Weakly convex and convex domination numbers of some products of graphs

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Abstract

If $G = (V, E)$ is a simple connected graph and $a, b \in V$, then a shortest $(a - b)$ path is called a $(u - v)$-*geodesic*. A set $X \subseteq V$ is called *weakly convex* in G if for every two vertices $a, b \in X$ exists $(a - b)$- geodesic whose all vertices belong to X. A set X is *convex* in G if for every $a, b \in X$ all vertices from every $(a - b)$-geodesic belong to X. The *weakly convex domination number* of a graph G is the minimum cardinality of a weakly convex dominating set in G, while the *convex domination number* of a graph G is the minimum cardinality of a convex dominating set in G. In this paper we consider weakly convex and convex domination numbers of Cartesian product, join and corona of some classes of graphs.
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1 Definitions

Here we consider simple undirected and connected graphs $G = (V, E)$ with $|V| = n$. The *open neighbourhood* $N_G(v)$ of a vertex $v \in V$ is the set of all vertices adjacent to v and the *closed neighbourhood* $N_G[v]$ of a vertex $v \in V$ is the set $N_G(v) \cup \{v\}$. The degree of a vertex v is $d_G(v) = |N_G(v)|$ and a vertex of degree $n - 1$ is called *a universal vertex*. For a set $X \subseteq V$, the *neighbourhood* $N_G(X)$ is defined to be $\bigcup_{v \in X} N_G(v)$. A subset D of V is *dominating* if every vertex of $V - D$ has at least one neighbour in D. Let $\gamma(G)$ be the minimum cardinality of a dominating set of G.

The *distance* $d_G(u, v)$ between two vertices u and v in a connected graph G is the length of a shortest $(u - v)$ path in G. A $(u - v)$ path of length $d_G(u, v)$ is called a $(u - v)$-*geodesic*. Let us denote by $\mathcal{P}_G(u, v)$ the set of all $(u - v)$-geodesics in G.

A set $X \subseteq V$ is called *weakly convex* in G, if there exists $(a - b)$-geodesic whose vertices all belong to X. A set X is *convex* in G if for all $a, b \in X$ all vertices from every $(a - b)$-geodesic belong to X. A set $X \subseteq V$ is a *weakly convex dominating set* in G if X is weakly convex and dominating. Further, X is a *convex dominating set*, if it is convex and dominating. The *weakly convex domination number* of a graph G, denoted $\gamma_{wcon}(G)$, is the minimum cardinality of a weakly convex dominating set, while the *convex domination number* of a graph G, denoted $\gamma_{con}(G)$, is the minimum cardinality of a convex dominating set. The convex and weakly convex domination numbers were first introduced by Jerzy Topp [4]. This parameters were also considered in [5], [6] and [7].

2 Cartesian product

The *Cartesian product* of two graphs G_1, G_2 is the graph $G = G_1 \Box G_2$ with a vertex set $V(G) = V(G_1) \times V(G_2)$ and two vertices (u_1, u_2), (v_1, v_2) are incident in $G_1 \Box G_2$ if and only if we have one of the two possibilities:

a) $u_1 = v_1$ and $u_2v_2 \in E(G_1)$,

b) $u_2 = v_2$ and $u_1v_1 \in E(G_2)$.

For $v_i \in V(G_2)$, G_1^i denotes the subgraph of $G = G_1 \Box G_2$ induced by $V(G_1) \times \{v_i\}$ in G and we call G_1^i the *ith* copy of G_1 in $G_1 \Box G_2$. If $V(G_2) = \{v_1, \ldots, v_n\}$, then G_1^i and G_1^j $(1 \leq i, j \leq n)$ are *neighbouring copies* in $G_1 \Box G_2$ if $v_iv_j \in E(G_2)$.

In 1963 Vizing conjectured that $\gamma(G_1 \Box G_2) \geq \gamma(G_1)\gamma(G_2)$. In [2] was proven that the following Vizing–type inequality for the convex domination number is true.

Theorem 1 *[2] For connected graphs G_1 and G_2,*

$$\gamma_{con}(G_1)\gamma_{con}(G_2) \leq \gamma_{con}(G_1 \Box G_2).$$

The domination number of the Cartesian product of two paths have been intensively investigated, see for example [1, 3]. Nevertheless, the complexity of determining the value of $\gamma(P_m \Box P_n)$ remains unknown. In this paper we give and prove exact values for the convex and weakly convex domination number of $P_m \Box P_n$ for $m, n \geq 2$. Weakly convex and convex domination in torus, the cartesian product of a path and a cycle, was considered in [8].

If G is a graph $P_m \Box P_n$, where $V(G) = \{(x_i, y_j) : x_i \in V(P_m), y_j \in V(P_n), 1 \leq i \leq m, 1 \leq j \leq n\}$, then let A denotes the set $\{(x_1, y_1), (x_1, y_n), (x_m, y_1), (x_m, y_n)\}$. Each element of A we call an *extreme vertex*.

Let $G = P_m \Box P_2$. It is easy to observe that for $m = 2$ and $m = 3$, $\gamma_{wcon}(G) = \gamma_{con}(G) = 2$ and $\{(x_2, y_1), (x_2, y_2)\}$ is the minimum weakly convex and convex dominating set of G (see Fig. 1).

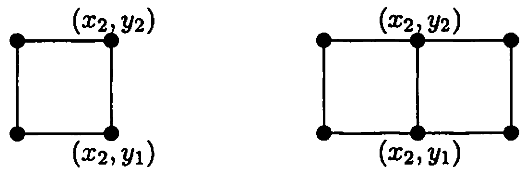

Figure 1: The Cartesian product $P_m \Box P_2$, $m = 2, 3$.

We follow with straightforward observations.

Observation 2 *If* (x_i, y_j) *and* (x_k, y_l) *are two distinct extreme vertices of* $G = P_m \Box P_n$ *with* $m, n > 3$, *then* $N_G[(x_i, y_j)] \cap N_G[(x_k, y_l)] = \emptyset$.

Observation 3 *If* $G = P_m \Box P_n$, *where* $V(G) = \{(x_i, y_j) : x_i \in V(P_m), y_j \in V(P_n), 1 \leq i \leq m, 1 \leq j \leq n\}$, *then*

$$|\mathcal{P}_G((x_i, y_j), (x_k, y_j))| = |\mathcal{P}_G((x_i, y_j), (x_i, y_l))| = 1$$

for every $1 \leq i, k \leq m$ *and* $1 \leq j, l \leq n$.

Observation 4 *If* G_1^i *and* G_1^j *are neighbouring copies of* G_1 *in* $G = G_1 \Box G_2$, *then every vertex of* $V(G_1^i)$ *has exactly one neighbour in* $V(G_1^j)$ *and similarly, if* G_2^i *and* G_2^j *are neighbouring copies of* G_2 *in* G, *then every vertex of* $V(G_2^i)$ *has exactly one neighbour in* $V(G_2^j)$.

Proposition 5 *For* $G = P_m \Box P_2$ *with* $m > 3$,

$$\gamma_{wcon}(G) = \gamma_{con}(G) = m.$$

Proof. Let $G = P_m \Box P_2$ where $V(G) = \{(x_i, y_j) : x_i \in V(P_m), y_j \in V(P_2), j \in \{1, 2\}, 1 \leq i \leq m\}$ and let $m > 3$. We show that $D = \{(x_i, y_1) : 1 \leq i \leq m\}$ (see Fig. 2) is a minimum convex and weakly convex dominating set of G. Since it is obvious that D is weakly convex and dominating in G, $\gamma_{wcon}(G) \leq \gamma_{con}(G) \leq m$. Now we suppose that D is not minimum weakly convex dominating set of G. Hence there exists a weakly convex dominating set $D' \subseteq V(G)$ such that $|D'| < |D| = m$. Then there exists i such that $(x_i, y_1), (x_i, y_2)$ do not belong to D'. Since the subgraph induced by D' in G is connected, $i = 1$ or $i = m$. Without loss of generality let $i = m$. D' is dominating, so $(x_{m-1}, y_1) \in D'$ and $(x_{m-1}, y_2) \in D'$. Moreover, at least one vertex from $N_G[(x_1, y_1)]$ belongs to D'. We consider three cases.

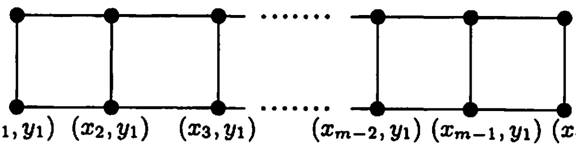

Figure 2: The Cartesian product $P_m \Box P_2, m > 3$

(a) If $(x_1, y_1) \in D'$, then Observation 3 implies that $(x_i, y_1) \in D'$ for each $1 \leq i \leq m - 1$ and hence $|D'| \geq m = |D|$, a contradiction.

(b) If $(x_2, y_1) \in D'$, then similarly like in Case 1, we have $(x_i, y_1) \in D'$ for $2 \le i \le m-1$. Moreover, at least one vertex from $N_G[(x_1, y_2)]$ belongs to D', so again $|D'| \ge m = |D|$, which is impossible.

(c) The case $(x_1, y_2) \in D'$ is similar to the Case (b) and hence is omitted.

Thus, D is a minimum convex and weakly convex dominating set of G. ■

Theorem 6 *For P_m and P_n with $m \ge n \ge 3$,*

$$\gamma_{con}(P_m \square P_n) = m(n-2).$$

Proof. Let $G = P_m \square P_n$, where $V(G) = \{(x_i, y_j) : x_i \in V(P_m), y_j \in V(P_n), 1 \le i \le m, 1 \le j \le n\}$ and let $m \ge n \ge 3$. We show that $D = \{(x_i, y_j) : 1 \le i \le m, 1 < j < n\}$ is a minimum convex dominating set of G. It is clear that D is convex and dominating in G. Suppose that D is not minimum. Then there exists a convex dominating set $D' \subseteq V(G)$ such that $|D'| < |D|$. We claim that $A \cap D' = \emptyset$, where A is the set of the extreme vertices of G. Without loss of generality, suppose that (x_1, y_1) belongs to D'. Since D' is dominating in G, $N_G[(x_m, y_n)] \cap D' \ne \emptyset$. We consider three cases.

(a) If (x_{m-1}, y_n) belongs to D', then the convexity of D' implies that $|D'| \ge (m-1)n$. Moreover, since $m \ge n \ge 3$, $|D'| \ge mn - n \ge mn - 2m = |D|$, a contradiction.

(b) If (x_m, y_{n-1}) belongs to D', then by the convexity of D' we have $|D'| \ge (n-1)m$, which is impossible.

(c) If (x_m, y_n) belongs to D', then the convexity of D' implies that $|D'| = mn$ and again $|D'| > |D|$, a contradiction.

Hence $A \cap D' = \emptyset$. Therefore, either (x_1, y_2) or (x_2, y_1) belongs to D' and similarly either (x_n, y_2) or (x_{n-1}, y_1) belongs to D'. Suppose $(x_2, y_1) \in D'$ and $(x_1, y_2) \notin D'$. Since $(x_1, y_1) \notin D'$ and (x_1, y_1) belongs to $\mathcal{P}_G((x_2, y_1), (x_1, y_{n-1}))$, $(x_1, y_{n-1}) \notin D'$. This implies that $(x_2, y_n) \in D'$ and since D' is convex, $(x_2, y_j) \in D'$, $j \in \{2, \ldots, n-1\}$.

Since $(x_2, y_n) \in D'$ and $(x_m, y_n) \notin D'$, and (x_m, y_n) belongs to $\mathcal{P}_G((x_2, y_n), (x_m, y_{n-1}))$, we obtain that $(x_m, y_{n-1}) \notin D'$. Hence $(x_{m-1}, y_n) \in D'$. Moreover, $(x_{m-1}, y_1) \in D'$, because (x_{m-1}, y_1) belongs to $\mathcal{P}_G((x_2, y_1), (x_{m-1}, y_n))$. The convexity of D' implies that $(x_i, y_j) \in D'$ for $1 < i < m$ and $1 \le j \le n$. Therefore $|D'| \ge n(m-2) = |D|$, a contradiction.

Similarly we can show that if $(x_1, y_2) \in D'$, then $|D'| \ge m(n-2) = |D|$, a contradiction. Thus $D = \{(x_i, y_j) : 1 \le i \le m, 1 < j < n\}$ is a minimum convex dominating set of $G = P_m \square P_n$ and therefore $\gamma_{con}(G) = |D| = m(n-2)$. ■

The following two results give exact values for weakly convex domination number of the Cartesian product of two paths.

Proposition 7 *For $m \geq 3$,*

$$\gamma_{wcon}(P_m \square P_3) = m.$$

Proof. Let $G = P_m \square P_3$, where $V(G) = \{(x_i, y_j) : x_i \in V(P_m), y_j \in V(P_3), 1 \leq i \leq m, 1 \leq j \leq 3\}$ and $m \geq 3$. It is sufficient to show that $D = \{(x_i, y_2) : 1 \leq i \leq m\}$ is a minimum weakly convex dominating set of G. Suppose $D' \subseteq V(G)$ is a weakly convex dominating set such that $|D'| < |D|$. Then there exists i such that $(x_i, y_1), (x_i, y_2)$ and (x_i, y_3) do not belong to D'. Moreover, D' is connected, so $i = 1$ or $i = m$. Without loss of generality, let $i = m$. Since D' is dominating, $(x_{m-1}, y_j) \in D'$ for $j = 1, 2, 3$. Further, at least one vertex from $N_G[(x_1, y_1)]$ belongs to D'. We consider three cases.

(a) If $(x_1, y_1) \in D'$, then Observation 3 implies that $(x_i, y_1) \in D'$ for $1 \leq i \leq m-1$ and hence $|D'| \geq m+1 > |D|$, a contradiction.

(b) If $(x_2, y_1) \in D'$, then similarly like in Case (a), $(x_i, y_1) \in D'$ for $2 \leq i \leq m-1$ and hence $|D'| \geq m = |D|$, which is impossible.

(c) If $(x_1, y_2) \in D'$, then by Observation 3, $(x_i, y_2) \in D'$ for $1 \leq i \leq m-1$ and hence $|D'| \geq m+1 > |D|$, which is a contradiction.

We conclude that $\gamma_{wcon}(G) = m$. ■

Theorem 8 *For P_m and P_n with $m \geq n > 3$,*

$$\gamma_{wcon}(P_m \square P_n) = (m-2)(n-2) + 4.$$

Proof. Let D be a minimum weakly convex dominating set of $G = P_m \square P_n$ where $V(G) = \{(x_i, y_j) : x_i \in V(P_m), y_j \in V(P_n), 1 \leq i \leq m, 1 \leq j \leq n\}$ and $m \geq n > 3$. By Observation 2, no extreme vertex is in D, so (x_1, y_1) and (x_m, y_1) have a neighbour in D. Observe that if $v \in N_G[(x_1, y_1)]$ and $w \in N_G[(x_m, y_1)]$, then every shortest $(v-w)$ path contains (x_2, y_1) or (x_2, y_2). Also, if $a \in N_G[(x_1, y_n)]$ and $b \in N_G[(x_m, y_n)]$, then every shortest $(a-b)$ path contains (x_2, y_{n-1}) or (x_2, y_n). Therefore, since D is weakly convex, vertices (x_2, y_j) belong to D for $2 \leq j \leq n-1$.

Similarly we can justify that vertices (x_{m-1}, y_j) belong to D for $2 \leq j \leq n-1$. Hence, since D is weakly convex, from Observation 3, $(x_i, y_j) \in D$ for $2 \leq i \leq m-1$ and $2 \leq j \leq n-1$.

Since the extreme vertices are dominated, Observation 2 implies that $|D| \geq (m-2)(n-2)+4$ and thus $\gamma_{wcon}(G) \geq (m-2)(n-2)+4$.

On the other hand, $D' = \{(x_i, y_j) : 2 \leq i \leq m-1, 2 \leq j \leq n-1\} \cup \{(x_2, y_n), (x_{m-1}, y_1), (x_1, y_2), (x_m, y_{n-1})\}$ is a weakly convex dominating set of G of cardinality $(n-2)(m-2)+4$ and hence $\gamma_{wcon}(G) = (n-2)(m-2)+4$. ∎

Next results consider weakly convex and convex domination numbers in the Cartesian product of graphs with a universal vertex.

Proposition 9 *Let $G = G_1 \Box G_2$ and $n(G_2) \leq n(G_1)$. Then*

- $\gamma_{wcon}(G) \geq n(G_2)$
- $\gamma_{con}(G) = \gamma_{wcon}(G) = n(G_2)$ *if and only if G_1 has a universal vertex.*

Proof. Let $G = G_1 \Box G_2$, $n(G_2) \leq n(G_1)$ and suppose $\gamma_{wcon}(G) < n(G_2)$. Let D be a minimum weakly convex dominating set of G. Clearly, $|D| < n(G_2) \leq n(G_1)$. Hence there exists a vertex $(u_i, v_j) \in V(G)$ such that $V(G_1^j) \cap D = \emptyset$ and $V(G_2^i) \cap D = \emptyset$. Thus, (u_i, v_j) has no neighbour in D, which implies that D is not dominating, a contradiction. Therefore $\gamma_{con}(G) \geq \gamma_{wcon}(G) \geq n(G_2)$.

If u is a universal vertex of G_1, then it is easy to see that $\{u\} \times V(G_2)$ is a minimum convex dominating set of G and thus $\gamma_{con}(G) = \gamma_{wcon}(G) = n(G_2)$.

On the other hand, let D be a minimum convex dominating set of G and $\gamma_{con}(G) = |D| = n(G_2)$. We claim, that D contains exactly one vertex from each copy of G_1. If not, then there exists a copy G_1^i such that $V(G_1^i) \cap D = \emptyset$. Hence Observation 4 implies that $|D| \geq n(G_1)$ and thus $n(G_1) = n(G_2) = |D|$. Moreover, since D is dominating, G_2 has a universal vertex. In this situation we can exchange G_1 and G_2 and we may claim that D contains exactly one vertex from each copy of G_1. Further, since D is convex, there exists a copy G_2^i such that $D = V(G_2^i)$. Since D is dominating in G, u_i is a universal vertex in G_1. ∎

The following result is an immediate cosequence of the Proposition 9.

Corollary 10 *Denote by K_p a complete graph and by $K_{p,q}$ a complete bipartite graph. Then*

(a) $\gamma_{con}(K_p \Box K_q) = \gamma_{wcon}(K_p \Box K_q) = \min\{p, q\}$.

(b) $\gamma_{con}(K_p \Box K_{1,q}) = \gamma_{wcon}(K_p \Box K_{1,q}) = \min\{p, q+1\}$.

(c) $\gamma_{con}(K_{1,p} \Box K_{1,q}) = \gamma_{wcon}(K_{1,p} \Box K_{1,q}) = \min\{p+1, q+1\}$.

Now we consider Cartesian product of a complete graph K_m and path $P_n = (v_1, \dots, v_n)$ for $m, n \geq 3$.

Theorem 11 *If* $G = K_m \square P_n$*, where* $m, n \geq 3$*, then*

$$\gamma_{wcon}(G) = \gamma_{con}(G) = n.$$

Proof. Let $G = K_m \square P_n$, where $m, n \geq 3$, and $P_n = (y_1, \dots, y_n)$. Moreover, let D be a minimum weakly convex dominating set of G. Suppose $|D| < n$. Then there exists a copy K_m^i, $1 \leq i \leq n$ such that $V(K_m^i) \cap D = \emptyset$. Since D is weakly convex, $i = 1$ or $i = n$. Without loss of generality let $i = 1$.

Since $N_G(V(K_m^1)) = V(K_m^2)$, Observation 4 implies that $V(K_m^2) \subseteq D$. If $V(K_m^n) \cap D = \emptyset$, then again by Observation 4, $V(K_m^{n-1}) \subseteq D$ and since D is weakly convex, $V(K_m^i) \subseteq D$, where $2 \leq i \leq n-1$. Thus $|D| \geq m(n-2) \geq 3n - 6 \geq n$, a contradiction.

If $V(K_m^n) \cap D \neq \emptyset$, then $(x_i, y_n) \in D$ for some $i = 1, \dots, m$. Since D is weakly convex and since $V(K_m^2) \subseteq D$, we also have $(x_i, y_j) \in D$ for $2 \leq j \leq n$ and thus $|D| \geq m + n - 2 \geq n$, a contradiction.

We conclude that $|D| \geq n$ and for this reason $\gamma_{con}(G) \geq \gamma_{wcon}(G) \geq n$. On the other hand, all vertices of any copy of P_n in G form a convex dominating set of G and hence $\gamma_{wcon}(G) = \gamma_{con}(G) = n$. ∎

Our next result gives the exact value for the Cartesian product of a graph K_m and a cycle C_n.

Theorem 12 *If* $G = K_m \square C_n$*, where* $m \geq 4$ *and* $n \geq 3$*, then*

$$\gamma_{wcon}(G) = \gamma_{con}(G) = n.$$

Proof. Let $G = K_m \square C_n$, where $m \geq 4$ and $n \geq 3$. If $m \geq n$, then the result follows from Proposition 9.

Assume $4 \leq m < n$. Let D be a minimum weakly convex dominating set of G. Suppose $|D| < n$. Then there exists a copy K_m^i, $1 \leq i \leq n$ such that $V(K_m^i) \cap D = \emptyset$. Without loss of generality let $i = 3$. Since D is dominating, (x_1, y_3) has a neighbour in D. Without loss of generality let $(x_1, y_4) \in D$. Since D is weakly convex and $n > 4$, $(x_1, y_2) \notin D$. By the same reason, $(x_l, y_2) \notin D$ for $l \in \{2, \dots, m\}$. Now, since D is dominating, $(x_l, y_k) \in D$ for $l \in \{2, \dots, m\}$ and $k \in \{1, 4\}$ (see Fig. 3). Further, $d_G((x_1, y_1), (x_1, y_4)) = 3$, so the weakly convexity of D implies that $n \leq 6$. Thus, we have only three possibilities: $G = K_4 \square C_5$, $G = K_4 \square C_6$ and $G = K_5 \square C_6$. However, since $|D| \geq 8 > 6$, we conclude that our assumption $|D| < n$ lead us to a contradiction.

∎

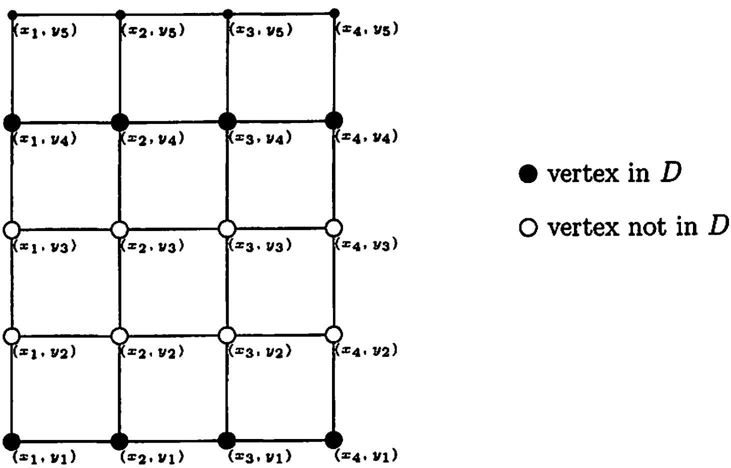

Figure 3: Main case of $K_m \Box C_n$ (some edges are omitted).

Now we consider the Cartesian product of a path P_n and a cycle C_m. The *ith* copy C_m^i of a cycle C_m in $G = C_m \Box P_n$, where $i \in \{1, \dots, n\}$, we call an *extremal copy* of C_m in G if $i = 1$ or $i = n$.

Lemma 13 *Let D be a minimum weakly convex dominating set of $G = C_m \Box P_n$, $m \geq 4$, such that $|V(C_m^i) \cap D| \leq m - 3$ for some $i \in \{1, \dots, n\}$. Then C_m^i is an extremal copy of C_m in G.*

Proof. Let D be a minimum weakly convex dominating set of $G = C_m \Box P_n$, where $m \geq 4$ and $|V(C_m^i) \cap D| \leq m - 3$ for some $i \in \{1, \dots, n\}$. Suppose C_m^i is not an extremal copy of C_m in G. In this situation there exists a set A of consecutive vertices of C_m^i such that $|A| \geq 3$ and $A \cap D = \emptyset$. (If the vertices of A would not be consecutive, then D would not be weakly convex in G.) Without loss of generality, let $\{(x_1, y_i), (x_2, y_i), (x_3, y_i)\} \subseteq A$. Let us denote $B = \{(x_2, y_{i-1}), (x_2, y_{i+1})\}$. Since (x_2, y_i) is dominated, $B \cap D \neq \emptyset$ and since D is weakly convex in G, $|B \cap D| = 1$.

If $(x_2, y_{i-1}) \in D$, then obviously $(x_2, y_{i+1}) \notin D$. Notice that every shortest $((x_2, y_{i-1}), (x_3, y_{i+1}))$-path contains at least one vertex from $C = \{(x_2, y_i), (x_3, y_i)\}$. Thus, since $C \cap D = \emptyset$, $(x_3, y_{i+1}) \notin D$. Similarly we may justify that $(x_1, y_{i+1}) \notin D$ and $(x_2, y_{i+1}) \notin D$. If $i = n - 1$, then $d_G(x_2, y_{i+1}) = 3$ and thus (x_2, y_{i+1}) has no neighbour in D, a contradiction. If $i \leq n - 2$, then the vertices (x_2, y_{i+1}) and (x_2, y_i) belong to the shortest $((x_2, y_{i+2}), (x_2, y_{i-1}))$-path. Therefore, since D is weakly convex in G, $(x_2, y_{i+2}) \notin D$. But then again (x_2, y_{i+1}) is not dominated by D, a contradiction.

The case when $(x_2, y_{i+1}) \in D$ is similar to the previous one and thus is omitted. We conclude that C_m^i is an extremal copy of C_m in G. ■

Theorem 14 *If G is the Cartesian product of the cycle C_m and the path P_n, where $m > 6$ and $n > 2$, then*

$$\gamma_{con}(G) = \gamma_{wcon}(G) = (n-2)m.$$

Proof. Let D be a minimum weakly convex dominating set of $G = C_m \Box P_n$ where $m > 6$ and $n > 2$. For $1 \le i \le n$ let C_m^i be the *ith* copy of C_m. Since the weakly convex domination number of a cycle C_k on $k > 6$ vertices equals k, $|V(C_m^i) \cap D| = m$ or $|V(C_m^i) \cap D| \le m-3$. (If $|V(C_m^i) \cap D| \in \{m-1, m-2\}$, then D would not be weakly convex in G.)

By Lemma 13, if C_m^i is not an extremal copy C_m in G, then $|V(C_m^i) \cap D| = m$. Thus, $V(C_m^j) \subseteq D$ for $1 < j < n$ and hence $\gamma_{con}(G) \ge \gamma_{wcon}(G) = |D| \ge m(n-2)$.

On the other hand, notice that the set $\{(x_i, y_j) : 1 \le i \le m, 1 < j < n\}$ is a convex dominating set of G and thus $\gamma_{con}(G) = \gamma_{wcon}(G) = m(n-2)$. ■

Theorem 15 *If G is the Cartesian product of the cycle C_m and the path P_n, where $m \in \{4, 5\}$ and $n \ge m$, then*

$$\gamma_{con}(G) = (m-2)n.$$

Proof. Let $G = C_m \Box P_n$ where $m \in \{4, 5\}$ and $n \ge m$. It is easy to see that $D = \{(x_i, y_j) : 1 \le i \le m-2, 1 \le j \le n\}$ is a convex dominating set of G and thus $\gamma_{con}(G) \le (m-2)n$. Suppose there exists a convex dominating set D' in G with $|D'| < |D|$. Since $|D'| < (m-2)n$, there exists an index j such that $|V(C_m^j) \cap D'| \le m-3$. Without loss of generality, $\{(x_k, y_j) : 1 \le k \le 3\} \subseteq V(G) - D'$.

Let $1 < j < n$. Since D' is dominating, $(x_2, y_{j-1}) \in D'$ or $(x_2, y_{j+1}) \in D'$. Assume $(x_2, y_{j-1}) \in D'$. Then the convexity of D' implies that $\{(x_k, y_j) : 1 \le k \le m\} \subseteq V(G) - D'$ and since D' is connected, we conclude that $\{(x_k, y_l) : 1 \le k \le m, j \le l \le n\} \subseteq V(G) - D'$. However, then (x_1, y_{j+1}) does not have a neighbour in D', which contradicts that D' is dominating. The case when $(x_2, y_{j+1}) \in D'$ is similar and thus is omitted.

Let $j = 1$ or $j = n$, say $j = 1$. Then $(x_2, y_2) \in D'$ dominates (x_2, y_1). Now the convexity of D' implies that $\{(x_k, y_1) : 1 \le k \le m\} \subseteq V(G) - D'$. Observe, that since D' is dominating, $\{(x_k, y_2) : 1 \le k \le m\} \subseteq D'$ and at least one vertex of C_m^{n-1} belongs to D'. Since D' is convex, we conclude that $\{(x_k, y_l) : 1 \le k \le m, 2 \le l \le n-1\} \subseteq D'$. For this reason, $|D'| \ge m(n-2) = mn - 2m \ge mn - 2n = (m-2)n = |D|$, a contradiction.

We conclude that $\gamma_{con}(G) = (m-2)n$. ■

Theorem 16 *If G is the Cartesian product of the cycle C_6 and the path P_n, where $n \geq 3$, then*

$$\gamma_{con}(G) = 6n - 12.$$

Proof. Let $G = C_6 \Box P_n$ where $n \geq 3$. It is easy to see that $D = \{(x_i, y_j) : 1 \leq i \leq 6, 2 \leq j \leq n-1\}$ is a convex dominating set of G and thus $\gamma_{con}(G) \leq 6(n-2) = 6n - 12$. Suppose there exists a convex dominating set D' in G with $|D'| < |D|$. Since $|D'| < 6(n-2)$, there exists an index $j \in \{2, \ldots, n-1\}$ such that $|V(C_m^j) \cap (V(G) - D')| \geq 1$. However D' is convex, so $|V(C_m^j) \cap (V(G) - D')| \geq 4$ and the vertices of $V(C_m^j) \cap (V(G) - D')$ induce a path in G. Without loss of generality, $\{(x_k, y_j) : 1 \leq k \leq 4\} \subseteq V(G) - D'$. Now, by the similar reasoning as in previous proof we conclude that D' is not dominating in G or $|D'| \geq |D|$, a contradiction. Thus D is a minimum convex dominating set of G. ∎

3 Other graph products

The *join* of graphs G_1 and G_2 is the graph $G = G_1 + G_2$, such that $V(G) = V(G_1) \cup V(G_2)$ and $E(G) = E(G_1) \cup E(G_2) \cup \{uv | u \in V(G_1)$ and $v \in V(G_2)\}$. This definition imply the following.

Observation 17 *For two connected graphs G_1 and G_2*

$$\gamma_{con}(G_1 + G_2) = \gamma_{wcon}(G_1 + G_2) = \begin{cases} 1 & \text{if } G_1 \text{ or } G_2 \text{ has a universal vertex,} \\ 2 & \text{otherwise.} \end{cases}$$

The *corona* $G \circ H$ is the graph formed from a copy of G and $|V(G)|$ copies of H by joining the *ith* vertex of G to every vertex in the *ith* copy of H.

Let $V(G) = \{x_i : i = 1, \ldots, n\}$ and let us denote by H_i the *i-th* copy of a graph H. The following observation is a consequence of the definition of the corona.

Observation 18 *If $G \circ H$ is a corona of connected graphs G and H, $x_i, x_j \in V(G)$ and $v_i \in V(H_i), v_j \in V(H_j)$ for $i \neq j$, then each $(v_i - v_j)$-path contains vertices x_i, x_j and each $(v_i - x_j)$-path contains x_i.*

Certainly, if $P = (v_0, \ldots, v_l)$ is the shortest $(v_0 - v_l)$-path in a connected graph G, then $v_i \neq v_j$ for every $i, j \in \{0, \ldots, l\}$.

Theorem 19 *If G and H are connected graphs, then $\gamma_{wcon}(G \circ H) = \gamma_{con}(G \circ H) = n(G)$.*

Proof. It suffices to justify that $V(G)$ is a minimum weakly convex dominating set and minimum convex dominating set of $G \circ H$. Since every vertex $x_i \in V(G)$, $(i = 1, \ldots, n)$ dominates itself and the *ith* copy H_i of a graph H, $V(G)$ is a dominating set of $G \circ H$.

By Observation 18, for every two vertices $x_i, x_j \in V(G)$, $V(\mathcal{P}_{G \circ H}(x_i x_j)) \subseteq V(G)$ and thus $V(G)$ is weakly convex and convex in $G \circ H$.

Since every convex dominating set is a weakly convex dominating set and every weakly convex dominating set is a dominating set, $\gamma(G \circ H) \leq \gamma_{wcon}(G \circ H) \leq \gamma_{con}(G \circ H)$. Further, $\gamma(G \circ H) = n(G)$, so $V(G)$ is a minimum weakly convex dominating set and minimum convex dominating set of $G \circ H$. ■

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