Bicoloured ordered trees, non-nesting partitions and non-crossing partitions

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Abstract: Using a new way to label edges in a bicoloured ordered tree, we introduce a bijection between bicoloured ordered trees and non-nesting partitions. Consequently, enumerative results of non-nesting partitions are derived. Together with another bijection given before, we obtain a bijection between non-nesting partitions and non-crossing partitions specified with four parameters.

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1 Introduction

An ordered tree can be defined inductively as an unlabelled rooted tree whose principal subtrees (the subtrees obtained by removing the root) are ordered trees and have been assigned a linear order (from left to right) among themselves. A bicoloured ordered tree is an ordered tree in which even height vertices are assigned by one colour and odd height ones by the other, where height of a vertex is the distance from it to the root ([6]).

A partition of the set $[n] = \{1, 2, \ldots, n\}$ is a collection $\pi = \{B_1, B_2, \ldots, B_k\}$ of non-empty disjoint subsets of $[n]$, called blocks, whose union is $[n]$. A partition is called non-nesting if there do not exist four numbers $a < b < c < d$ such that a and d are consecutive elements of a block and b and c are both contained in another block. A partition is called non-crossing if there do not exist four numbers $a < b < c < d$ such that a and c are in one block of the partition and b and d are in another block. A (complete) matching on $[2n]$ is a partition of $[2n]$ into n two-element blocks.

Bijections between non-crossing partitions and ordered trees were presented ([3, 8]), a bijection between ordered trees and bicoloured ordered trees was presented ([6]), meanwhile there are also bijections between non-crossing partitions and non-nesting partitions ([1, 9, 10, 11]), which implies that there could be a direct connection between bicoloured ordered trees and non-nesting partitions. We aim to exhibit a bijection between them. Enumeration of non-crossing partitions, non-nesting partitions has been studied ([1, 4, 5, 9, 11]). Using two different labelling algorithms, Liu et al. ([7]) introduced two bijections between bicoloured ordered trees and non-crossing partitions, and derived some new enumerative results of non-crossing partitions. From the bijection given here, we get some enumerative results of non-nesting partitions; together with one bijection in [7], we obtain in effect a bijection between non-nesting partitions and non-crossing partitions specified with four parameters.

2 Bijections

In an ordered tree T, the number of subtrees of a vertex u is called the degree $d(u)$ of u. A vertex is called a leaf if its degree is 0, otherwise an internal vertex. Suppose any edge in an ordered tree has a direction leading away from the root. Then the in-degree of any vertex u is 1 (with the root as exception) and the out-degree of u is $d(u)$. We denote the directed edge flowing from u to v by $e =< u, v >$. For a vertex u with linearly ordered (from left to right) subtrees $T_1, T_2, \ldots, T_m$, whose roots are $v_1, v_2, \ldots, v_m$ respectively, we call u the parent of $v_i (1 \leq i \leq m)$, $v_i (1 \leq i \leq m)$ the children of u, v_m the rightmost child and for $2 \leq i \leq m$, v_i the closest older brother of v_{i-1}; define the claw subtree of T centered by u (denoted by $CT(u)$) to be the subgraph of T induced by the edge set $\{< v_0, u >\} \bigcup \{< u, v_1 >, < u, v_2 >, \ldots, < u, v_m >\}$, where $< v_0, u >$ is the possible edge flowing to u, that is $\{< v_0, u >\} = \emptyset$ when u is the root of T. For a non-root even height vertex u with parent v, the indicating edge of u is defined by the edge $< w, v >$ where w is the parent of v if u is the rightmost child of v, otherwise by $< v, u_0 >$ where u_0 is the closest older brother of u.

For a block B in a partition π, we denote by $|B|$ the number of elements in B. B is called singleton if $|B| = 1$, otherwise non-singleton. The smallest element in B is called the leader of B (denoted by $l(B)$). Two different blocks B_i and B_j are said to be adjacent if $|l(B_i) - l(B_j)| = 1$. A block run is a maximal sequence of blocks $B_{i_1}, B_{i_2}, \ldots, B_{i_t}$ such that any two consecutive blocks are adjacent. For example, $\pi_1 =\{\{1,3,7,11\},\{2,5,9\},\{4,8\},\{6\}, \{10,14\},\{12\},\{13\}\}$ is a non-nesting partition of [14] into 7 blocks 4 of which are non-singleton, with 5 block runs; $\pi_2 = \{\{1,10,11,14\},\{2\},\{3,4,7\}, \{5\},\{6\},\{8\},\{9\},\{12\},\{13\}\}$ is a non-crossing partition of [14] into 9 blocks 2 of which are non-singleton, with 4 block runs.

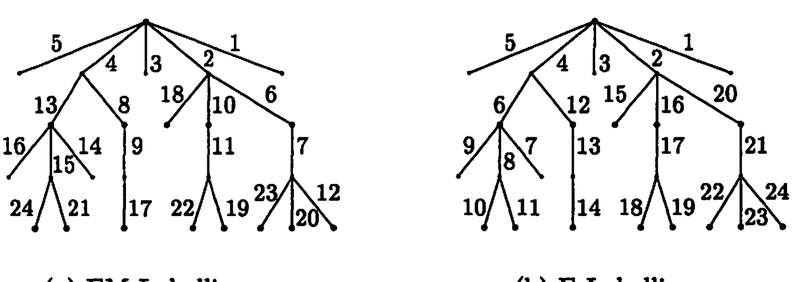
Using two different ways (E-Labelling and O-Labelling) to assign numbers to edges in a bicoloured ordered tree, Liu et al. ([7]) presented two bijections between bicoloured ordered trees and non-crossing partitions. In order to present a bijection between bicoloured ordered trees and non-nesting partitions, first we introduce another different labelling algorithm. The main difference between E-Labelling and that one given here is that even height vertices are selected to consider by different styles.

Given a bicoloured ordered tree T with $n+1-k$ even height vertices and k odd height ones, the procedure to attach numbers $1, 2, \ldots, n$ to the edges in T can be described as follows, where consecutive numbers are assigned to edges which may constitute a claw subtree.

Even-Height-Vertex-Centered-Modifying-Labelling (EM-Labelling): suppose u is the root of T, first label the edges in the claw subtree $CT(u)$ from right to left with the smallest not yet used number consecutively; next, other even height vertices are selected to consider inductively. Each time, among the remaining even height vertices whose indicating edges have been labelled, we find that one with the smallest indicating edge. Once an even height vertex (say w) is chosen, we label the edges in $CT(w)$ in a clockwise direction by beginning at the edge flowing to w, with the smallest not yet used number consecutively. That is, if the parent of w is v_0 and the linearly ordered (from right to left) children are $v_1, v_2, \ldots, v_m$, then first assign the smallest not yet used number, say i, to edge $< v_0, w >$ and consequently assign $i+1, i+2, \ldots, i+m$ to $< w, v_1 >, < w, v_2 >, \ldots, < w, v_m >$ respectively.

Now we give a brief review of the E-Labelling algorithm in [7]: traverse the tree in preorder (visit the root, then traverse its subtrees from left to right), whenever encountering an even height vertex u the first time, we label the edges in the claw subtree $CT(u)$ in a clockwise direction similarly.

Fig.1 is an illustration of the above two different labelling algorithms.



(a) EM-Labelling

(b) E-Labelling

Fig.1. Two labelling algorithms on edges of a tree T.

Theorem 2.1 *There is a bijection between the set of bicoloured ordered trees with k odd height vertices r of which are internal and $n+1-k$ even height ones s of which are internal and the set of non-nesting partitions of $[n]$ into k blocks r of which are non-singleton with s block runs.*

Proof. We first give the procedure to construct a non-nesting partition π from a bicoloured ordered tree T.
(1) Label the edges in T with numbers $1, 2, \ldots, n$ by the EM-Labelling algorithm.
(2) For each odd height vertex u with degree t, let the set of labels of those $t+1$ edges in $CT(u)$ be a block of the partition π.

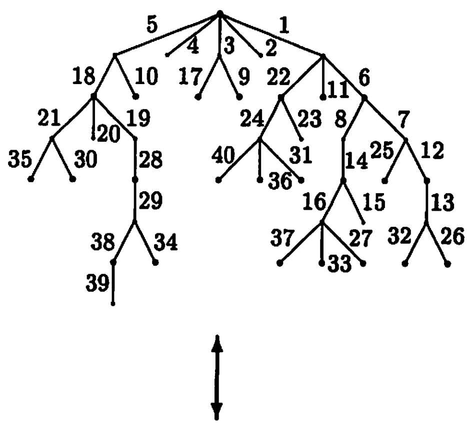
Since in T there are k odd height vertices r of which are internal, we obtain a partition of $[n]$ into k blocks r of which are non-singleton. For a block $\{p_0, p_1, \ldots, p_t\}$ ($p_0 < p_1 < \cdots < p_t$) in the partition π, suppose the edges in the corresponding claw subtree $CT(u)$ are $e_0 =< v_0, u >, e_1 =< u, v_1 >, \ldots, e_t =< u, v_t >$, where $v_1, \ldots, v_t$ are linearly ordered (from right to left) children of u. Then e_i ($0 \leq i \leq t$) must have been labelled by p_i. To prove that π is non-nesting, it suffices to show that for any number q_0, satisfying $p_i < q_0 < p_{i+1}$ ($0 \leq i \leq t-1$), there does not exist any other number q_1 such that $p_i < q_1 < p_{i+1}$ and q_0 and q_1 are in the same block. Without loss of generality, suppose $q_0 < q_1$, the block containing q_0 is $\{\ldots, q_0, q_1, \ldots\}$ ($\cdots < q_0 < q_1 < \cdots$), the edge being labelled by q_0 is $< w, x >$, and y is the even height vertex whose indicating edge is $< w, x >$. Obviously, if w is an even height vertex, then y is the rightmost child of x and $< x, y >$ is labelled by q_1; otherwise, x is the closest older brother of y and $< w, y >$ is labelled by q_1. By the EM-labelling algorithm, we have (i) firstly p_i is assigned to e_i, next q_0 to $< w, x >$; (ii) as e_i is the indicating edge of v_{i+1}, $< w, x >$ is the indicating edge of y and $p_i < q_0$, in the following steps e_{i+1} is labelled by p_{i+1} before the edges in $CT(y)$ are done. This leads to a conclusion that q_1 must be bigger than p_{i+1}. Moreover, to an even height internal vertex u with linearly ordered (from right to left) children $v_1, v_2, \ldots, v_m$, we have that if $< u, v_1 >$ is labelled by some number i, then $< u, v_2 >, < u, v_3 >, \ldots, < u, v_m >$ must have been labelled by $i+1, i+2, \ldots, i+m-1$ respectively, which will be leaders of different blocks $B_{j_1}, B_{j_2}, \ldots, B_{j_m}$ respectively, which result in a block run. Therefore the non-nesting partition obtained from T contains s block runs, which is the desired.

Conversely, to each block B in a partition π, a claw tree centered by some vertex (say u) with $|B|$ edges (denoted by T_B) shall be constructed where the edge flowing to u is labelled by $l(B)$ and other edges are labelled by the remaining elements in B increasingly from right ro left. That is, if $B = \{p_0, p_1, \ldots, p_t\}(p_0 < p_1 < \cdots < p_t)$, a claw tree T_B centered by u which has a parent v_0 and linearly ordered (from right to left) children

$v_1, v_2, \ldots, v_t$ would be constructed, where $< v_0, u >$ is labelled by p_0 and $< u, v_i >$ $(1 \leq i \leq t)$ by p_i. These corresponded claw trees may be put inductively in the suitable places to get the bicoloured ordered tree T as follows.
(1) Find the block B_1 in π containing number 1 and construct the corresponding claw tree T_{B_1}. Let T_{B_1} be a claw subtree of T such that their roots are identical.
(2) Find the block B_2 in π that contains the smallest remaining element, say x, and construct the claw tree T_{B_2}.
(A) If $x - 1$ has been assigned to some edge $< u, v >$ where u is an even height vertex, merge T_{B_2} and T_{B_1} by putting T_{B_2} on the left-hand side of $< u, v >$ and identifying the root of T_{B_2} with u. We shall call this operation a left-horizontal merge to T_{B_2}.
(B) Otherwise, i.e. $x - 1$ has been assigned to $< u, v >$ where v is an even height vertex, merge T_{B_2} and T_{B_1} by putting T_{B_2} underneath v and identifying the root of T_{B_2} with v. We shall call this operation a vertical merge to T_{B_2}.
(3) Repeat (2) until all blocks in π are considered.

Since a claw subtree with t edges is added corresponding to a block of t elements and an odd height vertex is added after either a left-horizontal merge or a vertical merge, when all k blocks in π are considered, we get a bicoloured ordered tree with k odd height vertices and $n+1-k$ even height vertices. Moreover, a singleton block leads to an odd height leaf and a non-singleton block to an odd height internal vertex. Since a left-horizontal merge is conducted to a claw subtree T_{B_i} if and only if $l(B_i) - 1$ is a leader of some other block $B_j (i \neq j)$, a block run corresponds to consecutive left-horizontal merges. Furthermore, to s different block runs, $s - 1$ vertical merges are needed, which lead to s even height internal vertices. This means that after $s - 1$ vertical merges and $k - s$ left-horizontal merges, the eventually obtained bicoloured ordered tree is the required. ∎

Example 2.2 *Fig. 2 shows a bicoloured ordered tree T with 24 even height vertices 8 of which are internal and 17 odd height vertices 11 of which are internal and the corresponding non-nesting partition π of* [40] *into 17 blocks 11 of which are non-singleton with 8 block runs, where the edges in T are assigned with numbers by the EM-Labelling algorithm.*



π={{1,6,11,22},{2},{3,9,17},{4},{5,10,18},{7,12,25},{8,14},{13,26,32},{15},
{16,27,33,37},{19,28},{20},{21,30,35},{23},{24,31,36,40},{29,34,38},{39}}

Fig. 2.

Lemma 2.3 *([2]) The number of bicoloured ordered trees with k odd height vertices r of which are internal and $n+1-k$ even height ones s of which are internal equals*

$$\frac{1}{n+1-k}\binom{k}{r}\binom{n+1-k}{s}\binom{n-1-k}{r-1}\binom{k-1}{s-1}.$$

From Theorem 2.1 and Lemma 2.3, we have

Corollary 2.4 *The number of non-nesting partitions of $[n]$ into k blocks r of which are non-singleton with s block runs equals*

$$\frac{1}{n+1-k}\binom{k}{r}\binom{n+1-k}{s}\binom{n-1-k}{r-1}\binom{k-1}{s-1}.$$

Summing the number in Corollary 2.4 over s and over r respectively, we obtain

Corollary 2.5 *The number of non-nesting partitions of $[n]$ into k blocks r of which are non-singleton equals*

$$\frac{1}{n+1-k}\binom{k}{r}\binom{n-1-k}{r-1}\binom{n}{k}.$$

Corollary 2.6 *The number of non-nesting partitions of* $[n]$ *into* k *blocks with* s *block runs equals*

$$\frac{1}{n+1-k}\binom{k-1}{s-1}\binom{n+1-k}{s}\binom{n-1}{k-1}.$$

Using the E-Labelling algorithm, Liu et al. presented the following bijection, whose proof is analogous to that of Theorem 2.1, with the main difference being that the "non-crossing" property had been stated instead.

Lemma 2.7 *([7]) There is a bijection between the set of bicoloured ordered trees with* k *odd height vertices* r *of which are internal and* $n+1-k$ *even height ones* s *of which are internal and the set of non-crossing partitions of* $[n]$ *into* k *blocks* r *of which are non-singleton with* s *block runs.*

For example, from the tree T in Example 2.2, we get a non-crossing partition:

$$\begin{aligned}\pi' = \{&\{1,20,26,27\},\{2\},\{3,18,19\},\{4\},\{5,6,17\},\{7,12\},\\ &\{8\},\{9,10,11\},\{13,14,16\},\{15\},\{21\},\{22,23,24,25\},\\ &\{28,36,37\},\{29,30\},\{31\},\{32,33,34,35\},\{38,39,40\}\},\end{aligned}$$

which contains 17 blocks 11 of which are non-singleton and 8 block runs.

Combining the bijections in Theorem 2.1 and Lemma 2.7, we establish a correspondence between non-nesting partitions and non-crossing partitions specified with several parameters.

Corollary 2.8 *There is a bijection between the set of non-nesting partitions of* $[n]$ *into* k *blocks* r *of which are non-singleton with* s *block runs and the set of non-crossing partitions of* $[n]$ *into* k *blocks* r *of which are non-singleton with* s *block runs.*

As a (complete) matching on $[2n]$ is a partition of $[2n]$ into n blocks each of which contains two elements exactly, from Corollary 2.4 we have

Corollary 2.9 *The number of non-nesting matchings on* $[2n]$ *with* s *block runs equals*

$$N_{n,s} = \frac{1}{n+1}\binom{n+1}{s}\binom{n-1}{s-1},$$

which is the famous Narayana number.

From Corollary 2.8, we have

Corollary 2.10 *There is a bijection between the set of non-nesting matchings on* $[2n]$ *with* s *blocks runs and the set of non-crossing matchings on* $[2n]$ *with* s *block runs.*

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