On total domination number of Cartesian product of directed cycles *

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Abstract: Let $\gamma_t(D)$ denote the total domination number of a digraph D and let $C_m\Box C_n$ denote the Cartesian product graph of C_m and C_n, where C_m denotes the directed cycle of length m, $m \leq n$. In [On domination number of Cartesian product of directed cycles, Information Processing Letters 110 (2010) 171-173.], Liu et al. determined the domination number of $C_2\Box C_n$, $C_3\Box C_n$ and $C_4\Box C_n$. In this paper, we determine the exact values of $\gamma_t(C_m\Box C_n)$ when at least one of m and n is even, or n is odd and $m = 1, 3, 5$ or 7.

Keywords: Cartesian product; Total domination number

1 Introduction

We use [11] for terminology and notations not defined here, consider finite digraphs and always assume the digraph without loop and multiple arcs. In particular, denote $V(D)$ the vertex set and $A(D)$ the arc set of D. A set $S \subseteq V(D)$ is a *total dominating* set if for any $v \in V(D)$, there exists $u \in S$ such that $uv \in A(D)$. For two vertices $u, v \in V(G)$, we say u *dominates* v if $uv \in A(D)$. The *total domination number* of D, denoted by $\gamma_t(D)$, is the minimum cardinality of a total dominating set of D. A total dominating set S is called a $\gamma_t(D)$-set if $|S| = \gamma_t(D)$.

Let $D_1 = (V_1, A_1)$ and $D_2 = (V_2, A_2)$ be two digraphs with vertex sets $V_1 = \{x_1, x_2, \cdots, x_s\}$ and $V_2 = \{y_1, y_2, \cdots, y_t\}$ and arc sets A_1 and A_2, respectively. The *Cartesian product graph* $D = D_1\Box D_2$ has vertex set $V = V_1 \times V_2$, and $(x, y)(x', y') \in A(D_1\Box D_2)$ if and only if either $xx' \in A_1$ and $y = y'$, or $x = x'$ and $yy' \in A_2$. The subdigraph D_1^y of $D_1\Box D_2$ has vertex set $V_1^y = \{(x, y)|$ for any $x \in V_1$, fixed $y \in V_2\}$, and arc set $A_1^y = \{(x, y)(x', y)|xx' \in A_1\}$. It is clear that

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$D_1^y = (V_1^y, A_1^y) \cong D_1$. Similarly, The subdigraph D_2^x of $D_1 \Box D_2$ has vertex set $V_2^x = \{(x,y) |$ for any $y \in V_2$, fixed $x \in V_1\}$, and arc set $A_2^x = \{(x,y)(x,y') | yy' \in A_2\}$. It is clear that $D_2^x = (V_2^x, A_2^x) \cong D_2$.

Total domination in graphs was introduced by Cockayne, Dawes, and Hedetniemi [1], which is now well studied in graph theory. The literature on this subject has been surveyed and detailed in the two books by Haynes, Hedetniemi, and Slater [2, 3]. The decision problem to determine the domination number and total domination number of a graph remains NP-hard. Laskar, Pfaff, Hedetniemi, and Hedetniemi [4] constructed the first linear algorithm for computing the total domination number of a tree. Furthermore, there are many articles which obtained the upper bounds and the lower bounds on the total domination number of some special connected graphs [5, 6, 7, 8, 9, 10]. Recently, Liu et al. [12] determined the domination number of the Cartesian product graph of two directed cycles C_m, C_n when $m = 2, 3, 4$. In this paper, we determine the exact values of $\gamma_t(C_m \Box C_n)$ when at least one of m and n is even; and n is odd and $m = 1, 3, 5$ or 7.

2 Main results

We emphasize that $V(C_n) = \{0, 1, 2, \cdots, n-1\}$ and $A(C_n) = \{(i+1, i) | i = 0, 1, 2, \cdots, n-1\}$, considered modulo n, throughout this paper.

Lemma 2.1. *Let S be any total dominating set of $C_m \Box C_n$, then for any $i \in \{0, 1, \cdots, n-1\}$, $|S \cap C_m^i| + |S \cap C_m^{i+1}| \geq m$.*

Proof. Let S be a total dominating set of $C_m \Box C_n$. Suppose the vertices in $S \cap C_m^i$ induce s vertex-disjoint directed paths $P_1, P_2, \cdots, P_s$ in C_m^i. Denote (p_j, i) (resp. (p_j', i)) the vertex of P_j of outdegree (resp. indegree) 0, $1 \leq j \leq s$. It is not difficult to see that the vertices in $V' = V(C_m^i) \setminus \bigcup_{k=1}^{s} (V(P_k) \cup \{(p_k - 1, i)\}) \bigcup \{(p_k', i) | 1 \leq k \leq s\}$ is not dominated by $S \cap C_m^i$. Therefore we need at least $|V'|$ vertices in C_m^{i+1} to dominate the vertices in V', and so we have $|S \cap C_m^i| + |S \cap C_m^{i-1}| \geq \sum_{j=1}^{s} |V(P_j)| + |V(C_m)| - \sum_{j=1}^{s} |V(P_j)| = m$. □

Theorem 2.1. *If at least one of m and n is even, then $\gamma_t(C_m \Box C_n) = \frac{mn}{2}$.*

Proof. Without loss of generality, we assume that n is even. Let $S = V(C_m^1) \cup V(C_m^3) \cup \cdots V(C_m^{n-1})$. Clearly, S is a total dominating set of $C_m \Box C_n$. Thus, $\gamma_t(C_m \Box C_n) \leq \frac{mn}{2}$.

Let S be a total domination set of $C_m \Box C_n$ with minimum size. By Lemma 2.1, we have $(|S \cap C_m^0| + |S \cap C_m^1|) + (|S \cap C_m^1| + |S \cap C_m^2|) + \cdots + (|S \cap C_m^{n-1}| + |S \cap C_m^0|) \geq mn$. That is, $|S| \geq \frac{mn}{2}$. Thus, we complete the proof. □

Next, we consider the case when both m and n are odd. We determine the value of $\gamma_t(C_m \Box C_n)$ when n is odd and $m = 1, 3, 5$ or 7 in the following.

Assume $m = 1$, we immediately obtain the following theorem.

Theorem 2.2. *Assume that $n \geq 3$ is odd. Then $\gamma_t(C_1 \Box C_n) = n$.*

Theorem 2.3. *Assume that $n \geq 3$ is odd. Then*

$$\gamma_t(C_3 \Box C_n) = \begin{cases} \frac{3n+1}{2}, & \text{if } n \equiv 1, 5 (\text{mod } 6); \\ \frac{3n+3}{2}, & \text{if } n \equiv 3 (\text{mod } 6). \end{cases}$$

Proof. Let S be a $\gamma_t(C_3 \Box C_n)$-set. By Lemma 2.1, we have $(|S \cap C_3^0| + |S \cap C_3^1|) + (|S \cap C_3^1| + |S \cap C_3^2|) + \cdots + (|S \cap C_3^{n-1}| + |S \cap C_3^0|) \geq 3n$. That is, $|S| \geq \frac{3n}{2}$. Note that both 3 and n are odd, $|S| \geq \frac{3n+1}{2}$.

If $n \equiv 1, 5(\text{mod } 6)$, we define a vertex set S_1 as follows: S_1 consists of vertices $(0, i)$ and $(1, i)$, $i \equiv 0(\text{mod } 6)$; $(1, i)$, $i \equiv 1(\text{mod } 6)$; $(1, i)$ and $(2, i)$, $i \equiv 2(\text{mod } 6)$; $(2, i)$, $i \equiv 3(\text{mod } 6)$; $(0, i)$ and $(2, i)$, $i \equiv 4(\text{mod } 6)$; $(0, i)$, $i \equiv 5(\text{mod } 6)$. Clearly, S_1 is a total domination set of $C_3 \Box C_n$ and $|S_1| = 9 \cdot \frac{n-1}{6} + 2 = \frac{3n+1}{2}$ if $n \equiv 1(\text{mod } 6)$, $|S_1| = 9 \cdot \frac{n-5}{6} + 8 = \frac{3n+1}{2}$ if $n \equiv 5(\text{mod } 6)$. Therefore, $\gamma_t(C_3 \Box C_n) = \frac{3n+1}{2}$ when $n \equiv 1, 5$ (mod 6).

If $n \equiv 3(\text{mod } 6)$, suppose there exists a $\gamma_t(C_3 \Box C_n)$-set S' such that $|S'| = \frac{3n+1}{2}$. We distinguish two cases.

Case 1. For any $i \in \{0, 1, \cdots, n-1\}$, $|C_3^i \cap S'| < 3$. By Lemma 2.1, there are $\frac{n+1}{2}$ C_3^i's with $|C_3^i \cap S'| = 2$, and there are $\frac{n-1}{2}$ C_3^i's with $|C_3^i \cap S'| = 1$. Furthermore, there exists an integer $j \in \{0, 1, \cdots, n-1\}$ such that $|C_3^j \cap S'| = |C_3^{j+1} \cap S'| = 2$, and for any other $k \in \{0, 1, \cdots, n-1\} \setminus \{j\}$, if $|C_3^k \cap S'| = 2$ (resp. $|C_3^k \cap S'| = 1$), then $|C_3^{k+1} \cap S'| = 1$ (resp. $|C_3^{k+1} \cap S'| = 2$). Without loss of generality, let $|C_3^0 \cap S'| = |C_3^{n-1} \cap S'| = 2$ and $C_3^0 \cap S' = \{(0,0), (1,0)\}$, we have $|C_3^1 \cap S'| = |C_3^3 \cap S'| = \cdots = |C_3^{n-2} \cap S'| = 1$, $|C_3^0 \cap S'| = |C_3^2 \cap S'| = \cdots = |C_3^{n-1} \cap S'| = 2$. It follows that $C_3^1 \cap S' = \{(1, 1)\}$. Similarly, $C_3^2 \cap S' = \{(1, 2), (2, 2)\}$, $C_3^3 \cap S' = \{(2, 3)\}$, $C_3^4 \cap S' = \{(0, 4), (2, 4)\}$, $C_3^5 \cap S' = \{(0, 5)\}$, $C_3^6 \cap S' = \{(0, 6), (1, 6)\}, \cdots, C_3^{n-1} \cap S' = \{(1, n-1), (2, n-1)\}$ (Note that $n \equiv 3(\text{mod } 6)$). Clearly, no vertex in $C_3^{n-1} \cap S'$ and $C_3^0 \cap S'$ can dominate $(2, n-1)$. This contradicts that S' is a $\gamma_t(C_3 \Box C_n)$-set.

Case 2. There exists an integer $i \in \{0, 1, \cdots, n-1\}$, such that $|C_3^i \cap S'| = 3$. Without loss of generality, let $|C_3^0 \cap S'| = 3$. We have $|S'| = |C_3^0 \cap S'| + (|C_3^1 \cap S'| + |C_3^2 \cap S'|) + (|C_3^3 \cap S'| + |C_3^4 \cap S'|) + \cdots + (|C_3^{n-2} \cap S'| + |C_3^{n-1} \cap S'|) \geq 3 + 3 \cdot \frac{n-1}{2} = \frac{3n+3}{2}$, a contradiction.

Hence, $\gamma_t(C_3 \Box C_n) > \frac{3n+1}{2}$ when $n \equiv 3(\text{mod } 6)$. It is easy to see that $S_2 = S_1 \cup \{(0, n-1)\}$ is a $\gamma_t(C_3 \Box C_n)$-set, and $|S_2| = \frac{3n+3}{2}$. Therefore, $\gamma_t(C_3 \Box C_n) = \frac{3n+3}{2}$ when $n \equiv 3(\text{mod } 6)$. □

Theorem 2.4. *Assume that $n \geq 5$ is odd. Then*

$$\gamma_t(C_5 \Box C_n) = \begin{cases} \frac{5n+1}{2}, & \text{if } n \equiv 1,3,7,9 (\text{mod } 10); \\ \frac{5n+5}{2}, & \text{if } n \equiv 5 (\text{mod } 10). \end{cases}$$

Proof. Let S be a $\gamma_t(C_5 \Box C_n)$-set. By Lemma 2.1, we have $(|S \cap C_5^0| + |S \cap C_5^1|) + (|S \cap C_5^1| + |S \cap C_5^2|) + \cdots + (|S \cap C_5^{n-1}| + |S \cap C_5^0|) \geq 5n$. That is, $|S| \geq \frac{5n}{2}$. Note that both 5 and n are odd, then $|S| \geq \frac{5n+1}{2}$.

If $n \equiv 3, 7 (\text{mod } 10)$, we define a vertex set S_1 as follows: S_1 consists of vertices $(0,i)$, $(1,i)$ and $(2,i)$, $i \equiv 0 (\text{mod } 10)$; $(2,i)$ and $(3,i)$, $i \equiv 1 (\text{mod } 10)$; $(0,i)$, $(3,i)$ and $(4,i)$, $i \equiv 2 (\text{mod } 10)$; $(0,i)$ and $(1,i)$, $i \equiv 3 (\text{mod } 10)$; $(1,i)$, $(2,i)$ and $(3,i)$, $i \equiv 4 (\text{mod } 10)$; $(3,i)$ and $(4,i)$, $i \equiv 5 (\text{mod } 10)$; $(0,i)$, $(1,i)$ and $(4,i)$, $i \equiv 6 (\text{mod } 10)$; $(1,i)$ and $(2,i)$, $i \equiv 7 (\text{mod } 10)$; $(2,i)$, $(3,i)$ and $(4,i)$, $i \equiv 8 (\text{mod } 10)$; $(0,i)$ and $(4,i)$, $i \equiv 9 (\text{mod } 10)$. Clearly, S_1 is a total domination set of $C_5 \Box C_n$ and $|S_1| = 25 \cdot \frac{n-3}{10} + 8 = \frac{5n+1}{2}$ if $n \equiv 3 (\text{mod } 10)$, $|S_1| = 25 \cdot \frac{n-7}{10} + 18 = \frac{5n+1}{2}$ if $n \equiv 7 (\text{mod } 10)$.

If $n \equiv 1, 9 (\text{mod } 10)$, we define a vertex set S_2 as follows: S_2 consists of vertices $(0,i)$, $(1,i)$ and $(3,i)$, $i \equiv 0 (\text{mod } 10)$; $(1,i)$ and $(3,i)$, $i \equiv 1 (\text{mod } 10)$; $(1,i)$, $(3,i)$ and $(4,i)$, $i \equiv 2 (\text{mod } 10)$; $(1,i)$ and $(4,i)$, $i \equiv 3 (\text{mod } 10)$; $(1,i)$, $(2,i)$ and $(4,i)$, $i \equiv 4 (\text{mod } 10)$; $(2,i)$ and $(4,i)$, $i \equiv 5 (\text{mod } 10)$; $(0,i)$, $(2,i)$ and $(4,i)$, $i \equiv 6 (\text{mod } 10)$; $(0,i)$ and $(2,i)$, $i \equiv 7 (\text{mod } 10)$; $(0,i)$, $(2,i)$ and $(3,i)$, $i \equiv 8 (\text{mod } 10)$; $(0,i)$ and $(3,i)$, $i \equiv 9 (\text{mod } 10)$. Clearly, S_2 is a total domination set of $C_5 \Box C_n$ and $|S_2| = 25 \cdot \frac{n-1}{10} + 3 = \frac{5n+1}{2}$ if $n \equiv 1 (\text{mod } 10)$, $|S_2| = 25 \cdot \frac{n-9}{10} + 23 = \frac{5n+1}{2}$ if $n \equiv 9 (\text{mod } 10)$.

Therefore, $\gamma_t(C_5 \Box C_n) = \frac{5n+1}{2}$ when $n \equiv 1, 3, 7, 9 (\text{mod } 10)$.

Suppose $n \equiv 5 (\text{mod } 10)$ and let S' be a $\gamma_t(C_5 \Box C_n)$-set with $|S'| = \frac{5n+1}{2}$. Similar to the proof of Theorem 2.3, there exists at least one vertex can not be dominated by the vertices in S'. Hence, $\gamma_t(C_5 \Box C_n) > \frac{5n+1}{2}$.

Now we show $\gamma_t(C_5 \Box C_n) > \frac{5n+3}{2}$ by contradiction. Suppose there exists a $\gamma_t(C_5 \Box C_n)$-set S'' such that $|S''| = \frac{5n+3}{2}$ when $n \equiv 5 (\text{mod } 10)$. We distinguish three cases.

Case 1. There exists some integer $i \in \{0, 1, \cdots, n-1\}$ such that $|C_5^i \cap S''| = 5$.

Without loss of generality, let $|C_5^0 \cap S''| = 5$. We have $|S''| = |C_5^0 \cap S''| + (|C_5^1 \cap S''| + |C_5^2 \cap S''|) + \cdots + (|C_5^{n-2} \cap S''| + |C_5^{n-1} \cap S''|) \geq 5 + 5 \cdot \frac{n-1}{2} = \frac{5n+5}{2} > \frac{5n+3}{2}$, a contradiction.

From now on, we assume that there is no integer i such that $|C_5^i \cap S''| = 5$.

Case 2. There exists some integer $i \in \{0, 1, \cdots, n-1\}$ such that $|C_5^i \cap S''| = 4$.

Without loss of generality, let $|C_5^0 \cap S''| = 4$. Clearly, $\frac{5n+3}{2} = |S''| = |C_5^0 \cap S''| + (|C_5^1 \cap S''| + |C_5^2 \cap S''|) + (|C_5^3 \cap S''| + |C_5^4 \cap S''|) + \cdots + (|C_5^{n-2} \cap S''| + |C_5^{n-1} \cap S''|) \geq 4 + 5 \cdot \frac{n-1}{2} = \frac{5n+3}{2}$. Hence, we have $|C_5^t \cap S''| + |C_5^{t+1} \cap S''| = 5$

for $t \in \{1, 3, \cdots, n-2\}$. Combining with Lemma 2.1, we can distinguish four subcases as follows (For any other subcase, it is not difficult to see that the subcase is same as one of the Subcases 2.1, 2.2, 2.3, 2.4 by picking a suitable C_5^i as the new C_5^0).

Subcase 2.1. $|C_5^0 \cap S''| = |C_5^2 \cap S''| = \cdots = |C_5^{n-1} \cap S''| = 4, |C_5^1 \cap S''| = |C_5^3 \cap S''| = \cdots = |C_5^{n-2} \cap S''| = 1$.

Without loss of generality, let $C_5^0 \cap S'' = \{(0,0),(1,0),(2,0),(3,0)\}$. Then $C_5^1 \cap S'' = \{(3,1)\}$, $C_5^2 \cap S'' = \{(0,2),(1,2),(3,2),(4,2)\}$, $\cdots$, $C_5^{n-1} \cap S'' = \{(1,n-1),(2,n-1),(3,n-1),(4,n-1)\}$ (note that $n \equiv 5 (\bmod 10)$), but $(4, n-1)$ can not be dominated by the vertices in $C_5^0 \cap S''$ and $C_5^{n-1} \cap S''$, a contradiction.

Subcase 2.2. There exists an even integer $i \in \{0, 1, \cdots, n-3\}$ such that $|C_5^0 \cap S''| = |C_5^2 \cap S''| = \cdots = |C_5^i \cap S''| = 4, |C_5^1 \cap S''| = |C_5^3 \cap S''| = \cdots = |C_5^{i-1} \cap S''| = 1, |C_5^{i+1} \cap S''| = |C_5^{i+3} \cap S''| = \cdots = |C_5^{n-2} \cap S''| = 2, |C_5^{i+2} \cap S''| = |C_5^{i+4} \cap S''| = \cdots = |C_5^{n-1} \cap S''| = 3$.

Without loss of generality, let $C_5^0 \cap S'' = \{(0,0),(1,0),(2,0),(3,0)\}$. Then $C_5^1 \cap S'' = \{(3,1)\}$, $\cdots$, $C_5^i \cap S'' = \{(\frac{3i}{2}, i), (\frac{3i}{2}+1, i), (\frac{3i}{2}+2, i), (\frac{3i}{2}+3, i)\}$. It follows that $\{(\frac{3i}{2}+3, i+1)\} \subseteq C_5^{i+1} \cap S''$, and then $C_5^{i+2} \cap S'' \subseteq \{(\frac{3i}{2}+3, i+2), (\frac{3i}{2}+4, i+2), (\frac{3i}{2}+5, i+2), (\frac{3i}{2}+6, i+2)\}$, $\cdots$, $C_5^{n-1} \cap S'' \subseteq \{(\frac{3(n-1)}{2}, n-1), (\frac{3(n-1)}{2}+1, n-1), (\frac{3(n-1)}{2}+2, n-1), (\frac{3(n-1)}{2}+3, n-1)\} = \{(1,n-1),(2,n-1),(3,n-1),(4,n-1)\}$ (note that $n \equiv 5 (\bmod 10)$), but $(4, n-1)$ can not be dominated by the vertices in $C_5^0 \cap S''$ and $C_5^{n-1} \cap S''$, a contradiction.

By the argument similar to that of Subcase 2.2, the contradiction in each of the following two subcases is easy to obtain. So we do not give the detailed proof here.

Subcase 2.3. There exists an even integer $i \in \{0, 1, \cdots, n-3\}$ such that $|C_5^0 \cap S''| = |C_5^2 \cap S''| = \cdots = |C_5^i \cap S''| = 4, |C_5^1 \cap S''| = |C_5^3 \cap S''| = \cdots = |C_5^{i-1} \cap S''| = 1, |C_5^{i+1} \cap S''| = |C_5^{i+3} \cap S''| = \cdots = |C_5^{n-2} \cap S''| = 3, |C_5^{i+2} \cap S''| = |C_5^{i+4} \cap S''| = \cdots = |C_5^{n-1} \cap S''| = 2$.

Subcase 2.4. There exist $i, j \in \{0, 1, \cdots, n-3\}$ such that $i < j$ and $|C_5^0 \cap S''| = |C_5^2 \cap S''| = \cdots = |C_5^i \cap S''| = 4, |C_5^1 \cap S''| = |C_5^3 \cap S''| = \cdots = |C_5^{i-1} \cap S''| = 1, |C_5^{i+1} \cap S''| = |C_5^{i+3} \cap S''| = \cdots = |C_5^{j-1} \cap S''| = |C_5^{j+2} \cap S''| = |C_5^{j+4} \cap S''| = \cdots = |C_5^{n-1} \cap S''| = 2, |C_5^{i+2} \cap S''| = |C_5^{i+4} \cap S''| = \cdots = |C_5^j \cap S''| = |C_5^{j+1} \cap S''| = |C_5^{j+3} \cap S''| = \cdots = |C_5^{n-2} \cap S''| = 3$.

So we complete the proof of Case 2.

Case 3. There exists no $i \in \{0, 1, \cdots, n-1\}$ such that $|C_5^i \cap S''| = 5$ or 4.

Since $|S''| = \frac{5n+3}{2}$, there exist exactly three integers $i, j, k \in \{0, 1, \cdots, n-1\}$ such that $i < j < k$ and $|C_5^i \cap S''| = |C_5^{i+1} \cap S''| = 3$, $|C_5^j \cap S''| = |C_5^{j+1} \cap S''| = 3$,

$|C_5^k \cap S''| = |C_5^{k+1} \cap S''| = 3$. Without loss of generality, let $i = 0$, and we distinguish three subcases: (1) $j = i+1$, $k = j+1$; (2) $j = i+1$, $k \neq j+1$ and $k \neq n-1$; (3) $j \neq i+1$, $k \neq j+1$ and $k \neq n-1$. (For any other subcase, it is not difficult to see that the subcase is same as one of the Subcases 3.1, 3.2, 3.3 by picking a suitable C_5^i as the new C_5^0.)

Subcase 3.1. $j = i+1$, $k = j+1$.

By Lemma 2.1, we have $|C_5^0 \cap S''| = |C_5^1 \cap S''| = |C_5^2 \cap S''| = |C_5^3 \cap S''| = 3$, $|C_5^4 \cap S''| = |C_5^6 \cap S''| = \cdots = |C_5^{n-1} \cap S''| = 2$, $|C_5^5 \cap S''| = |C_5^7 \cap S''| = \cdots = |C_5^{n-2} \cap S''| = 3$. Without loss of generality, we assume $C_5^0 \cap S'' = \{(0,0),(1,0),(2,0)\}$ or $\{(0,0),(1,0),(3,0)\}$.

If $C_5^0 \cap S'' = \{(0,0),(1,0),(2,0)\}$, we have $C_5^{n-1} \cap S'' = \{(0,n-1),(4,n-1)\}$, and then $C_5^{n-2} \cap S'' = \{(2,n-2),(3,n-2),(4,n-2)\}$, $\cdots$, $C_5^{n-11} \cap S'' = \{(0,n-11),(4,n-11)\}$, it is clear that $C_5^4 \cap S'' = \{(0,4),(4,4)\}$ (Note that $n \equiv 5 (\mathrm{mod}\ 10)$). Thus $C_5^3 \cap S'' = \{(2,3),(3,3),(4,3)\}$, and so $\{(1,2),(2,2)\} \subseteq C_5^2 \cap S''$. Since $C_5^0 \cap S'' = \{(0,0),(1,0),(2,0)\}$, $\{(2,1),(3,1)\} \subseteq C_5^1 \cap S''$. If $C_5^1 \cap S'' = \{(1,1),(2,1),(3,1)\}$, then at least one of $(3,1)$ and $(4,1)$ can not be dominated. If $C_5^1 \cap S'' = \{(0,1),(2,1),(3,1)\}$, then at least one of $(0,1)$ and $(3,1)$ can not be dominated. If $C_5^1 \cap S'' = \{(2,1),(3,1),(4,1)\}$, then at least one of $(0,1)$ and $(4,1)$ can not be dominated, a contradiction.

Similarly as above we can deduce the contradiction of the case $C_5^0 \cap S'' = \{(0,0),(1,0),(3,0)\}$.

Subcase 3.2. $j = i+1$, $k \neq j+1$ and $k \neq n-1$.

By Lemma 2.1, we have $|C_5^0 \cap S''| = |C_5^1 \cap S''| = |C_5^2 \cap S''| = 3$, $|C_5^4 \cap S''| = |C_5^6 \cap S''| = \cdots = |C_5^k \cap S''| = |C_5^{k+1} \cap S''| = |C_5^{k+3} \cap S''| = \cdots = |C_5^{n-2} \cap S''| = 3$, $|C_5^3 \cap S''| = |C_5^5 \cap S''| = \cdots = |C_5^{k-1} \cap S''| = |C_5^{k+2} \cap S''| = \cdots = |C_5^{n-1} \cap S''| = 2$. Without loss of generality, we assume $C_5^0 \cap S'' = \{(0,0),(1,0),(2,0)\}$ or $\{(0,0),(1,0),(3,0)\}$.

If $C_5^0 \cap S'' = \{(0,0),(1,0),(2,0)\}$, we have $C_5^{n-1} \cap S'' = \{(0,n-1),(4,n-1)\}$, and then $C_5^{n-2} \cap S'' = \{(2,n-2),(3,n-2),(4,n-2)\}$, $\cdots$, $C_5^{k+1} \cap S'' = \{(n-k-1,k+1),(n-k,k+1),(n-k+1,k+1)\}$. It follows that $\{(n-k-2,k),(n-k-1,k)\} \subseteq C_5^k \cap S''$, and $\{(n-k,k-2),(n-k+1,k-2)\} \subseteq C_5^{k-2} \cap S''$, $\cdots$, $\{(n-4,2),(n-3,2)\} = \{(1,2),(2,2)\} \subseteq C_5^2 \cap S''$ (Note that $n \equiv 5 (\mathrm{mod}\ 10)$). Since $C_5^0 \cap S'' = \{(0,0),(1,0),(2,0)\}$, $\{(2,1),(3,1)\} \subseteq C_5^1 \cap S''$. If $C_5^1 \cap S'' = \{(1,1),(2,1),(3,1)\}$, then at least one of $(3,1)$ and $(4,1)$ can not be dominated. If $C_5^1 \cap S'' = \{(0,1),(2,1),(3,1)\}$, then at least one of $(0,1)$ and $(3,1)$ can not be dominated. If $C_5^1 \cap S'' = \{(2,1),(3,1),(4,1)\}$, then at least one of $(0,1)$ and $(4,1)$ can not be dominated.

Similarly, the other case also contains a contradiction.

Subcase 3.3. $j \neq i+1$, $k \neq j+1$ and $k \neq n-1$.

By the argument similar to that of Subcase 3.2, we can always find some vertex which can not be dominated.

Therefore, S'' is not a $\gamma_t(C_5\Box C_n)$-set. And $\gamma_t(C_5\Box C_n) > \frac{5n+3}{2}$ when $n \equiv 5 (\mathrm{mod}\ 10)$. It is easy to see that $S_3 = S_1 \cup \{(0, n-1), (4, n-1)\}$ is a $\gamma_t(C_5\Box C_n)$-set, and $|S_3| = \frac{5n+5}{2}$. Therefore, $\gamma_t(C_5\Box C_n) = \frac{5n+5}{2}$ when $n \equiv 5 (\mathrm{mod}\ 10)$. $\Box$

Theorem 2.5. *Assume that $n \geq 7$ is odd. Then*

$$\gamma_t(C_7\Box C_n) = \begin{cases} \frac{7n+1}{2}, & \text{if } n \equiv 1,3,5,9,11,13 (\mathrm{mod}\ 14); \\ \frac{7n+7}{2}, & \text{if } n \equiv 7 (\mathrm{mod}\ 14). \end{cases}$$

Proof. Let S be any $\gamma_t(C_7\Box C_n)$-set. By Lemma 2.1, we have $(|S \cap C_7^0| + |S \cap C_7^1|) + (|S \cap C_7^1| + |S \cap C_7^2|) + \cdots + (|S \cap C_7^{n-1}| + |S \cap C_7^0|) \geq 7n$. That is, $|S| \geq \frac{7n}{2}$. Since $7, n$ are both odd, $|S| \geq \frac{7n+1}{2}$.

If $n \equiv 3, 11 (\mathrm{mod}\ 14)$, we define a vertex set S_1 as follows: S_1 consists of vertices $(0,i)$, $(1,i)$, $(2,i)$ and $(3,i)$, $i \equiv 0 (\mathrm{mod}\ 14)$; $(3,i)$, $(4,i)$ and $(5,i)$ $i \equiv 1 (\mathrm{mod}\ 14)$; $(0,i)$, $(1,i)$, $(5,i)$ and $(6,i)$, $i \equiv 2 (\mathrm{mod}\ 14)$; $(1,i)$, $(2,i)$ and $(3,i)$, $i \equiv 3 (\mathrm{mod}\ 14)$; $(3,i)$, $(4,i)$, $(5,i)$ and $(6,i)$, $i \equiv 4 (\mathrm{mod}\ 14)$; $(0,i)$, $(1,i)$ and $(6,i)$, $i \equiv 5 (\mathrm{mod}\ 14)$; $(1,i)$, $(2,i)$, $(3,i)$ and $(4,i)$, $i \equiv 6 (\mathrm{mod}\ 14)$; $(4,i)$, $(5,i)$ and $(6,i)$, $i \equiv 7 (\mathrm{mod}\ 14)$; $(0,i)$, $(1,i)$, $(2,i)$ and $(6,i)$, $i \equiv 8 (\mathrm{mod}\ 14)$; $(2,i)$, $(3,i)$ and $(4,i)$, $i \equiv 9 (\mathrm{mod}\ 14)$; $(0,i)$, $(4,i)$, $(5,i)$ and $(6,i)$, $i \equiv 10 (\mathrm{mod}\ 14)$; $(0,i)$, $(1,i)$ and $(2,i)$, $i \equiv 11 (\mathrm{mod}\ 14)$; $(2,i)$, $(3,i)$, $(4,i)$ and $(5,i)$, $i \equiv 12 (\mathrm{mod}\ 14)$; $(0,i)$, $(5,i)$ and $(6,i)$, $i \equiv 13 (\mathrm{mod}\ 14)$. It can be seen that S_1 is a total domination set of $C_7\Box C_n$, $|S_1| = 49 \cdot \frac{n-3}{14} + 11 = \frac{7n+1}{2}$ if $n \equiv 3 (\mathrm{mod}\ 14)$ and $|S_1| = 49 \cdot \frac{n-11}{14} + 39 = \frac{7n+1}{2}$ if $n \equiv 11 (\mathrm{mod}\ 14)$.

If $n \equiv 5, 9 (\mathrm{mod}\ 14)$, we define a vertex set S_2 as follows: S_2 consists of vertices $(0,i)$, $(1,i)$, $(3,i)$ and $(4,i)$, $i \equiv 0 (\mathrm{mod}\ 14)$; $(1,i)$, $(4,i)$ and $(5,i)$ $i \equiv 1 (\mathrm{mod}\ 14)$; $(1,i)$, $(2,i)$, $(5,i)$ and $(6,i)$, $i \equiv 2 (\mathrm{mod}\ 14)$; $(2,i)$, $(3,i)$ and $(6,i)$, $i \equiv 3 (\mathrm{mod}\ 14)$; $(0,i)$, $(3,i)$, $(4,i)$ and $(6,i)$, $i \equiv 4 (\mathrm{mod}\ 14)$; $(0,i)$, $(1,i)$ and $(4,i)$, $i \equiv 5 (\mathrm{mod}\ 14)$; $(1,i)$, $(2,i)$, $(4,i)$ and $(5,i)$, $i \equiv 6 (\mathrm{mod}\ 14)$; $(2,i)$, $(5,i)$ and $(6,i)$, $i \equiv 7 (\mathrm{mod}\ 14)$; $(0,i)$, $(2,i)$, $(3,i)$ and $(6,i)$, $i \equiv 8 (\mathrm{mod}\ 14)$; $(0,i)$, $(3,i)$ and $(4,i)$, $i \equiv 9 (\mathrm{mod}\ 14)$; $(0,i)$, $(1,i)$, $(4,i)$ and $(5,i)$, $i \equiv 10 (\mathrm{mod}\ 14)$; $(1,i)$, $(2,i)$ and $(5,i)$, $i \equiv 11 (\mathrm{mod}\ 14)$; $(2,i)$, $(3,i)$, $(5,i)$ and $(6,i)$, $i \equiv 12 (\mathrm{mod}\ 14)$; $(0,i)$, $(3,i)$ and $(6,i)$, $i \equiv 13 (\mathrm{mod}\ 14)$. Clearly, S_2 is a total domination set of $C_7\Box C_n$, $|S_2| = 49 \cdot \frac{n-5}{14} + 18 = \frac{7n+1}{2}$ if $n \equiv 5 (\mathrm{mod}\ 14)$ and $|S_2| = 49 \cdot \frac{n-9}{14} + 32 = \frac{7n+1}{2}$ if $n \equiv 9 (\mathrm{mod}\ 14)$.

If $n \equiv 1, 13 (\mathrm{mod}\ 14)$, we define a vertex set S_3 as follows: S_3 consists of vertices $(0,i)$, $(1,i)$, $(3,i)$ and $(5,i)$, $i \equiv 0 (\mathrm{mod}\ 14)$; $(1,i)$, $(3,i)$ and $(5,i)$ $i \equiv 1 (\mathrm{mod}\ 14)$; $(1,i)$, $(3,i)$, $(5,i)$ and $(6,i)$, $i \equiv 2 (\mathrm{mod}\ 14)$; $(1,i)$, $(3,i)$ and $(6,i)$, $i \equiv 3 (\mathrm{mod}\ 14)$; $(1,i)$, $(3,i)$, $(4,i)$ and $(6,i)$, $i \equiv 4 (\mathrm{mod}\ 14)$; $(1,i)$, $(4,i)$ and $(6,i)$, $i \equiv 5 (\mathrm{mod}\ 14)$; $(1,i)$, $(2,i)$, $(4,i)$ and $(6,i)$, $i \equiv 6 (\mathrm{mod}\ 14)$; $(2,i)$, $(4,i)$ and $(6,i)$, $i \equiv 7 (\mathrm{mod}\ 14)$; $(0,i)$, $(2,i)$, $(4,i)$ and $(6,i)$, $i \equiv 8 (\mathrm{mod}\ 14)$; $(0,i)$, $(2,i)$ and $(4,i)$, $i \equiv 9 (\mathrm{mod}\ 14)$; $(0,i)$, $(2,i)$, $(4,i)$ and $(5,i)$, $i \equiv 10 (\mathrm{mod}\ 14)$; $(0,i)$, $(2,i)$ and $(5,i)$, $i \equiv 11 (\mathrm{mod}\ 14)$; $(0,i)$, $(2,i)$, $(3,i)$ and $(5,i)$, $i \equiv 12 (\mathrm{mod}\ 14)$; $(0,i)$,

$(3, i)$ and $(5, i)$, $i \equiv 13 (\text{mod } 14)$. Clearly, S_3 is a total domination set of $C_7 \Box C_n$, $|S_3| = 49 \cdot \frac{n-13}{14} + 46 = \frac{7n+1}{2}$ if $n \equiv 13 (\text{mod } 14)$ and $|S_3| = 49 \cdot \frac{n-1}{14} + 4 = \frac{7n+1}{2}$ if $n \equiv 1 (\text{mod } 14)$.

If $n \equiv 7 (\text{mod } 14)$, by an argument similar to the proof of Theorem 2.4, there exists no $\gamma_t(C_7 \Box C_n)$-set S' such that $|S'| = \frac{7n+1}{2}$ or $\frac{7n+3}{2}$ or $\frac{7n+5}{2}$. Hence, $\gamma_t(C_7 \Box C_n) \geq \frac{7n+7}{2}$ when $n \equiv 7 (\text{mod } 14)$. It is easy to see that $S_4 = S_1 \cup \{(0, n-1), (5, n-1), (6, n-1)\}$ is a $\gamma_t(C_7 \Box C_n)$-set, and $|S_4| = \frac{7n+7}{2}$. Therefore, $\gamma_t(C_7 \Box C_n) = \frac{7n+7}{2}$ when $n \equiv 7 (\text{mod } 14)$. □

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