

Mixed connectivity of Cartesian graph products and bundles*

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Abstract

Mixed connectivity is a generalization of vertex and edge connectivity. A graph is $(p, 0)$ -connected, $p > 0$, if the graph remains connected after removal of any $p - 1$ vertices. A graph is (p, q) -connected, $p \geq 0, q > 0$, if it remains connected after removal of any p vertices and any $q - 1$ edges. Cartesian graph bundles are graphs that generalize both covering graphs and Cartesian graph products. It is shown that if graph F is (p_F, q_F) -connected and graph B is (p_B, q_B) -connected, then Cartesian graph bundle G with fibre F over the base graph B is $(p_F + p_B, q_F + q_B)$ -connected. Furthermore, if $q_F, q_B > 0$, then G is also $(p_F + p_B + 1, q_F + q_B - 1)$ -connected. Finally, let graphs $G_i, i = 1, \dots, n$, be (p_i, q_i) -connected and let k be the number of graphs with $q_i > 0$. The Cartesian graph product $G = G_1 \square G_2 \square \dots \square G_n$ is $(\sum p_i, \sum q_i)$ -connected, and, for $k \geq 1$, it is also $(\sum p_i + k - 1, \sum q_i - k + 1)$ -connected.

Keywords: vertex connectivity, edge connectivity, mixed connectivity, Cartesian graph bundle, Cartesian graph product, interconnection network, fault tolerance.

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1 Introduction

Graph products and bundles are among frequently studied interconnection network topologies. For example, the meshes, tori, hypercubes and some of their generalizations are Cartesian products. It is less known that some well-known topologies are Cartesian graph bundles, i.e. some twisted hypercubes [10, 13] and multiplicative circulant graphs [25]. Other graph products, sometimes under different names, have been studied as interesting network topologies [9, 22, 25].

In the design of large interconnection networks several factors have to be taken into account. A usual constraint is that each processor can be connected to a limited number of other processors and the delays in communication must not be too long. Furthermore, an interconnection network should be fault tolerant, because practical communication networks are exposed to failures of network components. Both failures of nodes and failures of connections between them happen and it is desirable that a network is robust in the sense that a limited number of failures does not break down the whole system. A lot of work has been done on various aspects of network fault tolerance, see, for example, the survey [8] and more recent papers [16, 26, 28]. In particular the fault diameter with faulty vertices which was first studied in [20] and the edge fault diameter has been determined for many important networks recently [11, 12, 21, 27]. In particular, the (vertex) fault diameter and the edge fault diameter of Cartesian graph products and Cartesian graph bundles was studied recently [2, 3, 4, 5]. Usually either only edge faults or only vertex faults are considered, while the case when both edges and vertices may be faulty is studied rarely. For example, [16, 26] consider Hamiltonian properties assuming a combination of vertex and edge faults. In recent work on fault diameter of Cartesian graph products and bundles [2, 3, 4, 5], analogous results were found for both fault diameter and edge fault diameter. However, the proofs for vertex and edge faults in [2, 3, 4, 5] are independent, and our effort to see how results in one case may imply the others was not successful. A natural question is whether it is possible to design a uniform theory that would enable unified proofs or provide tools to translate results for one type of faults to the other. It is therefore of interest to study general relationships between invariants under vertex and edge faults. Some basic results on edge, vertex and mixed fault diameters for general graphs appear in [6]. In order to study the fault diameters of graph products and bundles under mixed faults, it is important to understand the generalized connectivities.

Here, we study mixed connectivity which generalizes both vertex and edge connectivity. It is known that Cartesian graph bundle with fibre F over the base graph B is $(\kappa(F) + \kappa(B))$ -connected and $(\lambda(F) + \lambda(B))$ -edge connected [2, 5]. In this paper we generalize these results to mixed

connectivity of Cartesian graph bundles. More precisely, assuming that the fibre F is (p_F, q_F) -connected and the base graph B is (p_B, q_B) -connected, then Cartesian graph bundle G with fibre F over the base graph B is $(p_F + p_B, q_F + q_B)$ -connected. Furthermore, if $q_F, q_B > 0$, then the Cartesian graph bundle is also $(p_F + p_B + 1, q_F + q_B - 1)$ -connected. As a corollary, mixed connectivity of the Cartesian product of finite number of factors is given.

The rest of the paper is organized as follows. In the next section some general definitions are given, and in Section 3 the mixed connectivity is defined and some basic facts are observed. Section 4 recalls definition of graph bundles and states the main result (Theorem 4.6) which is proved in the last section.

2 Preliminaries

Here we only recall some basic definitions to fix the notation, for other standard notions not defined here we adopt the usual terminology (see for example [1]). A *simple graph* $G = (V, E)$ is determined by a *vertex set* $V = V(G)$ and a set $E = E(G)$ of (unordered) pairs of vertices, called the set of *edges*. As usual, we will use the short notation uv for edge $\{u, v\}$. For an edge $e = uv$ we call u and v its *endpoints*. It is convenient to consider the union of *elements* of a graph, $S(G) = V(G) \cup E(G)$. Given $X \subseteq S(G)$ then $S(G) \setminus X$ is a subset of elements of G . However, note that in general $S(G) \setminus X$ may not induce a graph. As we need notation for subgraphs with some missing (faulty) elements, we will formally define $G \setminus X$, the subgraph of G after deletion of X , as follows:

Definition 2.1 *Let $X \subseteq S(G)$, and $X = X_E \cup X_V$, where $X_E \subseteq E(G)$ and $X_V \subseteq V(G)$. Then $G \setminus X$ is the subgraph of $(V(G), E(G) \setminus X_E)$ induced on vertex set $V(G) \setminus X_V$.*

A *walk* between x and y is a sequence of vertices and edges $v_0, e_1, v_1, e_2, v_2, \dots, v_{k-1}, e_k, v_k$ where $x = v_0, y = v_k$, and $e_i = v_{i-1}v_i$ for each i . The *length* of a walk W , denoted by $\ell(W)$, is the number of edges in W . A walk with all vertices distinct is called a *path*, and the vertices v_0 and v_k are called the *endpoints* of the path. A path P in G , defined by a sequence $x = v_0, e_1, v_1, e_2, v_2, \dots, v_{k-1}, e_k, v_k = y$ can alternatively be seen as a subgraph of G with $V(P) = \{v_0, v_1, v_2, \dots, v_k\}$ and $E(P) = \{e_1, e_2, \dots, e_k\}$. Note that the reverse sequence gives rise to the same subgraph. Hence we use P for a path either from x to y or from y to x . A graph is *connected* if there is a path between each pair of vertices, and is *disconnected* otherwise.

The *connectivity* (or *vertex connectivity*) of a connected graph G , $\kappa(G)$, is the minimum cardinality over all vertex-separating sets in G . As the

complete graph K_n has no vertex-separating sets, we define $\kappa(K_n) = n - 1$. We say that G is k -connected (or k -vertex connected) for any $k \leq \kappa(G)$. The edge connectivity of a connected graph G , $\lambda(G)$, is the minimum cardinality over all edge-separating sets in G . A graph G is said to be k -edge connected for any $k \leq \lambda(G)$. In other words, the edge connectivity $\lambda(G)$ of a connected graph G is the smallest number of edges whose removal disconnects G , and the (vertex) connectivity $\kappa(G)$ of a connected graph G (other than a complete graph) is the smallest number of vertices whose removal disconnects G . It is well-known that (see, for example, [1], page 224) $\kappa(G) \leq \lambda(G) \leq \delta_G$, where δ_G is the smallest vertex degree of G . Thus if a graph G is k -connected, then it is also k -edge connected. The reverse does not hold in general. For later reference recall that by Menger's theorems (see, for example, [1], pages 230,234) we know that in a k -connected graph G there are at least k vertex disjoint paths between any two vertices in G , and if G is k -edge connected then there are at least k edge disjoint paths between any two vertices in G .

3 Mixed connectivity

Considering a graph with faulty vertices and faulty edges at the same time we can generalize both vertex and edge connectivity. We start with an observation that can be proved easily.

Proposition 3.1 *Let $p, q > 0$. If a graph G remains connected after removal of any p vertices and any $q - 1$ edges, then G also remains connected after removal of any $p - 1$ vertices and any q edges.*

Proof: Let H be any subgraph of G after removal of $p - 1$ vertices and $q - 1$ edges. By assumption, the graph H remains connected after removal of any vertex. Hence H is 2-connected which implies that H is 2-edge connected. In other words, H remains connected after removal of any edge which in turn implies that G remains connected after removal of any $p - 1$ vertices and any q edges. \square

If a graph G remains connected after removal of any p vertices (and any $q - 1$ edges), then $p < \kappa(G)$. By repeated application of Proposition 3.1, the graph G also remains connected after removal of any $p + q - 1$ edges, hence $p + q \leq \lambda(G)$. Now we formally define mixed-connectivity.

Definition 3.2

(1) *Let $p > 0$. Graph G is $(p, 0)$ -connected, if G remains connected after removal of any $p - 1$ vertices.*

(2) *Let $q > 0$. Graph G is (p, q) -connected, if G remains connected after removal of any p vertices and any $q - 1$ edges.*

Clearly, if G is (p, q) -connected graph, then G is (p', q') -connected for any $p' \leq p$ and any $q' \leq q$, and if $q > 0$ then (p, q) -connected graph is also $(p + 1, 0)$ -connected. For any (p, q) -connected graph we have $p + q \leq \lambda(G) \leq \delta_G$, thus each vertex of a (p, q) -connected graph has at least $p + q$ neighbors, and hence a (p, q) -connected graph has at least $p + q + 1$ vertices.

Mixed connectivity is a generalization of vertex and edge connectivity: a graph G is $(p, 0)$ -connected for all $p \leq \kappa(G)$ and is not $(p, 0)$ -connected for $p > \kappa(G)$. Furthermore, G is $(0, q)$ -connected for all $q \leq \lambda(G)$ and is not $(0, q)$ -connected for $q > \lambda(G)$. In particular, any graph G is $(\kappa(G), 0)$ -connected and $(0, \lambda(G))$ -connected.

The next statement follows directly from Proposition 3.1.

Corollary 3.3 *If $p > 0$ and graph G is (p, q) -connected then G is $(p - 1, q + 1)$ -connected.*

Hence for $p > 0$ we have a chain of implications

(p, q) -C \implies $(p - 1, q + 1)$ -C $\implies \dots \implies (1, p - 1 + q)$ -C $\implies (0, p + q)$ -C
 where (i, j) -C stands for " G is (i, j) -connected".

Corollary 3.3 is for $q = 0$ a generalization of well-known proposition that any k -connected graph is also k -edge connected. If G is $(k, 0)$ -connected, then it is also $(k - i, i)$ -connected for any $i \leq k$, and hence also $(0, k)$ -connected.

If for graph G $\kappa(G) = \lambda(G) = k$, then G is (i, j) -connected exactly when $i + j \leq k$. However, if $2 \leq \kappa(G) < \lambda(G)$, the question whether G is (i, j) -connected for $1 \leq i < \kappa(G) < i + j \leq \lambda(G)$ is not trivial. The example below shows that in general knowing $\kappa(G)$ and $\lambda(G)$ is not enough to decide whether G is (i, j) -connected.

Example 3.4 *For graphs on Fig. 1 we have $\kappa(G_1) = \kappa(G_2) = 2$ and $\lambda(G_1) = \lambda(G_2) = 3$. Both graphs are $(2, 0)$ -connected $\implies (1, 1)$ -connected $\implies (0, 2)$ -connected and $(0, 3)$ -connected. Graph G_1 is not $(1, 2)$ -connected, while graph G_2 is.*

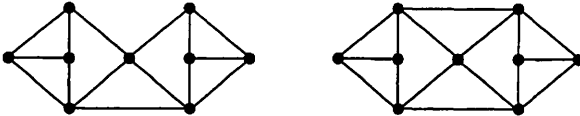


Figure 1: Graphs G_1 and G_2 from Example 3.4.

Let G be any connected graph, $\kappa(G) < \lambda(G)$. All known mixed connectivities for graph G are summarized in Diagram 1 (Fig. 2).

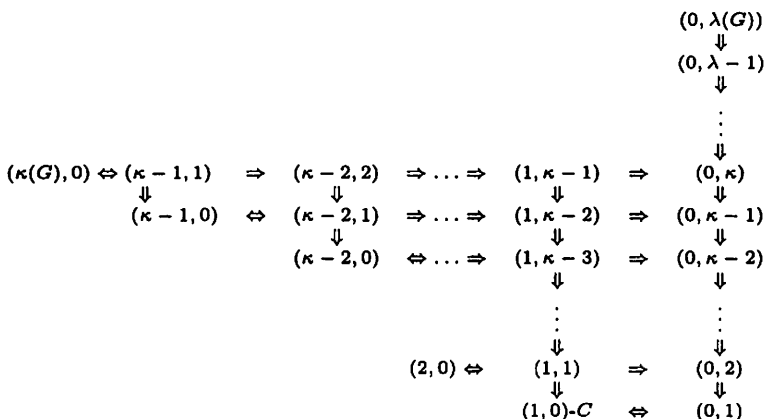


Figure 2: Diagram 1

Both edge connectivity and connectivity (or vertex connectivity) of a graph can be computed in polynomial time. More precisely, according to [19], there is an algorithm with time complexity $O(n^4)$ for vertex connectivity [15] and edge connectivity can be computed within $O(m + \lambda(G)n \log \frac{n}{\lambda(G)})$ time [14]. Hence given a graph G , there are polynomial time algorithms to decide $(i, 0)$ and $(0, j)$ -connectivities for all i, j . Therefore, it is interesting to ask

Problem. Let G be a graph and $1 \leq i < \kappa(G) < i + j \leq \lambda(G)$. Is there a polynomial algorithm to decide whether G is (i, j) -connected?

If graph G is (i, j) -connected for some $1 \leq i < \kappa(G) < i + j \leq \lambda(G)$ then the upper part of diagram can be updated with known mixed connectivities as in Diagram 2 (Fig.3).

Two extreme cases are:

(1) If graph G is $(\kappa(G) - 1, \lambda(G) - \kappa(G) + 1)$ -connected then the diagram of mixed connectivities is maximal because this connectivity implies all possible mixed connectivities for any connected graph. Namely, if graph G is $(\kappa(G) - 1, \lambda(G) - \kappa(G) + 1)$ -connected then graph G is (i, j) -connected for all $i < \kappa(G)$ and all $i + j \leq \lambda(G)$.

(2) If G is not $(1, \kappa(G))$ -connected then the diagram of mixed connectivities is minimal, i.e. the connectivities from Diagram 1 are all connectivities of G .

In general, at most $(\kappa(G) - 1) \times (\lambda(G) - \kappa(G))$ different connectivities may have to be checked to complete the diagram of all connectivities of G .

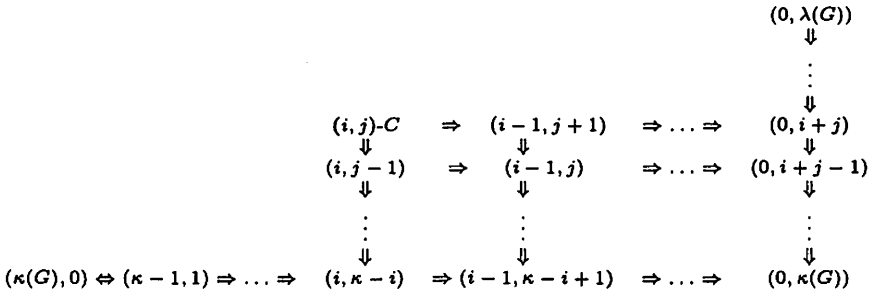


Figure 3: Diagram 2

4 Mixed connectivity of Cartesian graph bundles

A Cartesian graph bundle is a generalization of graph cover and the Cartesian graph product. Let G_1 and G_2 be graphs. The Cartesian product of graphs G_1 and G_2 , $G = G_1 \square G_2$, is defined on the vertex set $V(G_1) \times V(G_2)$. Vertices (u_1, v_1) and (u_2, v_2) are adjacent if either $u_1 u_2 \in E(G_1)$ and $v_1 = v_2$ or $v_1 v_2 \in E(G_2)$ and $u_1 = u_2$. For further reading on graph products we recommend [17].

Definition 4.1 *Let B and F be graphs. A graph G is a Cartesian graph bundle with fibre F over the base graph B if there is a graph map $p : G \rightarrow B$ such that for each vertex $v \in V(B)$, $p^{-1}(\{v\})$ is isomorphic to F , and for each edge $e = uv \in E(B)$, $p^{-1}(\{e\})$ is isomorphic to $F \square K_2$.*

More precisely, the mapping $p : G \rightarrow B$ maps graph elements of G to graph elements of B , i.e. $p : V(G) \cup E(G) \rightarrow V(B) \cup E(B)$. In particular, here we also assume that the vertices of G are mapped to vertices of B and the edges of G are mapped either to vertices or to edges of B . We say an edge $e \in E(G)$ is *degenerate* if $p(e)$ is a vertex. Otherwise we call it *nondegenerate*. The mapping p will also be called the *projection* (of the bundle G to its base B). Note that each edge $e = uv \in E(B)$ naturally induces an isomorphism $\varphi_e : p^{-1}(\{u\}) \rightarrow p^{-1}(\{v\})$ between two fibres. It may be interesting to note that while it is well-known that a graph can have only one representation as a product (up to isomorphism and up to the order of factors) [17], there may be many different graph bundle representations of the same graph [31]. Here we assume that the bundle

representation is given. Note that in some cases finding a representation of G as a graph bundle can be found in polynomial time [18, 29, 30, 31, 32, 33]. For example, one of the easy classes is the Cartesian graph bundle over a triangle-free base [18]. Graph bundles were first studied in [23, 24]. Note that a graph bundle over a tree T (as a base graph) with fibre F is isomorphic to the Cartesian product $T \square F$ (not difficult to see, appears already in [23]), i.e. we can assume that all isomorphisms φ_e are identities.

Example 4.2 Let $F = K_2$ and $B = C_3$. On Fig. 4 we see two nonisomorphic bundles with fibre F over the base graph B . Informally, one can say that bundles are “twisted products”.

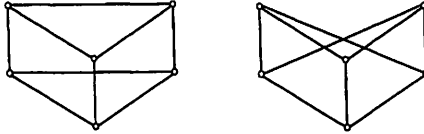


Figure 4: Nonisomorphic bundles from Example 4.2

Example 4.3 It is less well-known that graph bundles also appear as computer topologies. A well-known example is the twisted torus on Fig. 5. Cartesian graph bundle with fibre C_4 over base C_4 is the ILLIAC IV architecture [7], a famous supercomputer that inspired some modern multicomputer architectures. It may be interesting to note that the original design was a graph bundle with fibre C_8 over base C_8 , but due to high cost a smaller version was built [34].

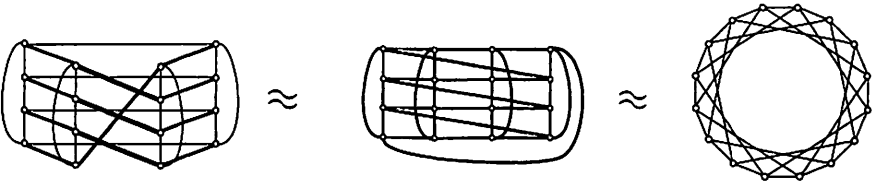


Figure 5: Twisted torus: Cartesian graph bundle with fibre C_4 over base C_4 .

Let G be a Cartesian graph bundle with fibre F over the base graph B . The fibre of vertex $x \in V(G)$ is denoted by F_x , formally, $F_x = p^{-1}(\{p(x)\})$.

We will also use notation $F(u)$ for the fibre of the vertex $u \in V(B)$, i.e. $F(u) = p^{-1}(\{u\})$. Note that $F_x = F(p(x))$. Let $u, v \in V(B)$ be distinct vertices, Q be a path from u to v in B , and $x \in F(u)$. Then the *lift of the path Q to the vertex $x \in V(G)$* , \tilde{Q}_x , is the path from $x \in F(u)$ to a vertex in $F(v)$, such that $p(\tilde{Q}_x) = Q$ and $\ell(\tilde{Q}_x) = \ell(Q)$. Let $x, x' \in F(u)$. Then \tilde{Q}_x and $\tilde{Q}_{x'}$ have different endpoints in $F(v)$ and are disjoint paths if and only if $x \neq x'$. We will also use notation \tilde{Q} for lifts of path Q to any vertex in $F(u)$.

In previous work [2, 5] on vertex and edge fault diameters of Cartesian graph bundles, propositions about vertex and edge connectivity of Cartesian graph bundles have been proved. In terms of mixed connectivity, they read as follows.

Proposition 4.4 [2] *If graph F is $(p_F, 0)$ -connected and graph B is $(p_B, 0)$ -connected, then Cartesian graph bundle with fibre F over the base graph B is $(p_F + p_B, 0)$ -connected.*

Proposition 4.5 [5] *If graph F is $(0, q_F)$ -connected and graph B is $(0, q_B)$ -connected, then Cartesian graph bundle with fibre F over the base graph B is $(0, q_F + q_B)$ -connected.*

A natural generalization of these two propositions would be that if graph F is (p_F, q_F) -connected and graph B is (p_B, q_B) -connected, then Cartesian graph bundle with fibre F over the base graph B is $(p_F + p_B, q_F + q_B)$ -connected. This is indeed true, but we can prove a slightly stronger statement (see Theorem 4.6). Roughly speaking, consider the maximum allowed number of faulty elements of graphs F , B and G . If $q_F = 0$ or $q_B = 0$, then generalization assures connectivity of graph bundle G with one more faulty element (vertex or edge) as the sum of faulty vertices and faulty edges in graphs F and B .

We prove that besides the sum of allowed faulty elements of the fibre and the base, one additional faulty element is allowed. We also show that whenever applicable, the extra faulty element can be a node. In particular this improves the result on edge connectivity. Namely, Theorem 4.6 is for $p_F = p_B = 0$ stronger than Proposition 4.5. If graph F is $(0, q_F)$ -connected and graph B is $(0, q_B)$ -connected, then by Theorem 4.6, Cartesian graph bundle with fibre F over the base graph B is $(1, q_F + q_B - 1)$ -connected, while Proposition 4.5 assures only $(0, q_F + q_B)$ -connectivity.

Theorem 4.6 *Let G be a Cartesian graph bundle with fibre F over the base graph B , graph F be (p_F, q_F) -connected and graph B be (p_B, q_B) -connected. Then Cartesian graph bundle G is:*

1. $(p_F + p_B, q_F + q_B)$ -connected.

2. for $q_F, q_B > 0$ also $(p_F + p_B + 1, q_F + q_B - 1)$ -connected.

As the Cartesian product is a Cartesian graph bundle where all the isomorphisms between the fibres are identities, the statement about mixed connectivity of Cartesian graph products of a finite number of factors follows easily from Theorem 4.6. Let $G = G_1 \square G_2 \square \dots \square G_k$, and let $G_i, i = 1, \dots, k$, be (p_i, q_i) -connected, and $q_i > 0$. Then, by induction, G is $(\sum p_i + k - 1, \sum q_i - k + 1)$ -connected. Therefore

Corollary 4.7 *Let graphs $G_i, i = 1, \dots, n$, be (p_i, q_i) -connected and let k be the number of graphs with $q_i > 0$. Then the Cartesian graph product $G = G_1 \square G_2 \square \dots \square G_n$ is:*

1. $(\sum p_i, \sum q_i)$ -connected, and
2. $(\sum p_i + k - 1, \sum q_i - k + 1)$ -connected, for $k \geq 1$.

5 Proof of the main theorem

Let G be any connected graph. Then for any $p < \lambda(G)$ a graph G is $(p + 1, 0)$ -connected if and only if G is $(p, 1)$ -connected. Therefore it is enough to prove Theorem 4.6 only for the case when $q_F, q_B > 0$. For example, if graphs F and B are $(p_F, 0)$ -connected and $(p_B, 0)$ -connected, then graphs are also $(p_F - 1, 1)$ -connected and $(p_B - 1, 1)$ -connected respectively. By Theorem 4.6 (2) a Cartesian graph bundle G is $(p_F + p_B - 1, 1)$ -connected, hence G is $(p_F + p_B, 0)$ -connected.

Hence from now on we can assume $q_F, q_B > 0$. We will show that the Cartesian graph bundle with fibre F over the base graph B without maximum allowed number of faulty elements is connected. Denote the set of faulty vertices by $X \subseteq V(G)$, $|X| = p_F + p_B + 1$, and the set of faulty edges by $Y \subseteq E(G)$, $|Y| = q_F + q_B - 2$. We have to show that $G \setminus (X \cup Y)$ is a connected graph.

For each vertex $v \in V(B)$, fibre $F(v) \setminus (X \cup Y)$ is either a connected or a disconnected subgraph of $G \setminus (X \cup Y)$. This wording is convenient and we will use it although it is not formally clean because $G \setminus (X \cup Y)$ is of course most likely not a graph bundle, and by saying that the fibre $F(v)$ is connected or disconnected we are in fact referring to the properties of the subgraph $F(v) \setminus (X \cup Y)$.

It is not difficult to see that a graph $G \setminus (X \cup Y)$ always contains at least one connected fibre, and it may contain some disconnected fibres. We will prove Theorem 4.6 by proving two lemmas. With Lemma 5.1 we show that in $G \setminus (X \cup Y)$ any vertex of a disconnected fibre is connected with some vertex of connected fibre. With Lemma 5.2 we show that there is

a path between any two connected fibres in $G \setminus (X \cup Y)$. Both lemmas together assure that all pairs of vertices in $G \setminus (X \cup Y)$ are connected by paths, which implies Theorem 4.6.

The *weights* which are used in proofs of lemmas are defined as follows:

- (*faulty*) *vertex-weight* of vertex $v \in V(B)$, $w_X(v)$, is the number of faulty vertices in fibre $F(v)$, $w_X(v) = |F(v) \cap X|$;
- (*faulty*) *edge-weight* of vertex $v \in V(B)$, $w_Y(v)$, is the number of faulty (degenerate) edges in fibre $F(v)$, $w_Y(v) = |F(v) \cap Y|$;
- (*faulty*) *edge-weight* of path $Q \subseteq B$, $w_Y(Q)$, is the number of faulty (nondegenerate) edges on lifts of path Q , $w_Y(Q) = |p^{-1}(Q) \cap Y_N|$, where Y_N is the set of faulty nondegenerate edges in Y , $Y_N \subseteq Y$, $p(Y_N) \subseteq E(B)$, $p(Y \setminus Y_N) \subseteq V(B)$.

Lemma 5.1 *Let G be a Cartesian graph bundle with fibre F over the base graph B , graph F be (p_F, q_F) -connected, graph B be (p_B, q_B) -connected. Let $X \subseteq V(G)$ and $Y \subseteq E(G)$ be sets of faulty vertices and faulty edges with maximal allowed number of elements, and let $x \in V(G) \setminus X$ be any vertex of a disconnected fibre in $G \setminus (X \cup Y)$. Then in $G \setminus (X \cup Y)$ there exists a (neighboring) connected fibre and there is a path between vertex x and a vertex of connected fibre.*

Proof: Let graph F be (p_F, q_F) -connected, graph B be (p_B, q_B) -connected, $q_F, q_B > 0$, let $X \subseteq V(G)$ and $Y \subseteq E(G)$ be sets of faulty vertices and faulty edges, $|X| = p_F + p_B + 1$, $|Y| = q_F + q_B - 2$. Let $x \in V(G) \setminus X$ and $F_x \setminus (X \cup Y)$ be a disconnected fibre in $G \setminus (X \cup Y)$. Then either $w_X(p(x)) > p_F$ or $w_X(p(x)) + w_Y(p(x)) \geq p_F + q_F$. We distinguish two cases.

1. Suppose $w_X(p(x)) + w_Y(p(x)) \geq p_F + q_F$.

Fibre F_x contains at least $p_F + q_F$ faulty elements, so outside of fibre F_x there are at most $p_B + q_B - 1$ faulty elements. As vertex $p(x)$ has at least $p_B + q_B$ neighbors in B , there is a neighbor v , $e = p(x)v$, with weights $w_X(v) + w_Y(v) = 0$ and $w_Y(e) = 0$. Hence fibre $F(v) \subseteq G \setminus (X \cup Y)$ is connected and the lift \tilde{e}_x avoids $X \cup Y$. So x is adjacent to a vertex in $F(v)$ as needed.

2. Now assume that $w_X(p(x)) > p_F$.

Fibre F_x contains at least $p_F + 1$ faulty vertices, so outside of fibre F_x there are at most p_B faulty vertices (and $q_F + q_B$ faulty edges).

Let $X_B = \{v \in V(B) \setminus \{p(x)\} \mid w_X(v) > 0\}$ and $b = |X_B|$. Then $b \leq p_B$.

As there are at least $p_B + q_B$ neighbors of vertex $p(x)$ in B , therefore there are at least $p_B + q_B - b \geq 1$ neighbors v_i in $B \setminus X_B$ with weights $w_X(v_i) = 0$.

If there is a neighbor v of vertex $p(x)$ in $B \setminus X_B$ ($v = v_i$ for some i), $e = p(x)v$, with edge-weights $w_Y(v) = w_Y(e) = 0$, then $F(v) \subseteq G \setminus (X \cup Y)$ is a connected fibre and the lift \tilde{e}_x avoids $X \cup Y$. So x is adjacent to a vertex in $F(v)$ as needed.

Now suppose that for every neighbor v_i of vertex $p(x)$ in $B \setminus X_B$, $e_i = p(x)v_i$, $w_Y(v_i) + w_Y(e_i) > 0$. Then let v be any neighbor ($v = v_i$ for any i), $e = p(x)v$, in $B \setminus X_B$. Outside of $p^{-1}(\{e\})$ there are at least b faulty vertices (because we eliminate set X_B) and $p_B + q_B - b - 1$ faulty edges (in other neighbors), together $p_B + q_B - 1$ faulty elements. Therefore there are at most $p_F + q_F$ faulty elements in $p^{-1}(\{e\})$. As fibre $F_x \subset p^{-1}(\{e\})$ has at least $p_F + 1$ faulty vertices, there are at most $q_F - 1$ faulty edges in $F(v) \subset p^{-1}(\{e\})$, hence $F(v) \setminus Y \subseteq G \setminus (X \cup Y)$ is connected fibre.

There are at least $p_F + q_F$ neighbors of vertex x in F_x . Denote the neighbors by s_i , $i = 1, \dots, p_F + q_F$, and let $e_i = xs_i$. There are $p_F + q_F + 1$ vertex disjoint paths in $p^{-1}(\{e\})$ with one endpoint x and another endpoint in connected fibre $F(v)$: $x, e_i, s_i, \tilde{e}, v_i$ and x, \tilde{e}, v' , where \tilde{e} is a lift of the edge $e = p(x)v$ and v_i, v' are different vertices in fibre $F(v)$. As there are more vertex disjoint paths than faulty elements, at least one of these paths avoids $X \cup Y$. \square

Lemma 5.2 *Let G be a Cartesian graph bundle with fibre F over the base graph B , graph F be (p_F, q_F) -connected, graph B be (p_B, q_B) -connected. Let $X \subseteq V(G)$ and $Y \subseteq E(G)$ be sets of faulty vertices and faulty edges with maximal allowed number of elements. Then for any two connected fibres $F_x \setminus (X \cup Y)$ and $F_y \setminus (X \cup Y)$ there is a path with endpoints in fibres F_x and F_y that avoids faulty elements.*

Proof: Let graph F be (p_F, q_F) -connected, graph B be (p_B, q_B) -connected, $q_F, q_B > 0$, let $X \subseteq V(G)$ and $Y \subseteq E(G)$ be sets of faulty vertices and faulty edges, $|X| = p_F + p_B + 1$, $|Y| = q_F + q_B - 2$.

Let $X_B = \{v \in V(B) \setminus \{p(x), p(y)\} \mid w_X(v) > 0\}$ and $b = |X_B|$.

1. First assume $b < p_B$.

Graph $B \setminus X_B$ is $(p_B - b, q_B)$ -connected, hence $B \setminus X_B$ is $(0, p_B + q_B - b)$ -connected, and therefore there are at least $p_B + q_B - b \geq q_B + 1 \geq 2$ edge disjoint paths between $p(x)$ and $p(y)$ in B that avoid X_B ; one of them may be an edge. There are at least $p_B + q_B - b - 1 \geq q_B \geq 1$

edge disjoint paths (with lengths more than 1) that internally avoid $p(X)$.

- (A) Suppose there is a path Q between $p(x)$ and $p(y)$ (with length more than 1) that internally avoids $p(X)$ with edge-weights $w_Y(Q) + w_Y(v) = 0$, where $v \in Q$ is neighbor of vertex $p(x)$. Then $F(v) \subseteq G \setminus (X \cup Y)$ is connected fibre. There are at least $p_F + q_F + 1$ lifts of edge $e = p(x)v \subset Q$ with different endpoints in fibres F_x and $F(v)$. As these lifts contain at most p_F faulty endpoints in fibre F_x , there is a lift \tilde{e} that avoids faulty elements. Similarly, there is a lift of the path $vp(y) \subset Q$ that avoids faulty elements, and as $F(v) \subseteq G \setminus (X \cup Y)$ is connected fibre, there is a path between fibres F_x and F_y that avoids faulty elements.
- (B) Now suppose that for each path Q_i between $p(x)$ and $p(y)$ (with length more than 1) that internally avoids $p(X)$, the sum of edge-weights is $w_Y(Q_i) + w_Y(v_i) > 0$, where $v_i \in Q_i$ is the neighbor of vertex $p(x)$ along Q_i . Let Q be one of the shortest paths between $p(x)$ and $p(y)$ in $B \setminus X_B$. Then outside $p^{-1}(Q)$ there are at least b faulty vertices and $p_B + q_B - b - 1$ faulty edges, together at least $p_B + q_B - 1$ faulty elements. Therefore $p^{-1}(Q)$ contains at most $p_F + q_F$ faulty elements. As there are at least $p_F + q_F + 1$ lifts of path Q between fibres F_x and F_y , at least one of them avoids faulty elements.

2. Now assume $b \geq p_B$.

Let $X'_B \subseteq X_B$ be any subset with $|X'_B| = p_B$. Graph $B \setminus X'_B$ is $(0, q_B)$ -connected, so there are $q_B \geq 1$ edge disjoint paths between $p(x)$ and $p(y)$ in $B \setminus X'_B$.

- (A) If there is a path Q between $p(x)$ and $p(y)$ in $B \setminus X'_B$ with edge-weight $w_Y(Q) = 0$, then the lifts of path Q do not contain faulty edges. Outside $p^{-1}(Q)$ there are at least p_B faulty vertices. Therefore $p^{-1}(Q)$ contains at most $p_F + 1$ faulty vertices. As there are at least $p_F + q_F + 1$ lifts of path Q between fibres F_x and F_y , at least one of them avoids faulty elements.
- (B) Finally assume that there is no path with edge-weight 0 between $p(x)$ and $p(y)$ in $B \setminus X'_B$. Let Q be any path between $p(x)$ and $p(y)$ in $B \setminus X'_B$. Outside $p^{-1}(Q)$ there are at least p_B faulty vertices and $q_B - 1$ faulty edges, together at least $p_B + q_B - 1$ faulty elements. Therefore $p^{-1}(Q)$ contains at most $p_F + q_F$ faulty elements. As there are at least $p_F + q_F + 1$ lifts of path Q between fibres F_x and F_y , at least one of them avoids faulty elements. \square

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