Colorful Paths in Vertex Coloring of Graphs

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Abstract. Let G be a graph with a vertex coloring. A colorful path is a path with $\chi(G)$ vertices, in which the vertices have different colors. A colorful path starting at vertex v is a colorful v-path. We show that for every graph G and given vertex v of G, there exists a proper vertex coloring of G with a colorful path starting at v. Let G be a connected graph with maximum degree $\Delta(G)$ and $|V(G)| \geq 2$. We prove that there exists a proper $(\chi(G) + \Delta(G) - 1)$-coloring of G such that for every $v \in V(G)$, there is a colorful v-path.

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I. Introduction

Throughout this paper all graphs are without loops and multiple edges. Let G be a graph. We denote the minimum degree and the maximum degree of G by $\delta(G)$ and $\Delta(G)$, respectively. A *k-coloring* of a graph G is a labeling $c : V(G) \longrightarrow \{1, \ldots, k\}$. A k-coloring is called *proper* if adjacent vertices have different labels. A graph G is *k-colorable* if it has a proper k-coloring. The *chromatic number* $\chi(G)$ is the least k such that G is k-colorable.

Let G be a graph with a proper vertex-coloring. A path in G *represents all colors* if all colors appear on this path. A path in G is a *rainbow path* if its vertices receive different colors. A rainbow path starting at vertex v

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is called a *rainbow v-path*. A rainbow path in G with $\chi(G)$ vertices is a *colorful path*. A colorful path starting at a vertex v is a *colorful v-path*. For every $v \in V(G)$ and $A \subseteq V(G)$, $N(v)$ denotes the neighborhood of v in G and $N(A) = \cup_{v \in A} N(v)$. Let $c : V(G) \longrightarrow \{1, \ldots, k\}$ be a vertex coloring of G. Then for every $S \subseteq V(G)$, we define $c(S) = \{c(v) \mid v \in S\}$.

We collect some results concerning paths representing all colors in a proper vertex-coloring.

Let G be a graph with a vertex k-coloring. In [2] the complexity of the existence of a rainbow path of length $k-1$ in G is investigated. Also, the following question was studied:

How much time is needed to find a rainbow path of length $k-1$ in G, if one exists, or all pairs of vertices connected by rainbow paths of length $k-1$ in G?

The next theorem, due to Fung and Lin concerns the existence of colorful paths.

Theorem 1.[5, 8] *Let G be a graph. In any proper $\chi(G)$-coloring of G, there exists a colorful path.*

The following theorem, due to Li and Lin, is related to the existence of paths starting at a given vertex and representing all colors.

Theorem 2.[7, 8] *Let G be a connected graph and let x be an arbitrary vertex of G. In any proper vertex-coloring of G with colors $1, \ldots, \chi(G)$, there exists a path of some length starting at x which represents all $\chi(G)$ colors.*

The following problem was posed in [8].

Problem 1. *Let G be a connected graph. Does there exist a proper $\chi(G)$-coloring of G such that every vertex of G is on a colorful path?*

The next result shows that Problem 1 has an affirmative answer for $\chi(G) = 3$.

Theorem 3. *If G is a connected graph with $\chi(G) = 3$, then there exists a proper 3-coloring of G such that every vertex of G is on a colorful path.*

Proof. Let c be a proper 3-coloring of G with colors $\{1, 2, 3\}$. Assume that there is no colorful path containing v, for some $v \in V(G)$. We may assume $c(v) = 1$. Since v is contained in no colorful path, all its neighbors have the same color, say 2, and all vertices of distance 2 from v have color 1. Now, change $c(v)$ to 3. Clearly, v is on a colorful path. We note that since v is contained in no colorful path, by changing the color of v, all vertices of G contained in a colorful path, remain to be on a colorful path. □

Now, we propose the following conjecture which is stronger than Problem 1, see Problem 514 (BCC22.10) of [4].

Conjecture. *If G is a connected graph, $G \neq C_7$, then there exists a proper $\chi(G)$-coloring of G such that for every vertex v of G, there is a colorful v-path.*

In [1], the authors using DFS trees in G showed that if $G \neq C_7$ is a connected graph, then there is a $(\Delta(G) + 1)$-coloring of G with a colorful v-path, for every $v \in V(G)$. The proof is rather long.

In the following theorem by a short proof we show that Conjecture is true if one can use $\chi(G) + \Delta(G) - 1$ colors in the proper coloring of G.

Theorem 4. *If G is a connected graph with at least 2 vertices, then there exists a proper $(\chi(G) + \Delta(G) - 1)$-coloring of G such that for every $v \in V(G)$, there is a colorful v-path.*

Proof. If $\chi(G) = 2$, then clearly the assertion holds. Thus assume that $\chi(G) \geq 3$. Let $r = \chi(G) + \Delta(G) - 1$.

First we claim that if H is a subgraph of G with a proper r-coloring such that for every $v \in V(H)$, there is a rainbow v-path with $\chi(G)$ colors, then G has also a proper r-coloring with the desired property.

Let $v \in V(G) \setminus V(H)$ and v has at least a neighbor in H, say u. We know that there is a rainbow u-path in H with $\chi(G)$ colors, say P. Let S_1 be the set of all $\chi(G) - 1$ colors appeared in the first $\chi(G) - 1$ vertices of P. Moreover let $S_2 = c(N(v))$. Obviously, $|S_1 \cup S_2| \leq \chi(G) + \Delta(G) - 2$. Thus we can properly color v by a color not contained in $S_1 \cup S_2$ to obtain a proper r-coloring for $H \cup \{v\}$ with the desired property. By repeating this procedure we find a proper r-coloring of G with the desired property and the claim is proved.

Now, let G be a connected graph with the minimum number of vertices such that G has no proper r-coloring with the desired property. If there exists $v \in V(G)$ such that $d(v) \leq \chi(G)-2$, then clearly, $\chi(G \setminus \{v\}) = \chi(G)$. Hence $G \setminus \{v\}$ has a connected component H such that $\chi(H) = \chi(G)$. By assumption, H has a proper $(\chi(G) + \Delta(H) - 1)$-coloring (and so a proper r-coloring) such that for every $v \in V(H)$, there is a colorful v-path in H. Now, by the claim we find a proper r-coloring of G with the desired property, a contradiction.

Therefore, assume $\delta(G) \geq \chi(G) - 1$. By Exercise 2.1.5 of [3], G has a cycle with at least $\chi(G)$ vertices, say C. There are two cases:

Case 1. $|V(C)| \leq r$. If we assign different colors to the vertices of C, then by the claim we obtain a proper r-coloring of G with the desired property, a contradiction.

Case 2. $|V(C)| > r$. By the greedy coloring algorithm it is easy to see that $\chi(G) \leq \Delta(G) + 1$. Thus $r = \chi(G) + \Delta(G) - 1 \geq 2\chi(G) - 2$. In this case consider a path, P with r vertices in G. If we assign r different colors to the vertices of P, then since $r \geq 2\chi(G) - 2$, for every vertex v of P, there is a rainbow v-path in P with $\chi(G)$ colors. Now, by the claim the proof is complete. □

The next result is an improvement of Theorem 4 in the case $\chi(G) = 3$.

Theorem 5. *If G is a connected graph and $\chi(G) = 3$, then there exists a proper 4-coloring of G such that for every $v \in V(G)$, there is a colorful v-path.*

Proof. In a proper vertex coloring of G a vertex v is called *bad* if there is no colorful v-path. First we claim that there exists a 3-coloring of G such that the set of all bad vertices forms an independent set. Let c be a 3-coloring of G with the minimum number of bad vertices and u, v be two bad vertices such that $uv \in E(G)$, $c(u) = 1$ and $c(v) = 2$. Since u and v are bad vertices $c(N(u)) = \{2\}$ and $c(N(v)) = \{1\}$. Moreover, since u is a bad vertex, the color of all vertices with distance 2 of u is 1. Similarly, the color of all vertices with distance 2 of v is 2. Now, we change the color of v to 3 to obtain a new 3-coloring of G. In this coloring the number of bad vertices decreases, a contradiction. Thus the claim is proved.

Let c_1 be a proper 3-coloring of G in which the set of all bad vertices is independent. Let X_1 be the set of all bad vertices in c_1 and assume that $a_1 \in X_1$. In coloring c_1 change the color of a_1 to 4 and call the new coloring by c_2. There exists $b_1 \in N(a_1)$ such that $d(b_1) > 1$. Since the color of every vertex of $V(G) \setminus \{a_1\}$ is 1, 2 or 3, we conclude that a_1 is not a bad vertex in this coloring. Clearly, in the new coloring no vertex of distance 2 of a_1 is a bad vertex. Now, let X_2 be the set of all bad vertices in the coloring c_2. If $X_2 = \varnothing$, then we are done. Otherwise, choose $a_2 \in X_2$ and change the color of a_2 to 4 and call this coloring by c_3. Since the distance between a_1 and a_2 is more than 2, a_2 is not a bad vertex in c_3. Repeating this procedure, we obtain a proper 4-coloring of G with no bad vertex and the proof is complete. □

Now, we are in a position to prove that the local version of the conjecture is true.

Theorem 6. *If G is a connected graph and $v \in V(G)$, then there exists a proper $\chi(G)$-coloring of G with a colorful v-path.*

Proof. By contradiction, suppose that G is a connected graph with the minimum number of vertices for which there is a vertex v such that in every proper $\chi(G)$-coloring of G, there is no colorful v-path. Assume that c is a proper $\chi(G)$-coloring containing the longest rainbow v-path. Let $P = v_1v_2\cdots v_t(v_1 = v)$ be one of the longest rainbow v-paths in c. Assume that $c(v_i) = c_i$, for $i = 1, \ldots, t$. Obviously, $c(N(v_t)) \subseteq \{c_1, \ldots, c_t\}$. There are two cases:

Case 1. There exists $u \in V(G)\setminus\{v_1, \ldots, v_t\}$ such that $v_tu \in E(G)$. Let $c(u) = c_j$, for some $j, 1 \leq j \leq t-1$. Let $A_1 = \{v\}$. For $i = 2, \ldots, t$ define:

$$A_i = \{x \in N(A_{i-1}) \mid c(x) = c_i\}.$$

Assume that $c_{t+1} \notin \{c_1, \ldots, c_t\}$ is an arbitrary color. Now, define a new proper $\chi(G)$-coloring c', as follows:

$$c'(x) = \begin{cases} c_{j+1} & if\ x \in A_j\setminus\{u\} \\ c_{k+1} & if\ x \in A_k, k = j+1, \ldots, t \\ c(x) & otherwise. \end{cases}$$

Since $c_{t+1} \notin c(N(A_t))$, it is not hard to see that c' is a proper $\chi(G)$-coloring for G. In this coloring the path $v = v_1, v_2, \ldots, v_t, u$ is a rainbow v-path with $t+1$ vertices which is a contradiction.

Case 2. $N(v_t) \subseteq \{v_1, v_2, \ldots, v_{t-1}\}$. Clearly, $G\backslash\{v_t\}$ is a connected graph and $\chi(G\backslash\{v_t\}) = \chi(G)$. Since $|V(G\backslash\{v_t\})| < |V(G)|$, there exists a proper $\chi(G)$-coloring of $G\backslash\{v_t\}$ with a colorful v-path. Since $d(v_t) \le t-1$, one can extend this coloring to a proper $\chi(G)$-coloring of G to obtain a colorful v-path in G, a contradiction. □

In the following theorem we study the existence of rainbow paths in 3-coloring of bipartite graphs.

Theorem 7. *Let G be a bipartite graph such that no connected component of G is a complete bipartite graph. Then there exists a proper 3-coloring of G such that for every vertex v, there exists a rainbow v-path with three vertices.*

Proof. Let G be a bipartite graph with two parts X and Y. Clearly, it suffices to prove the theorem for a connected bipartite graph. First we would like to give the idea of the proof. Decompose the vertices of G into subsets

$$Y_1, X_1, Y_2, X_2, \ldots \qquad (1)$$

such that $X_i \subseteq X$, $Y_i \subseteq Y$, for $i = 1, 2, \ldots$ and every vertex in each subset has at least one neighbor in the previous subset and also in the next subset in the sequence (1). We color the vertices of the sequence alternately by $1, 2, 0, 1, 2, 0, \ldots$. That is, we color all vertices of Y_1 by 1, all vertices of X_1 by 2 and so on. Since G is not a complete bipartite graph, there exists a vertex $v \in X$ such that $N(v) \neq Y$. Assume that v has the minimum degree among all vertices with this property. Let

$$X_1 = \{\, x \in X \mid N(x) \subseteq N(v) \,\},\ Y_1 = \{\, y \in N(v) \mid N(y) \subseteq X_1 \,\}.$$

Now, for each i, $i \ge 2$, define:

$$X_i = \{\, x \in X\backslash X_{i-1} \mid N(x) \cap Y_i \neq \varnothing \,\},\ Y_i = \{\, y \in Y\backslash Y_{i-1} \mid N(y) \cap X_{i-1} \neq \varnothing \,\}.$$

By the minimality of the degree of v, for every $x \in X_1$, we have $N(x) = N(v)$. Also one can see that $N(v) = Y_1 \cup Y_2$. First we claim that Y_2, X_2 and Y_3 are not empty. If $Y_2 = \varnothing$, then for every $y \in Y_1$, $N(y) \subseteq X_1$ and moreover $N(X_1) \subseteq Y_1$ and this implies that G is not connected, a contradiction. Thus $Y_2 \neq \varnothing$. If $X_2 = \varnothing$, then $N(Y_2) \subseteq X_1$ and so $Y_2 \subseteq Y_1$, a contradiction. Hence $X_2 \neq \varnothing$. If $Y_3 = \varnothing$, then $N(X_2) \subseteq N(v)$. Since

$Y \backslash N(v) \neq \varnothing$, G is not connected, a contradiction. Therefore $Y_3 \neq \varnothing$. Furthermore, for every $x \in X_2$, $N(x) \cap Y_3 \neq \varnothing$, since otherwise $x \in X_1$, a contradiction.

Now, we provide a proper 3-coloring of G with the desired property. For each i, color all vertices of X_i and Y_i by $2i$ and $2i-1 \pmod 3$, respectively.

For every $x \in X_1$, note that $N(x) = N(v)$. Hence there exists $z \in N(x) \cap Y_2$. Also there exists $w \in N(z) \cap X_2$ and so we obtain a rainbow x-path with three vertices. Now, if $Y_1 \neq \varnothing$ and $y \in Y_1$, then noting that $v \in N(y)$ and $N(v) \cap Y_2 \neq \varnothing$, we obtain a rainbow y-path with three vertices. For every $y \in Y_2$, there exists a vertex $u \in N(y) \cap X_2$. As we saw before $N(u) \cap Y_3 \neq \varnothing$. Thus we obtain a rainbow y-path with three vertices. Let $i \geq 2$ and $x \in X_i$. Then there are two vertices z and w such that $z \in N(x) \cap Y_i$ and $w \in N(z) \cap X_{i-1}$. Clearly, xzw is a rainbow x-path. Similarly, for every $y \in Y_i$, $i \geq 3$, there exists a rainbow y-path with three vertices. The proof is complete. □

A proper vertex coloring of a graph G is called a *dynamic coloring*, if for every vertex v of degree at least 2, the neighbors of v receive at least two different colors. A dynamic coloring with k colors is called a *dynamic k-coloring*. The smallest integer k such that G has a dynamic k-coloring is called the *dynamic chromatic number* of G and denoted by $\chi_2(G)$. In [6], it has been proved that if G is a graph and $\Delta(G) \leq 3$, then $\chi_2(G) \leq 4$ with the only exception that $G = C_5$, in which case $\chi_2(C_5) = 5$. Also if $\Delta(G) \geq 4$, then $\chi_2(G) \leq \Delta(G) + 1$.

We close this paper with the following theorem.

Theorem 8. *If G is a connected graph with $\chi(G) \geq 4$, then there is a $(\Delta(G)+1)$-coloring of G such that every vertex of G is on a rainbow path with 4 vertices.*

Proof. Let G be a graph with minimum number of vertices for which the assertion does not hold. If G has a vertex v of degree 1, then $G \setminus \{v\}$ has a $(\Delta(G)+1)$-coloring with the desired property. Now, by a suitable coloring of v we obtain a desired coloring for G.

Thus we assume that $\delta(G) \geq 2$. By a result in [6], there exists a proper dynamic coloring of G, say c, using $\Delta(G)+1$ colors. Let $v \in V(G)$ and there is no rainbow path with 4 vertices containing v. Suppose that

$c(v) = 1$. Since c is a dynamic coloring we may assume that $\{2,3\} \subseteq c(N(v))$. Let $u_1, u_2 \in N(v)$, $c(u_1) = 2$, $c(u_2) = 3$. Thus $c(N(u_1)) = \{1,3\}$ and $c(N(u_2)) = \{1,2\}$. There exists $w \in N(u_1)$ such that $c(w) = 3$. If $|c(N(v))| \geq 3$, then we find a rainbow path with 4 vertices containing v, a contradiction. Hence assume that $|c(N(v))| = 2$. If $1 \in c((N(u_1) \cup N(u_2))\backslash\{v\})$, then by changing the color of v to 4, we find a proper dynamic coloring of G with a rainbow path with 4 vertices containing v. Thus assume that $c(N(u_1) \setminus \{v\}) = \{3\}$, $c(N(u_2) \setminus \{v\}) = \{2\}$. Let $u_3 \in N(u_2) \setminus \{v\}$. If there exists $z \in N(u_3) \setminus \{v\}$ such that $c(z) = 1$, then by changing the color of v to 4, we find a proper dynamic coloring of G with a rainbow path of order 4 containing v. Otherwise, since c is a proper dynamic coloring, $u_3v \in E(G)$. In this case we change the color of u_2 to 4, to obtain a proper dynamic coloring with the desired property for v.

By repeating this procedure we find a proper dynamic coloring with the desired property, a contradiction. □

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