generalized butterfly graph is not super vertex-graceful when t is odd; 2. B_n^0 is not super vertex-graceful when $n \equiv 1, 2(mod4)$.

2. Preliminary

Definition 2.1. Let a graph $G(V, E)$ that has p vertices and q edges, if there is a bijection f from V to $\{0, \pm 1, \pm 2, \cdots, \pm \frac{|V|-1}{2}\}$ when $|V|$ is odd and from V to $\{\pm 1, \pm 2, \cdots, \pm \frac{|V|}{2}\}$ when $|V|$ is even such that the induced edge labeling g defined by $g(uv) = f(u) + f(v)$ $(u, v \in V,\ uv \in E)$ over all edges uv is a bijection from E to $\{0, \pm 1, \pm 2, \cdots, \pm \frac{|E|-1}{2}\}$ when $|E|$ is odd and from E to $\{\pm 1, \pm 2, \cdots, \pm \frac{|E|}{2}\}$ when $|E|$ is even, then graph $G(V, E)$ is called *super vertex − graceful graph.*

Definition 2.2. *Butterfly graph* is obtained by two even cycles of the same order sharing a common vertex with an arbitrary number of pendant edges attached at the common vertex.

Definition 2.3. *The generalizedbutterfly graph* is obtained by two cycles of the same order sharing a common vertex with an arbitrary number of pendant edges attached at the common vertex. The generalized butterfly graph is denoted by B_n^t, and illustrated in Fig.1. On B_n^t, the edges $u_1u_2, u_2, u_3, \cdots, u_nu_1, u_1u_{n+2}, u_{n+2}u_{n+3}, \cdots, u_{2n}u_1$ are denoted by $e_1, e_2, \cdots, e_n, e_{n+1}, \cdots, e_{2n}$, the edges $u_1v_1, u_1v_2, \cdots, u_1v_t$ are denoted by $e_{11}, e_{12}, \cdots, e_{1t}$.

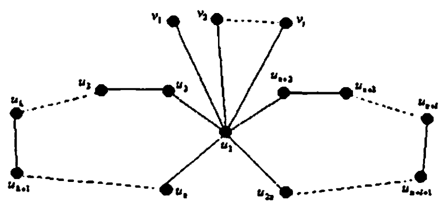

Fig.1. generalized butterfly graph B_n^t

Definition 2.4. *The* $C_3^{(t)}$ *graph* is obtained by the one-point union of t C_3. The $C_3^{(t)}$ Graph is illustrated in Fig.2. $C_3^{(t)}$is also call *friendshipgraph* or *Ductch* $t-$ *windmill.*

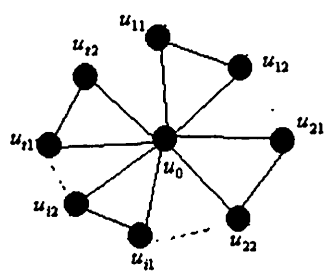

Fig.2. $C_3^{(t)}$ graph

3. Super Vertex-Graceful Graph

Theorem 3.1. If $t(t > 0)$ is even, then the generalized butterfly graph B_n^t is vertex-graceful.

Before we prove the theorem, first, we divide B_n^t into two part, one is B_n^2, another is star that the center vertex is u_1 and the others are $v_3, v_4, \cdots, v_t$; next, we define $f(u_1) = 0$ from beginning to end in the proof; the third, according to the symmetrical characteristic of B_n^2, we only give the labelings of $u_2, u_3, \cdots, u_n, v_1$, the labelings of $u_{n+2}, u_{n+3}, \cdots, u_{2n}, v_2$ are opposite number of the labelings of $u_2, u_3, \cdots, u_n, v_1$.

Lemma 3.1. When n is even, B_n^2 is super vertex-graceful.
Proof we consider the following two cases.
Case 1. $n \equiv 0(mod4)$. When $n = 4$, we define f as follows:
$f : \{u_2, u_3, u_4, v_1\} \to \{4, 1, -2, 3\}$,
it is easy to see that B_4^2 is super vertex-graceful.

When $n > 4$, let $f : \{u_2, u_n, v_1\} \to \{n, n-2, n-1\}$; for the vertices $u_3, u_4, \cdots, u_{\frac{n}{2}}$, if the vertex subscripts are odd, we label these vertices in proper order: $1, 2, \cdots, \frac{n}{4} - 1$, if the vertex subscripts are even, we label these vertices in proper order: $-(n-3), -(n-4), \cdots, -(\frac{3n}{4} - 1)$; let $f(u_{\frac{n}{2}+1}) = \frac{3n}{4} - 2$. For the vertices $u_{\frac{n}{2}+2}, u_{\frac{n}{2}+3}, \cdots, u_{n-1}$, if the vertex subscripts are odd, we label these vertices in proper order: $-\frac{n}{4}, -(\frac{n}{4} + 1), \cdots, -(\frac{n}{2} - 2)$, if the vertex subscripts are even, we label these vertices in proper order: $\frac{3n}{4} - 3, \frac{3n}{4} - 4, \cdots, \frac{n}{2}, -(\frac{n}{2} - 1)$. On the above rule, we can get the labeling set of the vertices $u_1, u_2, \cdots, u_n, v - 1$ is $\{0, n, 1, -(n-3), 2, -(n-4), \cdots, \frac{n}{4} - 1, -(\frac{3n}{4} - 1)\} \cup \{\frac{3n}{4} - 2, -\frac{n}{4}, \frac{3n}{4} - 3, -(\frac{n}{4}+1), \cdots, \frac{n}{2}, -(\frac{n}{2}-2), -(\frac{n}{2}-1), n-2, n-1\}$, depend upon definition 2.1 and the labelings of $u_{n+2}, u_{n+3}, \cdots, u_{2n}, v_2$ are opposite number of the labelings of $u_2, u_3, \cdots, u_n, v_1$, we can know that the vertex labelings of B_n^2 satisfy the demand from definition 2.1.

Now, we discuss edge labelings. By labeling the vertices defined above and the edge labeling rules, the labelings of the edges that are related to the vertices $u_2, u_3, \cdots, u_n, v_1$ are in proper order: $\{n, n+1, -(n-4), -(n-5), \cdots, -\frac{n}{2}\} \cup \{-1\} \cup \{\frac{n}{2} - 2, \frac{n}{2} - 3, \cdots, 2, -(n-3), \frac{n}{2} - 1, n-2\} \cup \{n-1\}$, together with the labelings of the edges related to the vertices $u_{n+2}, u_{n+3}, \cdots, u_{2n}, v_2$, we can get clearly that these edge labelings on B_n^2 also satisfy the demand from definition 2.1, so B_n^2 is the super vertex-graceful at this time.
Case 2. $n \equiv 2(mod4)$. $n = 6$, we define f as follows:

$f:\{u_2,u_3,\cdots,u_6,v_1\}\rightarrow\{6,1,-3,2,-5,4\}$,
it is easy to see that B_6^2 is super vertex-graceful.

When $n>6$, let $f:\{u_2,u_n,v_1\}\rightarrow\{n,n-1,n-2\}$; for the vertices $u_3,u_4,\cdots,u_{\frac{n}{2}-1}$, if the vertex subscripts are odd, we label these vertices in proper order: $1,2,\cdots,\frac{n-2}{4}-1$, if the vertex subscripts are even, we label these vertices in proper order: $-(n-3),-(n-4),\cdots,-(\frac{3n+2}{4}-1)$; let $f(u_{\frac{n}{2}})=\frac{3n+2}{4}-2$. For the vertices $u_{\frac{n}{2}+1},u_{\frac{n}{2}+2},\cdots,u_{n-1}$, if the vertex subscripts are odd, we label these vertices in proper order: $\frac{3n+2}{4}-3,\frac{3n+2}{4}-4,\cdots,\frac{n}{2},-(\frac{n}{2}-1)$, if the vertex subscripts are even, we label these vertices in proper order: $-\frac{n-2}{4},-(\frac{n-2}{4}+1),\cdots,-(\frac{n}{2}-2)$. On the above rule, we can get the labeling set of the vertices $u_1,u_2,\cdots,u_n,v-1$ is $\{0,n,1,-(n-3),2,-(n-4),\cdots,\frac{n-2}{4}-1,-(\frac{3n+2}{4}-1)\}\cup\{\frac{3n+2}{4}-2,-\frac{n-2}{4},\frac{3n+2}{4}-3,-(\frac{n-2}{4}+1),\cdots,\frac{n}{2},-(\frac{n}{2}-2),-(\frac{n}{2}-1),n-1,n-2\}$, depend upon definition 2.1 and the labelings of $u_{n+2},u_{n+3},\cdots,u_{2n},v_2$ are opposite number of the labelings of $u_2,u_3,\cdots,u_n,v_1$, we can know that the vertex labelings of B_n^2 satisfy the demand from definition 2.1.

For the edge labelings, we can also get the same conclusion by the proof of case 1.

Lemma 3.2. When n is odd, B_n^2 is super vertex-graceful.

Proof With lemma 3.1, we consider the following two cases too.

Case 1. $n\equiv 1(mod4)$. We consider the following two subcases.

Subcase 1.1. $\frac{n-1}{4}$ is even, let $f:\{u_2,u_n,v_1\}\rightarrow\{n,-(n-1),n-2\}$; for the vertices $u_3,u_4,\cdots,u_{\frac{n+1}{2}}$, if the vertex subscripts are odd, we label these vertices in proper order: $1,2,\cdots,\frac{n-1}{4}$, if the vertex subscripts are even, we label these vertices in proper order: $-(n-3),-(n-4),\cdots,-(\frac{3n+1}{4}-1)$; let $f(u_{\frac{n+1}{2}+1})=\frac{3n+1}{4}-3$, $f(u_{\frac{n+1}{2}+2})=-(\frac{n-1}{4}+1)$, $f(u_{\frac{n+1}{2}+3})=\frac{3n+1}{4}-2$, $f(u_{\frac{n+1}{2}+4})=-(\frac{n-1}{4}+3)$, for the vertices $u_{\frac{n+1}{2}+5},u_{\frac{n+1}{2}+6},\cdots,u_{n-4}$, each four divide into a group, let $f(u_{\frac{n+1}{2}+5})=\frac{3n+1}{4}-5$, $f(u_{\frac{n+1}{2}+6})=-(\frac{n-1}{4}+2)$, $f(u_{\frac{n+1}{2}+7})=\frac{3n+1}{4}-4$, $f(u_{\frac{n+1}{2}+8})=-(\frac{n-1}{4}+5)$, the vertex labelings in the other groups is defined that the vertex labelings in front of adjacent group minus 2. For u_{n-3},u_{n-2},u_{n-1}, let their labelings are: $\frac{n-1}{2},-(\frac{n-1}{2}-2),\frac{n-1}{2}+1$, so there are: if $1\le i\le\frac{n+1}{2}$, $0\le f(u_i)\le\frac{n-1}{4}$, or equals n, or $-(n-3)\le f(u_i)\le-(\frac{3n+1}{4}-1)$; for the vertex u_i ($\frac{n+1}{2}+1\le i\le n$), except for the first group and $u_{n-3}, u_{n-2}, u_{n-1}, u_n$, in the other groups, the first vertex labeling difference 1 with the third vertex labeling, and them difference 2 with the corresponding vertex labels adjacent to its front group, so these labelings are different successively, the maximum value is $\frac{3n+1}{4} - 4$, the minimum value is $\frac{n-1}{2} + 2$; the second vertex labeling even, the forth vertex labeling is odd (except for $-(\frac{n-1}{2} - 2)$), the maximum value is $-(\frac{n-1}{4} + 2)$, the minimum value is $-(\frac{n-1}{2} - 1)$. With first group and the last four vertex labelings, we can know that these vertex labelings are different each other, and also different with the labelings of the vertex u_i $(\frac{n+1}{2} + 1 \leq i \leq n)$ and v_1, the labelings of $u_{n+2}, u_{n+3}, \cdots, u_{2n}, v_2$ are opposite number of the labelings of $u_2, u_3, \cdots, u_n, v_1$, so the vertex labelings satisfy demand from definition 2.1.

Now, we discuss the edge labelings, the labelings of the edges that are related to the vertices $u_1, u_2, \cdots, u_{\frac{n-1}{2}}$ are in proper order: $\{n-2, -(n-1), n, n+1, -(n-4), -(n-5), \cdots, -\frac{n-1}{2}$, the labelings of the edges that are related to the vertices $u_{\frac{n+3}{2}}, u_{\frac{n+5}{2}}, \cdots, u_{n-1}$ are in proper order: $\{n-3, \frac{n-1}{2}-3, \frac{n-1}{2}-2, \frac{n-1}{2}-4\} \cup \{\frac{n-1}{2}-7, \frac{n-1}{2}-6, \frac{n-1}{2}-5, \frac{n-1}{2}-8\} \cup \{\frac{n-1}{2}-7, \frac{n-1}{2}-6, \frac{n-1}{2}-5, \frac{n-1}{2}-8\}-11, \frac{n-1}{2}-8\}-10, \frac{n-1}{2}-9, \frac{n-1}{2}-12\} \cup \cdots \cup \{5,6,7,4\} \cup \{1,2,3,-(\frac{n-1}{2}-1)\}$, we can get clearly , regardless of the sign of these labelings, the labelings are $1, 2, \cdots, n+1$, together with the labelings of the edges related to the vertices $u_{n+2}, u_{n+3}, \cdots, u_{2n}, v_2$, B_n^2 is super vertex-graceful at this time.

Subcase 1.2. $\frac{n-1}{4}$ is odd, $n = 5$, we define f as follows: $f : \{u_2, u_3, u_4, u_5, v_1\} \rightarrow \{5, 1, -2, 4, 3\}$; $n = 13$, we define f as follows: $f : \{u_2, u_3, \cdots, u_{13}, v_1\} \rightarrow \{13, 1, -10, 2, -9, 3, -5, 8, -4, -6, 7, -12, 11\}$, we can know B_5^2, B_{13}^2 are super vertex-graceful.

$n > 13$, the labelings rule of the vertices $u_1, u_2, \cdots, u_{n-6}, u_n, v_1$ is the same as the labelings rule of $\frac{n-1}{4}$ is even, $f : \{u_{n-5}, u_{n-4}, \cdots, u_{n-1}\} \rightarrow \{\frac{n+3}{2}, -\frac{n-7}{2}, \frac{n-1}{2}, -\frac{n-3}{2}, \frac{n+1}{2}\}$, similar to subcase 1.1, we know that B_n^2 is super vertex-graceful at this time.

Case 2. $n \equiv 3(mod4)$. $n = 3$, we define f as follow: $f : \{u_2, u_3, v_1\} \rightarrow \{3, 1, 2\}$, it is easy to see that B_3^2 is super vertex-graceful.

When $n > 3$, let $f : \{u_2, v_1\} \rightarrow \{n, n-1\}$, for the vertices $u_3, u_4, \cdots, u_{\frac{n+1}{2}}$, if the vertex subscripts are odd, we label these vertices in proper order: $1, 2, \cdots, \frac{n-3}{4}$, the vertex subscripts are even, we label these vertices in proper order: $-(n-2), -(n-3), \cdots, -\frac{3n-1}{4}$; for the vertices $u_{\frac{n+3}{2}}, u_{\frac{n+5}{2}}, \cdots, u_{n-1}$, the vertex subscripts are even, we label these vertices in proper order: $-\frac{n+1}{4}, -\frac{n+5}{4}, \cdots, \frac{n-3}{2}$, the vertex subscripts are odd, we label these vertices in proper order: $\frac{3n-5}{4}, \frac{3n-9}{4}, \cdots, \frac{n+1}{2}, -\frac{n-1}{2}$. By the same argument in the case 1, we can know that these vertex labelings satisfy the demand from definition 2.1.

Now we discuss the edge labelings of B_n^2. the edge labelings that are related to the vertices $u_1, u_2, \cdots, u_n$,
v_1 are in proper order: $\{n, n+1, -(n-3), -(n-4), \cdots, -\frac{n+1}{2}, -1, \frac{n-3}{2}, \frac{n-5}{2}, \cdots, 2, -(n-2), n-1$; similarly, we can get the same conclusion by following the proof of the case 1.

Overall, B_n^2 is super vertex-graceful.

For B_n^t $(t(t>0)$ is even), according to the front rule that we divide B_n^t into two part, one is B_n^2, another is a star that the center vertex is u_1 and the others are $v_3, v_4, \cdots, v_t$; first, we modify: $f(u_2) = \frac{|V|-1}{2}$, $f(u_{n+2}) = -\frac{|V|-1}{2}$; next, for the star, we define the vertex labelings are: $\pm(\frac{|V|-1}{2}-1), \pm(\frac{|V|-1}{2}-2), \cdots, \pm(n+1)$. We can know that the star is super vertex-graceful, hence, the generalized butterfly graph B_n^t $(t(t>0)$ is even) is super vertex-graceful.

Theorem 3.2. When $n \equiv 0, 3(mod 4)$, B_n^0 is super vertex-graceful.

In the following proof, using the same method and Theorem 3.1, first, we define the labelings of vertex u_{1j} $(1 \le j \le n-1)$ on B_n^0, the labelings of vertex u_{2j} $(1 \le j \le n-1)$ is opposite number of the labelings of vertex u_{1j} $(1 \le j \le n-1)$, and $f(u_0) = 0$.

Lemma 3.3. When $n \equiv 0(mod 4)$, B_n^0 is super vertex-graceful.

Proof **Case 1.** $\frac{n}{4}$ is even. First, we consider the vertex u_{1j} $(1 \le j \le \frac{n}{2})$, if j is odd, we define the vertex u_{1j} $(1 \le j \le \frac{n}{2})$ in proper order: $n-1, -(n-2), -(n-3), \cdots, -\frac{3n}{4}$, if j is even, we define the vertex u_{1j} $(1 \le j \le \frac{n}{2})$ in proper order: $1, 2, \cdots, \frac{n}{4}$. For the vertex u_{1j} $(\frac{n}{2}+1 \le j \le n-4)$, each four divide into a group, let $f : \{u_{1(\frac{n}{2}+1)}, u_{1(\frac{n}{2}+2)}, u_{1(\frac{n}{2}+3)}, u_{1(\frac{n}{2}+4)} \to \{\frac{3n}{4}-2, -(\frac{n}{4}+2), \frac{3n}{4}-1, -(\frac{n}{4}+1)\}$, the vertex labelings in the other groups is defined that the vertex labelings in front of adjacent group minus 2, $f: \{u_{1n-3}, u_{1n-2}, u_{1n-1}\} \to \{\frac{n}{2}, -(\frac{n}{2}+1), \frac{n}{2}-1\}$. Thus, we can get that the labelings set of the vertex u_{ij} $(1 \le j \le n-1)$ is $\{n-1, 1, -(n-2), 2, -(n-3), \cdots, -\frac{3n}{4}, \frac{n}{4}\} \cup \{\frac{3n}{4}-2, -(\frac{n}{4}+2), \frac{3n}{4}-1, -(\frac{n}{4}+1)\} \cup \{\frac{3n}{4}-4, -(\frac{n}{4}+4), \frac{3n}{4}-3, -(\frac{n}{4}+3)\} \cup \cdots \cup \{\frac{n}{2}+2, -(\frac{n}{2}-2), \frac{n}{2}+3, -(\frac{n}{2}-3)\} \cup \{\frac{n}{2}, -(\frac{n}{2}+1), \frac{n}{2}-1\}$, we can get clearly , regardless of the sign of these labelings, they are different each other, and the maximum is $n-1$, the minimum is 1, together with the labelings of the vertex u_{2j} $(1 \le j \le n-1)$, these labelings satisfy the demand from definition 2.1.

Next, we consider the edge labeling, for the edge $u_0u_{11}, u_{1(j-1)}u_{1j}$ $(2 \le j \le \frac{n}{2})$, their labelings are in proper order: $n-1, n, -(n-3), -(n-4), \cdots, -\frac{n}{2}$, $g(u_{1\frac{n}{2}}u_{1(\frac{n}{2}+1)}) = n-2$, for the edge $u_{1j}u_{1(j+1)}$ $(\frac{n}{2}+1 \le j \le n-4$, their labelings are in proper order: $\{\frac{n}{2}-4, \frac{n}{2}-3, \frac{n}{2}-2, \frac{n}{2}-5\} \cup \{\frac{n}{2}-8, \frac{n}{2}-7, \frac{n}{2}-6, \frac{n}{2}-9\} \cup \cdots \cup \{4,5,6,3\}$, the labelings of edges $u_{1n-3}u_{1n-2}, u_{1n-2}u_{1n-1}, u_{1n-1}u_0$ are $1, 2, \frac{n}{2}-1$, together with the labelings of the edges $u_0u_{21}, u_{2j}u_{2j+1}$ $(1 \le j \le n-2), u_{2n-1}u_0$, these labelings satisfy the demand from definition 2.1, so $C_n^{(2)}$ is super vertex-graceful at this time.

Case 2. $\frac{n}{4}$ is odd, $n = 4$, let $f: \{u_{11}, u_{12}, u_{13}\} \to \{3, 1, -2\}$.

When $n > 4$, for the vertices $u_{11}, u_{12}, \cdots, u_{1\frac{n}{2}}$, the rule of their labelings are defined as its in the case 1, for the vertices $u_{1(\frac{n}{2}+1)}, u_{1(\frac{n}{2}+2)}, \cdots, u_{1n-2}$, the rule of their labelings are defined as $u_{1(\frac{n}{2}+1)}, u_{1(\frac{n}{2}+2)}, \cdots, u_{1n-4}$, let $f(u_{1n-1}) = \frac{n}{2}-1$, we can get the same conclusion by following the proof of the case 1.

Lemma 3.4. When $n \equiv 3(mod4)$, B_n^0 is super vertex-graceful.

Proof We consider the vertex u_{1j} $(1 \le j \le \frac{n-1}{2})$, if j is odd, we define the vertex u_{1j} $(1 \le j \le \frac{n}{2})$ in proper order: $n-1, -(n-2), -(n-3), \cdots, -\frac{3n-1}{4}$, if j is even, we define the vertex u_{1j} $(1 \le j \le \frac{n}{2})$ in proper order: $1, 2, \cdots, \frac{n-3}{4}$. For the vertex u_{1j} $(\frac{n+1}{2} \le j \le n-1)$, if j is odd, we define the vertex u_{1j} $(1 \le j \le \frac{n}{2})$ in proper order: $-\frac{n+1}{4}, -\frac{n+5}{4}, \cdots, -\frac{n-3}{2}$, if j is even, we define the vertex u_{1j} $(1 \le j \le \frac{n}{2})$ in proper order: $\frac{3n-1}{4}-1, \frac{3n-1}{4}-2, \cdots, \frac{n+1}{2}, -\frac{n-1}{2}$. Thus, we get the labeling set of the vertex u_{1j} $(1 \le j \le n-1)$ is $\{n-1, 1, -(n-2), 2, -(n-3), \cdots, \frac{n-3}{4}, -\frac{3n-1}{4}\} \cup$

$\{\frac{3n-1}{4}-1, -\frac{n+1}{4}, \frac{3n-1}{4}-2, -\frac{n+5}{4}, \cdots, \frac{n+1}{2}, -\frac{n-3}{2}, -\frac{n-1}{2}\}$, together with the labelings of the vertex u_{2j} $(1 \le j \le n-1)$, these labelings satisfy the demand from definition 2.1.

Next, we consider the edge labeling, for the edge $u_0u_{11}, u_{1(j-1)}u_{1j}$ $(2 \le j \le n-1)$, the set of these edges labelings is $\{n-1, n, -(n-3), -(n-4), \cdots, -\frac{n+1}{2}\} \cup \{-1\} \cup \{\frac{n-3}{2}, \frac{n-5}{2}, \cdots, 2, -(n-2), -\frac{n-1}{2}\}$, we can get that they are different each other in regardless of the sign of these labelings, and the maximum is n, the minimum is 1, the labelings of the edges $u_0u_{21}, u_{2(j-1)}u_{2j}$ $(2 \le j \le n-1$ are opposite number of the above edge labelings, so the edge labelings satisfy the demand from definition 2.1.

Overall, When $n \equiv 0, 3(mod4)$, B_n^0 is super vertex-graceful.

Theorem 3.3. $C_3^{(t)}$ is super vertex-graceful if and only if $t = 1, 2, 3, 5, 7$.

Proof Be base on the characteristic of $C_3^{(t)}$, if $C_3^{(t)}$ is super vertex-graceful, then $f(u_0) = 0$, otherwise it is easily obtained contradiction.

Suppose t is even, $C_3^{(t)}$ is super vertex-graceful, then the vertex labelings set of $C_3^{(t)}$ is $\{0, \pm1, \pm2, \cdots, \pm t\}$, the edge labelings set is $\{\pm1, \pm2, \cdots, \pm\frac{3t}{2}\}$. For u_{k1}, u_{k2}, the sign of $f(u_{k1})$ must be same as the sign of $f(u_{k2})$, otherwise, the labeling of $u_{k1}u_{k2}$ satisfy: $1 \le g(u_{k1}u_{k2}) \le t$, or $-t \le g(u_{k1}u_{k2}) \le -1$, thus, the $g(u_{k1}u_{k2})$ must be equal to the labeling of some edge u_0u_{m1}, or u_0u_{l2}, contradiction. Hence, there are $\frac{t}{2}$ C_3, the labelings of vertex u_{k1}, u_{k2} on them are $1, 2, \cdots, t$, and $t+1 \le g(u_{k1}u_{k2}) \le \frac{3t}{2}$, the sum of these labelings is $t+1+t+2+\cdots+t+\frac{t}{2} = 1+2+\cdots+\frac{t}{2}+\frac{t}{2}t$, by definition 2.1, there should be

$$1+2+\cdots+\frac{t}{2}+\frac{t}{2}t = 1+2+\cdots+t,$$

so we should find out $t = 2$.

Suppose t is odd, $C_3^{(t)}$ is super vertex-graceful, then the vertex labelings set of $C_3^{(t)}$ is $\{0, \pm1, \pm2, \cdots, \pm t\}$, the edge labelings set is $\{\pm1, \pm2, \cdots, \pm\frac{3t-1}{2}\}$. the discussion is similar to the above discussion, there is a pair of vertices u_{i1}, u_{i2}, their labelings contrary to each other (otherwise there is no edge label 0), suppose $f(u_{i1}) = -f(u_{i2}) = x$, to the remaining vertices, the signs of vertices u_{k1}, u_{k2} labelings are same, so there are $\frac{t-1}{2}C_3$, the labelings of vertex u_{k1}, u_{k2} on them are $1, 2, \cdots, t$, and $t+1 \le g(u_{k1}u_{k2}) \le$

$\frac{3t-1}{2}$, the sum of $u_{k1}u_{k2}$ labelings is $t+1+t+2+\cdots+t+\frac{t-1}{2} = 1+2+\cdots+\frac{t-1}{2}+\frac{t}{2}t-\frac{t}{2}$, by definition 2.1, there should be

$$1+2+\cdots+\frac{t-1}{2}+\frac{t}{2}t-\frac{t}{2} = 1+2+\cdots+t-x,$$

so we get $x = \frac{8t+1-t^2}{8}$, because of $1 \leq x \leq t$, thus

$$8 \leq 8t+1-t^2 \leq 8t,$$

$$1 \leq t \leq 7,$$

hence, $t = 1, 3, 5, 7$. Overall, $C_3^{(t)}$ is not super vertex-graceful expect for $t = 1, 2, 3, 5, 7$. Now, we define the super vertex-graceful labelings of $C_3^{(t)}$ when $t = 1, 2, 3, 5, 7$.

$t = 1, f : \{u_0, u_{11}, u_{12}\} \to \{0, 1, -1\}$;

$t = 2, f : \{u_0, u_{11}, u_{12}, u_{21}, u_{22}\} \to \{0, 1, 2, -1, -2\}$;

$t = 3, f : \{u_0, u_{11}, u_{12}, u_{21}, u_{22}, u_{31}, u_{32}\} \to \{0, 2, -2, 3, 1, -3, -1\}$;

$t = 5, f : \{u_0, u_{11}, u_{12}, u_{21}, u_{22}, \cdots, u_{51}, u_{52}\} \to$
$\{0, 2, -2, 5, 1, 4, 3, -5, -1, -4, -3\}$;

$t = 7, f : \{u_0, u_{11}, u_{12}, u_{21}, u_{22}, \cdots, u_{71}, u_{72}\} \to$
$\{0, 1, -1, 7, 3, 5, 4, 6, 2, -7, -3, -5, -4, -6, -2\}$.

Summarizing the conclusions above and doing some validation, we propose the following two conjectures:

Conjecture 1. The butterfly graph B_n^t (t is odd) is super vertex-graceful.

Conjecture 2. When $n \equiv 1, 2 (mod 4)$, B_n^0 is not super vertex-graceful.

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