Star coloring of Cartesian product of paths and cycles

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Abstract A star coloring of an undirected graph G is a proper vertex coloring of G such that any path on four vertices in G is not bicolored. The star chromatic number $\chi_s(G)$ of an undirected graph G is the smallest integer k for which G admits a star coloring with k colors. In this paper, the star chromatic numbers for some infinite subgraphs of Cartesian product of paths and cycles are established. In particular, we show that $\chi_s(P_i \Box C_j) = 5$ for $i, j \geq 4$ and $\chi_s(C_i \Box C_j) = 5$ for $i, j \geq 30$. We also show that $\chi_s(P_i \Box P_j \Box P_k) = 6$ for $i, j, k \geq 4$, $\chi_s(C_3 \Box C_3 \Box C_k) = 7$ for $k \geq 3$, $\chi_s(C_{4i} \Box C_{4j} \Box C_{4k} \Box C_{4\ell}) \leq 9$ for $i, j, k, \ell \geq 1$. Furthermore, we give the star chromatic numbers of d-dimensional hypercubes for $d \leq 6$.

1 Introduction

For a simple graph G, we denote by $V(G)$ and $E(G)$ the vertex set and edge set of G, respectively. Let G be a graph and $v \in V(G)$, the *open neighborhood* of v in G is denoted by $N(v)$. That is to say $N_(v) = \{u | uv \in E(G)\}$. A *proper k-coloring* of a graph G is an assignment of colors from $\{1, 2, \cdots, k\}$ to the vertices of G such that adjacent vertices receive distinct colors. The minimum k so that G has a proper k-coloring is called the *chromatic number* of G, denoted by $\chi(G)$. Let f be a proper-coloring of

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G. If G is assigned exactly with two colors under f, then we say G is *bicolored.* Moreover, we say G is S-colored under f if $S = \cup_{v \in V(G)} f(v)$. The *graph coloring problem* consists of finding the chromatic number of a graph, which is a well-studied **NP**-complete problem [1].

A *star coloring* of an undirected graph G is a proper vertex coloring of G such that any path on four vertices in G is not bicolored. A star coloring with k colors is called the *k-star-coloring.* The *star chromatic number* of an undirected graph G, denoted by $\chi_s(G)$, is the smallest integer k for which G admits a k-star-coloring. *The star coloring problem* consists of finding the minimum k such that a graph admits a k-star-coloring. Grünbaum [2] noted that the condition that the union of any two color classes induce a forest can be generalized to other bipartite graphs. Among other problems, he suggested requiring that the union of any two color classes induce a star forest, which is actually the star coloring problem. Later, it was well studied and has been widely investigated [3, 7].

A star coloring is thus a usual coloring with an additional condition: any path on four vertices must contain three vertices with mutually different colors. We add that star colorings are of similar nature as $L(p,q)$-labelings in which labels of adjacent vertices differ by at least p and labels of vertices at distance 2 differ by at least q. More precisely, star colorings are similar to $L(1,1)$-labelings, which were also considered in the study of the coloring of square of graphs [13]. Star colorings require that any path on four vertices must receive at least three different colors while $L(1,1)$-labelings require that any path on three vertices must receive three different colors. For more information on $L(p,q)$-labelings we refer to recent papers [12], in which graph products were studied.

Albertson et al. [9] showed that the star coloring problem is **NP**-complete even restricted to planar bipartite graphs. Fertin et. al. [4] determined the exact value of the star chromatic number of different families of graphs such as trees, cycles, complete bipartite graphs, outer planar graphs and 2-dimensional grids. They also provided bounds for the star chromatic numbers of other families of graphs, such as planar graphs, hypercubes, d-dimensional grids ($d \geq 3$), d-dimensional tori ($d \geq 2$), graphs with bounded treewidth and cubic graphs. However, the star coloring of planar graphs has attracted lots of attention. Fertin et. al. [4] linked star coloring to acyclic coloring and provided an upper bound 2304 for the star chromatic number of a planar graph. Later, this upper bound was pushed down to 80 by [8] and to 30 by [6]. However, Albertson et. al. [9] improved this bound to 20. Concerning the lower bound, Fertin et. al. [4] gave a planar graph with its star chromatic number 6.

The *Cartesian product* of graphs G and H is the graph $G \square H$ with the vertex set $V(G) \times V(H)$, and $(g,h)(g',h') \in E(G \square H)$ if either $gg' \in E(G)$ and $h = h'$, or $hh' \in E(H)$ and $g = g'$. The Cartesian product is commutative and associative, having the one vertex graph as a unit. The subgraph of $G \square H$ induced by $V(G) \times \{h\}$, where $h \in V(H)$, is isomorphic to G, called a *G-layer* (over h) and denoted G^h. For more information on the Cartesian product of graphs see [5]. The *path* P_n of length $n-1$ is the graph whose vertices are $0, 1, \ldots, n-1$ and for which two vertices are adjacent precisely if their difference is ± 1. For an integer $n \geq 3$, the *cycle of length* n is the graph C_n whose vertices are $0, 1, \ldots, n-1$ and whose edges are the pairs $i, i+1$, where the arithmetic is done modulo n. Let X_i be P_i or C_i. The subgraph X_i^k of $X_i \square C_j$ is the graph induced by the vertices of k-th column and we denote by $V(X_i^k)$ the vertex set $\{(0,k), (1,k), \cdots, (i-1,k)\}$. For convenience, when considering the Cartesian product of three graphs (resp. two graphs), we write the vertex (i,j,k) (resp. (i,j)) as ijk (resp. ij) and $f(i,j,k)$ (resp. $f(i,j)$) as $f(ijk)$ (resp. $f(ij)$).

In [4], the chromatic numbers of d-dimensional grid and tori were studied. Let Q_d be the d-dimensional hypercube, i.e., $Q_d = \underbrace{P_2 \square P_2 \square \cdots, \square P_2}_{d \text{ times}}$.

In [4], it was shown that

Theorem 1 $\frac{d+3}{2} \leq \chi_s(Q_d) \leq d+1$.

For any $n_i \geq 3$, $1 \leq i \leq d$, we denote by $TG_d = TG(n_1, n_2, \cdots, n_d)$ the toroidal d-dimensional grid having n_i vertices in dimension i, which is the Cartesian product of d cycles of length n_i, $1 \leq i \leq d$. In [4], Fertin et al. proved that

Theorem 2

$$d+2 \leq \chi_s(TG_d) \leq \begin{cases} 2d+1, & \textit{if } 2d+1 \textit{ devides each } n_i \\ 2d^2+d+1, & \textit{otherwise} \end{cases}$$

In this paper we are interested in the star coloring of Cartesian products of paths and cycles, primarily motivated with investigations in [4] where exact star chromatic numbers were determined for some specific products. More precisely, we study the star chromatic number in the infinite families of Cartesian product of paths and cycles. The chromatic numbers of Cartesian product of two cycles and some Cartesian product of three paths are determined. Some bounds for the chromatic numbers of Cartesian product of paths and cycles are provided. Moreover, the star chromatic numbers of d-dimensional hypercubes are given for $d \leq 6$.

2 Preliminaries

Lemma 1 *If G is a subgraph of H, then $\chi_s(G) \leq \chi_s(H)$.*

Given two integers r and s, let $S(r,s)$ denote the set of all nonnegative integer combinations of r and s:

$$S(r,s) = \{\alpha r + \beta s : \alpha, \beta \in Z^{+}\}$$

The following result of Sylvester [11] is useful to provide star colorings for infinite cases:

Lemma 2 *(Sylvester [11]) If $r, s > 1$ are relatively prime integers, then $t \in S(r,s)$ for all $t \geq (s-1)(r-1)$.*

A k-star-coloring f of $C_i \square C_j$ can be represented by a pattern with i rows and j columns. For example, the pattern

```
3 1 2 1
2 4 3 4
3 1 2 1
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is a 4-star-coloring f of $P_3 \square C_4$, where $f(00) = 3$, $f(10) = 2$, $f(20) = 3$, $f(01) = 1$, $f(11) = 4$, $f(20) = 1$, $f(20) = 2$, $f(12) = 3$, $f(22) = 4$, $f(03) = 1$, $f(13) = 4$ and $f(23) = 1$.

Definition 1 *Let $p, q, i, j \geq 3$ with $p \leq i$ and $q \leq j$ and a k-star-coloring f of $C_i \square C_j$ be represented by a pattern A. A coloring f is called a $(k;p,q)$-star-coloring (or simply $(k;p,q)$-coloring) of $C_i \square C_j$ if the following three conditions are fulfilled:*
(1) the first p rows of A induce a k-star-coloring of $C_p \square C_j$,
(2) the first q columns of A induce a k-star-coloring of $C_i \square C_q$,
(3) the upper left $p \times q$ pattern of A induces a k-star-coloring of $C_p \square C_q$.

For example, let A be the pattern depicted in Fig. 6. Note that the first 4 rows of A induce a 5-star-coloring of $C_4 \square C_{11}$, the first 4 columns of A induce a 5-star-coloring of $C_{11} \square C_4$, and the upper left 4×4 pattern of A induces a 5-star-coloring of $C_4 \square C_4$, we conclude that A is a $(5;4,4)$-star-coloring of $C_{11} \square C_{11}$.

Lemma 3
(1) Let $p, q, i, j \geq 3$ with $p \leq i, q \leq j$. If f is a $(k;p,q)$-star-coloring of $C_i \square C_j$. Then $\chi_s(C_s \square C_t) \leq k$ for any $s \in S(p,i)$ and $t \in S(q,j)$.
(2) Let $i, j, p \geq 3$ and f be a $(k;p,q)$-star-coloring of $C_i \square C_j$. Then $\chi_s(P_r \square C_t) \leq k$ for $r \geq 1$ and $t \in S(q,j)$.
(3) Let $j, p \geq 3, i \geq 1$ and f be a $(k;i,p)$-star-coloring of $P_i \square C_j$. Then $\chi_s(P_i \square C_t) \leq k$ for any $t \in S(q,j)$.

Proof. (1) Let A be the corresponding pattern of the $(k;p,q)$-coloring f of $C_i \Box C_j$. By repeating the topmost p rows for α times and i rows for β times of A, and the leftmost q columns for γ times and j columns for δ times of A, we obtain a k-star-coloring of $C_{p\alpha+i\beta} \Box C_{q\gamma+j\delta}$. Therefore, $\chi_s(C_{p\alpha+i\beta} \Box C_{q\gamma+j\delta}) \le k$ for integers $\alpha, \beta, \gamma, \delta$ which completes the proof.
(2) The proof is similar to part (1).
(3) The proof is similar to part (1). □

3 Results

3.1 Cartesian product of two graphs

Theorem 3 *(Fertin et el. [4])*
(1) $\chi_s(P_2 \Box P_2) = 3$, $\chi_s(P_2 \Box P_3) = 4$, *and for any* $m \ge 4$, $\chi_s(P_2 \Box P_m) = \chi_s(P_3 \Box P_m) = 4$.
(2) For any n *and* m *such that* $\min\{n,m\} \ge 4$, $\chi_s(P_n \Box P_m) = 5$.

Theorem 4 *Let* $j \ge 3$ *Then*

$$\chi_s(P_3 \Box C_j) = \begin{cases} 4, & j \equiv 0 \ (\ mod\ 2) \\ 5, & j \equiv 1 \ (\ mod\ 2) \end{cases}$$

Proof. Let $G = P_3 \Box C_j$ with $j \ge 3$. We first show that $\chi_s(G) \ge 4$. Otherwise, suppose to the contrary that f is a 3-star-coloring $\chi_s(G)$, we assume w.l.o.g., that $f(00) = 2$, $f(10) = 1$, $f(01) = 3$ and $f(11) = 2$. By considering the path $00 \to 10 \to 11 \to 21$, we have $f(21) = 3$. By considering the path $00 \to 01 \to 11 \to 21$, we have $f(21) = 2$, a contradiction.

Now, we assume f is a 4-star-coloring $\chi_s(G)$. By case analysis, we have that the vertices of each induced C_4 in G have distinct colors. Then we assume w.l.o.g., that $f(00) = 1$, $f(10) = 3$, $f(01) = 2$ and $f(11) = 4$.

If $f(20) = 2$, we have that $(f(0k),\ f(1k),\ f(2k))= (1,3,2)$ for even k and $(f(0k),\ f(1k),\ f(2k))= (2,4,1)$ for odd k. Therefore, if j is odd, $(f(0k), f(1k), f(2k)) = (1,3,2)$ for $k = 0, j-1$, which is impossible since P_3^0 and P_3^{j-1} are consecutive P_3-layers.

If $f(20) = 1$, we have that $(f(0k), f(1k),\ f(2k)) = (1,3,1)$ for even k and $(f(0k), f(1k), f(2k)) = (2,4,2)$ for odd k. Therefore, if j is odd, $(f(0k), f(1k), f(2k)) = (1,3,1)$ for $k = 0, j-1$, which is impossible since P_3^0 and P_3^{j-1} are consecutive P_3-layers

Therefore, the lower bounds are established.

We give the upper bounds by providing the colorings in the following two cases.
Case 1: j is even.

(a)

3	1	2	1
2	4	3	4
3	1	2	1

(b)

3	4	3	1	2	1	3	4	3	1	2	1	3	1	2	1
2	1	2	4	3	4	2	1	2	4	3	4	2	4	3	4
3	4	3	1	2	1	3	4	3	1	2	1	3	1	2	1

(c)

3	4	3	1	2	1
2	1	2	4	3	4
3	4	3	1	2	1

(d)

3	4	3	1	2	1	3	4	2	1	3	1	2	1
2	1	2	4	3	4	2	1	3	4	2	4	3	4
3	4	3	1	2	1	3	4	2	1	3	1	2	1

Fig. 1: (a) 4-star-coloring of $P_3\Box C_4$ (b) $(4;3,6)$-coloring of $P_3\Box C_{16}$ (c) 4-star-coloring of $P_3\Box C_6$ (d) $(4;3,6)$-coloring of $P_3\Box C_{14}$

The pattern depicted in Fig. 1 part (a) is a 4-star-coloring of $P_3\Box C_4$. It follows that $P_3\Box C_{4\alpha} \leq 4$ for integer $\alpha \geq 1$. The pattern depicted in Fig. 1 part (b) is a $(4;3,6)$-coloring of $P_3\Box C_{16}$. By Lemma 3, we have $P_3\Box C_{6\alpha+4} \leq 4$ for integer $\alpha \geq 2$. The pattern depicted in Fig. 1 part (c) is a 4-star-coloring of $P_3\Box C_6$. By Lemma 3, we have $P_3\Box C_{6\alpha} \leq 4$ for integer $\alpha \geq 1$. The pattern depicted in Fig. 1 part (c) is a $(4;3,6)$-coloring of $P_3\Box C_{14}$. By Lemma 3, we have $P_3\Box C_{6\alpha+2} \leq 4$ for integer $\alpha \geq 2$.
Case 2: j is odd.

(a)

4	5	2	5	1
2	1	4	1	3
4	3	1	2	1

(b)

4	5	2	5	1	2	3
2	1	4	1	3	4	1
4	3	1	2	1	5	3

(c)

4	5	2	5	1	4	1	5
2	1	4	1	3	2	5	3
4	3	1	2	1	5	4	1

(d)

4	5	2	5	1	4	1	3	5
2	1	4	1	3	2	5	4	3
4	3	1	2	1	4	1	2	1

(e)

4	5	2	5	1	4	1	5	2	5	1
2	1	4	1	3	5	2	1	4	1	3
4	3	1	2	1	4	1	3	1	2	1

Fig. 2: (a) 5-star-coloring of $P_3\Box C_5$ (b) $(5;3,5)$-coloring of $P_3\Box C_7$ (c) $(5;3,5)$-coloring of $P_3\Box C_8$ (d) $(5;3,5)$-coloring of $P_3\Box C_9$ (e) $(5;3,5)$-coloring of $P_3\Box C_{11}$

The necessary colorings are shown in Fig. 2 and the proof is similar to part (1). □

Theorem 5 *Let* $i, j \geq 4$ *Then* $\chi_s(P_i\Box C_j) = 5$.

Proof.
The lower bound follows from Theorem 3 and Lemma 1. The patterns depicted in Fig. 3 are $(5;4,4)$-colorings of $C_4\Box C_7$, $C_4\Box C_9$ and $C_4\Box C_{10}$,

respectively. Note that $S(4,7)\cup S(4,9)\cup S(4,10)$ is the set of integers more than 3 except 5 and 6.

```
    2415415         241542315         2415451415
(a) 1232132     (b) 123213432     (c) 1232132132
    2514514         251425154         2514541514
    3121321         312134521         3121323121
```

Fig. 3: (a) $(5;4,4)$-coloring of $C_4\Box C_7$ (b) $(5;4,4)$-coloring of $C_4\Box C_9$ (c) $(5;4,4)$-coloring of $C_4\Box C_{10}$

By Lemma 3, we have $\chi_s(P_i\Box C_j)\leq 5$ for $i\geq 4, j\geq 7$. The patterns shown in Fig. 4 and Fig. 5 show that both $C_4\Box C_5$ and $C_4\Box C_6$ admit 5-star-colorings, which provide the upper bounds for $\chi_s(P_i\Box C_j)$ for $j=5,6$.

3	5	2	5	4	5
5	4	5	3	5	2
1	2	1	4	1	3
4	1	3	1	2	1

Fig. 4: $C_4\Box C_6$

2	4	1	5	4	1	5
1	2	3	2	1	3	2
2	5	1	4	5	1	4
3	1	2	1	3	2	1

Fig. 5: $C_4\Box C_7$

□

Theorem 6 *Let* $i,j\geq 30$ *Then* $\chi_s(C_i\Box C_j)=5$.

Proof. The lower bound follows from Theorem 5 and Lemma 1. The pattern depicted in Fig. 6 induces a $(5;4,4)$-coloring of $C_{11}\Box C_{11}$. By Lemma 3, we have $\chi_s(C_i\Box C_j)\leq 5$ for $i,j\in S(4,11)$. Then by Lemma 2, we have $\chi_s(C_i\Box C_j)\leq 5$ for $i,j\geq(4-1)(11-1)=30$.

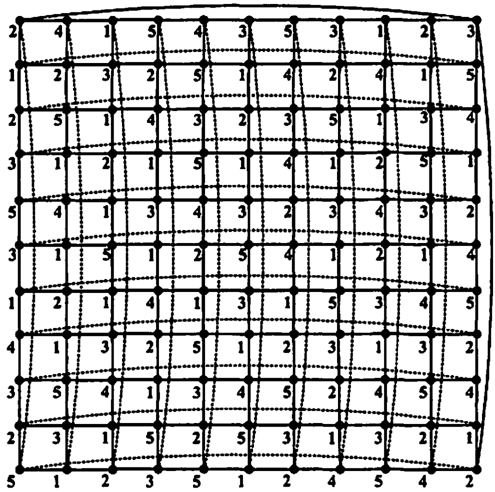

Fig. 6: (5; 4, 4)-coloring of $C_{11}\square C_{11}$

□

3.2 $P_i\square P_j\square P_k$

We now investigate some d-dimensional grids with $d \geq 3$. For larger $i, j, k \geq 3$, it is too complicated enough to analysis the star-coloring of $P_j\square P_j\square P_k$ by hand. Here we use the SAT reduction method adapted for the coloring problem, which was proposed in [14] for the distance-constrained labeling problem. Let $G = (V, E)$ be a graph and k a positive integer. For every $v \in V$ and every $i \in \{1, 2, \cdots, k\}$ introduce an atom $x_{v,i}$. Intuitively, this atom expresses that the vertex v is assigned the color i. Consider the following propositional formulas:

(1) for all $v \in V$,
$\vee_{i=1}^{k} x_{v,i}$

(2) for all $v \in V$, $1 \leq i < j \leq k$,
$\neg x_{v,i} \vee \neg x_{v,j}$

(3) for each path $s \to t \to u \to v$ of length 3 and $1 \leq i, j \leq k$,
$\neg x_{s,i} \vee \neg x_{t,j} \neg x_{u,i} \vee \neg x_{v,j}$

Clauses (1) and (2) ensure that each vertex is labeled with exactly one label. Clauses (3) guarantee that an obtained coloring satisfied that it is not bicolored in any path of length 3. Therefore, the above propositional formulas transform a star-coloring into a propositional satisfiability test (SAT). We can see that an obtained SAT instance is satisfiable if and only if G admits a k-star-coloring.

We solve the SAT instances transformed from k-star-coloring problems described above by using the software MiniSat [15]. As a result, we confirm that the graph $P_3 \square P_4 \square P_4$ and Q_5 admit no 5-star-coloring. Therefore, we have

Lemma 4
$\chi_s(P_2 \square P_2 \square P_4) \geq 5$,
$\chi_s(P_3 \square P_4 \square P_4) \geq 6$,
$\chi_s(Q_4) \geq 5$.
$\chi_s(Q_5) \geq 6$.

Theorem 7

(1) $\chi_s(P_2 \square P_2 \square P_k) = 4$ *for* $k = 2, 3$, $\chi_s(P_2 \square P_2 \square P_k) = 5$ *for* $k \geq 4$.

(2) $\chi_s(P_3 \square P_3 \square P_3) = 4$, $\chi_s(P_3 \square P_3 \square P_k) = 5$ *for* $k \geq 4$.

(3) $\chi_s(P_2 \square P_3 \square P_3) = 4$, $\chi_s(P_2 \square P_3 \square P_k) = 5$ *for* $k \geq 4$.

(4) $\chi_s(P_2 \square P_4 \square P_k) = 5$ *for* $k \geq 4$.

(5) $\chi_s(P_i \square P_j \square P_k) = 6$ *for* $i \geq 3, j \geq 4, k \geq 4$.

Proof. (1) We first show that $P_2 \square P_2 \square P_2$ admits no 3-star-coloring. Suppose to the contrary that f is a 3-star-coloring of $P_2 \square P_2 \square P_2$. We assume w.l.o.g. that $f(000) = f(110) = 1$, $f(100) = 2$ and $f(010) = 3$. Then $f(001) \in \{2, 3\}$. It can be seen that $f(001) \neq 2$, otherwise 110, 100, 000, 001 induce a bicolored P_4, a contradiction. Similarly, we also have $f(001) \neq 3$. Then we have $\chi_s(P_2 \square P_2 \square P_2) \geq 4$. The pattern depicted in Fig. 7 shows a 4-star-coloring of $P_2 \square P_2 \square P_3$, so $\chi_s(P_2 \square P_2 \square P_2) \leq \chi_s(P_2 \square P_2 \square P_3) \leq 4$.

```
2 3   1 4   2 3
3 1   4 2   3 1
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Fig. 7: $P_2 \square P_2 \square P_3$

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2 1   1 4   3 1   1 5
1 3   5 1   1 2   4 1
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Fig. 8: $P_2 \square P_2 \square C_4$

By Lemma 4, we have $\chi_s(P_2 \square P_2 \square P_k) \geq 5$ for $k \geq 4$. The pattern depicted in Fig. 8 shows a 5-star-coloring of $P_2 \square P_2 \square C_4$, so $\chi_s(P_2 \square P_2 \square P_k) \leq \chi_s(P_2 \square P_2 \square C_4) \leq 5$.

(2) Note that $\chi_s(P_3\square P_3\square P_3) \geq \chi_s(P_2\square P_2\square P_3) \geq 4$ and $\chi_s(P_3\square P_3\square P_4) \geq \chi_s(P_2\square P_2\square P_4) \geq 5$, we have the desired lower bounds. The pattern depicted in Fig. 9 shows a 4-star-coloring of $P_3\square P_3\square P_3$, so $\chi_s(P_3\square P_3\square P_3) \leq 4$. The pattern depicted in Fig. 10 shows a 5-star-coloring of $P_3\square P_3\square C_{12}$, so $\chi_s(P_3\square P_3\square P_k) \leq \chi_s(P_3\square P_3\square C_{12}) \leq 5$.

141	323	141
232	414	232
141	323	141

Fig. 9: $P_3\square P_3\square P_3$

234	151	423	151	342	151	234	151	423	151	342	151
515	342	515	234	515	423	515	342	515	234	515	423
234	151	423	151	342	151	234	151	423	151	342	151

Fig. 10: $P_3\square P_3\square C_{12}$

(3) Note that $\chi_s(P_2\square P_3\square P_3) \geq \chi_s(P_2\square P_2\square P_3) \geq 4$ and $\chi_s(P_2\square P_3\square P_4) \geq \chi_s(P_2\square P_2\square P_4) \geq 5$, we have the desired lower bounds. On the other hand, since $\chi_s(P_2\square P_3\square P_k) \leq \chi_s(P_3\square P_3\square P_k)$ for $k \geq 3$, we have the desired upper bounds.

(4) Note that $\chi_s(P_2\square P_4\square P_4) \geq \chi_s(P_2\square P_2\square P_4) \geq 5$, we have the desired lower bounds. The pattern depicted in Fig. 11 shows a 5-star-coloring of $P_2\square P_3\square C_{10}$, so $\chi_s(P_2\square P_4\square P_k) \leq \chi_s(P_2\square P_4\square C_{10}) \leq 5$.

4345	1214	3532	4143	5251	3435	2124	5352	1413	2521
5121	4353	2414	3525	1343	5212	4535	2141	3252	1434

Fig. 11: $P_2\square P_4\square C_{10}$

(5) The lower bound follows from Lemma 4. The pattern depicted in Fig. 17 shows a 6-star-coloring of $C_4\square C_4\square C_4$, so $\chi_s(P_i\square P_j\square P_k) \leq \chi_s(C_4\square C_4\square C_4) \leq 6$. □

3.3 $C_i\square C_j\square C_k$

Lemma 5 *$C_3\square C_3$ admits no 5-star-coloring.*

Proof. Suppose to the contrary that f is a 5-star-coloring of $C_3 \square C_3$. Then there are at least two vertices have the same color, and we assume w.l.o.g. that $f(00) = f(11) = 1$. It can be seen that $f(01) \neq f(10)$ and we assume w.l.o.g. that $f(01) = 2$ and $f(10) = 3$. Then we assume w.l.o.g. that $f(02) = 4$ and $f(12) = 5$. Then we have $f(22) \in \{2, 3\}$, $f(02) \in \{4, 5\}$ and $f(12) \in \{4, 5\}$. We have that if $(f(02), f(12)) \in \{4, 5\}$, $f(22)$ can not be either 2 or 3, a contradiction. □

Lemma 6 *Let $G = C_3 \square C_3$. If G admits a 6-star-coloring f, then for any distinct vertices $u, v, w \in V(G)$, the result $f(u) = f(v) = f(w)$ is impossible.*

Proof. Suppose to the contrary that there are three vertices that receive the same color in a 6-star-coloring of G. We assume w.l.o.g. that $f(00) = f(11) = f(22) = 1$. It can be seen that $f(x) \neq 1$ for any other vertex x. Moreover, if there exist two other vertices that receive the same color, we will obtain a bicolored P_4. Therefore, G needs 7 colors, a contradiction. □

By Lemma 5 and Lemma 6, it can be seen that the number of vertices with color i is either 1 or 2 for any i. Therefore, we have

Corollary 1 *If f is a 6-star-coloring of $C_3 \square C_3$, then for three different colors, each is assigned to two vertices and for the other three different colors, each is assigned to exactly one vertex.*

Let f and g be two star-colorings of a graph G. We say f and g are *equivalent* if there is an automorphism τ on $V(G)$ such that $f(u) = g(\tau(u))$ for all $u \in V(G)$. By Corollary 1 and case analysis, we have,

Lemma 7 *If f is a 6-star-coloring of $C_3 \square C_3$, then it is equivalent to either the graph depicted in Fig. 12 (a) or the graph depicted in Fig. 12 (b).*

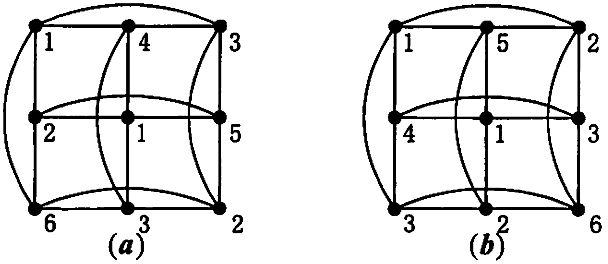

Fig. 12: Two 6-star-colorings in $C_3 \square C_3 \square C_3$.

Lemma 8 *If f is a 6-star-coloring of $C_3\Box C_3\Box C_3$, then there are exactly 27 induced C_4s being three-colored and 54 induced C_4s being four-colored.*

Proof. There are totally 81 induced C_4s in $C_3\Box C_3\Box C_3$, where among 9 $(C_3\Box C_3)$-layers, each has 9 induced C_4s. By Lemma 7, we can see that in each 6-star-coloring of $C_3\Box C_3$-layer there are 3 induced C_4s with three colors and 6 induced C_4s with four colors. Therefore, the proof is completed. $\Box$

The following lemma plays a key role in the proof of Theorem 8, where the technique of counting colored induced C_4s is used.

Lemma 9 *If f is a 6-star-coloring of $C_3\Box C_3\Box C_3$, then for any three different colors $\{c_1, c_2, c_3\}$, there exists at most one $\{c_1, c_2, c_3\}$-colored induced C_4.*

Proof. Let $G = C_3\Box C_3\Box C_3$. We now define nine $(C_3\Box C_3)$-layers of G as follows:

$$\begin{aligned}
H_1 =&G[\{000,010,020,001,011,021,002,012,022\}],\\
H_2 =&G[\{100,110,120,101,111,121,102,112,122\}],\\
H_3 =&G[\{200,210,220,201,211,221,202,212,222\}],\\
H_4 =&G[\{000,010,020,100,110,120,200,210,220\}],\\
H_5 =&G[\{001,011,021,101,111,121,201,211,221\}],\\
H_6 =&G[\{002,012,022,102,112,122,202,212,222\}],\\
H_7 =&G[\{000,100,200,001,101,201,002,102,202\}],\\
H_8 =&G[\{010,110,210,011,111,211,012,112,212\}],\\
H_9 =&G[\{020,120,220,021,121,221,022,122,222\}].
\end{aligned}$$

Suppose to the contrary that there are at least two $\{c_1, c_2, c_3\}$-colored induced C_4s in a 6-star-coloring of G for three different colors $\{c_1, c_2, c_3\}$, then f restricted to any $(C_3\Box C_3)$-layer is equivalent to either the colored graph depicted in Fig. 12 (a) or Fig. 12 (b). We apply these 6-star-colorings to the layer H_5 in G in the following two cases.
(a) $f(001) = f(111) = 1$, $f(101) = f(221) = 2$, $f(021) = f(211) = 3$, $f(011) = 4$, $f(121) = 5$, $f(201) = 6$.
(b) $f(001) = f(111) = 1$, $f(021) = f(211) = 2$, $f(121) = f(201) = 3$, $f(101) = 4$, $f(011) = 5$, $f(221) = 6$.
If we apply the case (a), we need to show that there are at most one copy of C_4 with colors $(1,2,1,4)$ or $(2,5,2,6)$ or $(4,3,2,3)$. For the case (b) it is similar. So there are totally six cases to consider. We here consider the case (a) and show that there is at most one induced C_4 with colors $(1,2,1,4)$, and we left to the readers for the other cases. Let F be an induced C_4

with colors $(1,2,1,4)$. By the definition of the star coloring, noting that $f(N(r)) \cap \{f(s)\} = \emptyset$ and $f(N(t)) \cap \{f(s)\} = \emptyset$ if $f(r) = f(t)$ for a path $r \rightarrow s \rightarrow t$, we have the following results.

•$f(000) \notin \{1,2,4\}$, $f(020) \notin \{2,3,4\}$, $f(022) \notin \{2,3,4\}$, $f(110) \notin \{1,2,4\}$
•$f(110) \notin \{1,2,4\}$, $f(112) \notin \{1,2,4\}$, $f(210) \notin \{2,3,4\}$, $f(212) \notin \{2,3,4\}$
•$f(100) \notin \{2,5,6\}$, $f(220) \notin \{2,5,6\}$, $f(102) \notin \{2,5,6\}$, $f(222) \notin \{2,5,6\}$.

From this result, it can be seen that H_1 and H_8 contain no F. If H_2 contains F, then $V(F) = \{100, 120, 102, 122\}$. Since $f(100) \neq 2$ and $f(102) \neq 2$ (because $f(101) = 2$), this case is impossible. If H_3 contains F, then $V(F) = \{200, 220, 202, 222\}$. Since $f(200) \neq 2$ and $f(202) \neq 2$ (because $f(221) = 2$), this case is impossible. For the same reason, H_i contains no F for $i \in \{4,6,7,9\}$. Since any C_4 lies in a $(C_3 \square C_3)$-layer, we can see that only one copy of F lies in H_5, which contradicts with the assumption. □

Theorem 8 $\chi_s(C_3 \square C_3 \square C_3) = 7$.

Proof. Let $G = C_3 \square C_3 \square C_3$. We will show that G admits no 6-star-coloring. Suppose to the contrary that f is a 6-star-coloring of G. By Lemma 9, any group of three colors lie in at most one copy of C_4. Since there are totally $20 = \binom{6}{3}$ combinations of three colors, we have there are at most 20 copies of induced C_4 with three colors, which contradicts with Lemma 8.

Therefore, such a 6-star-coloring does not exist and the lower bound is established. The pattern depicted in Fig. 13 shows a 7-star-coloring of G, which completes the proof.

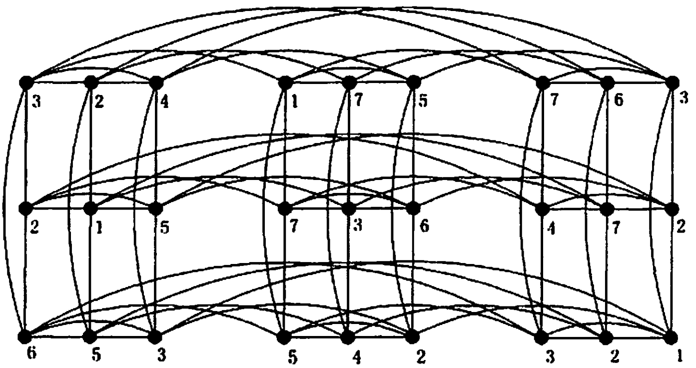

Fig. 13: A 7-star-coloring of $C_3 \square C_3 \square C_3$

□

Let f be a 6-star-coloring of $C_3 \square C_3 \square P_2$. By Lemma 7, f is equivalent to either the graph depicted in Fig. 12 (a) or the graph depicted in Fig.

12 (b). Then it is sufficient to consider the following two cases:
Case 1. $f(001) = f(111) = 1$, $f(101) = f(221) = 2$, $f(021) = f(211) = 3$, $f(011) = 4$, $f(121) = 5$, $f(201) = 6$.
Case 2. $f(001) = f(111) = 1$, $f(021) = f(211) = 2$, $f(121) = f(201) = 3$, $f(101) = 4$, $f(011) = 5$, $f(221) = 6$.
Then we are able to obtain the following result, which shows the structure of a 6-star-coloring of $C_3\square C_3\square P_2$.

Lemma 10 *Let f be a 6-star-coloring of $C_3\square C_3\square P_2$.*

(a) If $f(001) = f(111) = 1$, $f(101) = f(221) = 2$, $f(021) = f(211) = 3$, $f(011) = 4$, $f(121) = 5$, $f(201) = 6$, then $f(010) = 2$, $f(120) = f(200) = 4$, $f(\{000, 020\}) = f(\{110, 210\}) = \{5, 6\}$ and $f(\{100, 210\}) = \{1, 3\}$.

(b) If $f(001) = f(111) = 1$, $f(021) = f(211) = 2$, $f(121) = f(201) = 3$, $f(101) = 4$, $f(011) = 5$, $f(221) = 6$, then $f(000) \in \{2, 3, 6\}$. Moreover, (1) if $f(000) \neq 6$, then $f(220) = 1$, $f(120) = f(200) = 5$ and $f(020) = f(210) = 4$, and $\{000, 010, 100, 110\}$ form an induced C_4 with the color set $\{2, 3, 6\}$;
(2) if $f(000) = 6$, then $f(110) = 6$, and either $f(020) = f(210)$ or both $f(010) = f(220)$ and $f(200) = f(120)$ hold.

Proof.

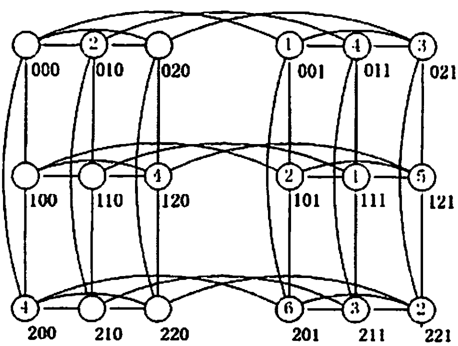

Fig. 14: 6-star-coloring of $C_3\square C_3\square P_2$

(a) With the coloring of the second $(C_3\square C_3)$-layer fixed as above, by a rather tedious case by case analysis (confirmed by the computer), we obtain that all five possible colorings of the first $(C_3\square C_3)$-layer. The results are presented in Fig. 15.

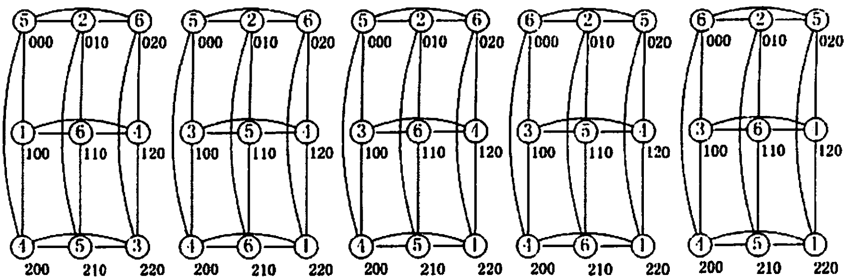

Fig. 15: 6-star-colorings of the first $(C_3\Box C_3)$-layer

From above, it can be see that Lemma 10 (a) holds.
(b) The proof is similar to (a) and it is omitted here. □

Now, by applying the possible colorings of $C_3\Box C_3\Box P_2$ to the second and third $(C_3\Box C_3)$-layers of $C_3\Box C_3\Box P_4$, it is easy to have that $C_3\Box C_3\Box P_4$ admits no 6-star-coloring. Therefore, we have

Corollary 2 $\chi_s(C_3\Box C_3\Box P_4) \geq 7$.

Theorem 9
Let $k \geq 4$. Then $\chi_s(C_3\Box C_3\Box C_k) = 7$.

Proof. The lower bounds follow from Corollary 2. The pattern depicted in Fig. 16 part (a) is a 7-star-coloring of $C_3\Box C_3\Box C_4$. This implies that $\chi_s(C_3\Box C_3\Box C_k) \leq 7$ for $k = 4q$ with $q \geq 1$. The pattern depicted in Fig. 16 part (b) is a 7-star-coloring of $C_3\Box C_3\Box C_9$. Moreover, the leftmost four 3×3 subpatterns induce a 7-star-coloring of $C_3\Box C_3\Box C_4$. This implies that $\chi_s(C_3\Box C_3\Box C_k) \leq 7$ for $k = 4q + 1$ with $q \geq 2$. Similarly, the patterns depicted in Fig. 16 part (c) and (d) confirm that $\chi_s(C_3\Box C_3\Box C_k) \leq 7$ for $k = 4q+2$ with $q \geq 1$ or $k = 4q+3$ with $q \geq 1$. The pattern depicted in Fig. 16 part (e) is a 7-star-coloring of $C_3\Box C_3\Box C_5$. Then the 7-star-coloring of $C_3\Box C_3\Box C_k$ for each $k \geq 3$ is established.

```
     3 4 6   2 1 3   7 6 5   4 2 7
(a)  7 5 1   6 2 4   1 7 2   6 4 3
     1 6 2   4 5 7   2 3 6   5 7 1

     541   326   645   137 | 542   276   713   354   167
(b)  612   731   267   354 | 671   734   576   241   375
     253   564   713   476 | 723   165   421   637   412

     5 7 2   4 6 1   7 2 3   6 1 5 | 4 6 7   3 2 4
(c)  1 4 5   2 5 3   1 7 4   3 6 7 | 1 5 2   7 3 6
     4 2 6   3 1 7   5 3 6   7 4 1 | 2 7 3   6 5 1

     3 1 4   4 2 6   5 3 1   6 2 5 | 3 1 4   5 4 7   6 2 5
(d)  5 3 2   1 6 7   2 5 6   1 4 3 | 7 6 5   3 2 6   1 7 4
     2 6 1   7 4 5   4 1 2   3 5 7 | 2 3 6   4 6 1   7 5 3

     4 7 2   6 5 1   4 2 5   7 6 3   2 1 5
(e)  1 2 6   7 3 4   5 7 6   1 4 5   3 5 7
     5 4 1   2 6 3   3 1 7   5 7 2   6 3 4
```

Fig. 16: 7-star-colorings of $C_3 \square C_3 \square C_k$ for $k \in \{4, 5, 6, 7, 9\}$

□

Proposition 1
(1) $\chi_s(C_4 \square C_4 \square C_4) = 6$.
(2) $\chi_s(C_6 \square C_6 \square C_6) = 6$.
(3) $\chi_s(C_4 \square C_4 \square C_6) = 6$.
(4) $\chi_s(C_4 \square C_6 \square C_6) = 6$.

Proof. By Lemma 4, we have $\chi_s(P_3 \square P_4 \square P_4) \geq 6$, and so all the lower bounds follow. The obtained 6-star-colorings of $C_4 \square C_4 \square C_4$, $C_6 \square C_6 \square C_6$, $C_4 \square C_4 \square C_6$ and $C_4 \square C_6 \square C_6$ are depicted in Figs. 17-20. Therefore, the assertion follows.

```
4 1 5 6   5 3 2 1   1 6 1 3   2 5 4 5
5 2 3 2   6 1 6 4   2 4 5 6   3 6 2 1
1 4 1 6   3 5 2 5   5 6 3 1   2 1 5 4
5 6 2 3   2 1 4 6   4 2 5 2   6 3 6 1
```

Fig. 17: 6-star-coloring of $C_4 \square C_4 \square C_4$

154154	325325	651651	263263	614614	432432
613613	246246	134134	542542	351351	526526
435435	521521	365365	216216	463463	142142
263263	614614	432432	154154	325325	651651
542542	351351	526526	613613	246246	134134
216216	463463	142142	435435	521521	365365

Fig. 18: 6-star-coloring of $C_6 \square C_6 \square C_6$

6 2 4 3	5 6 1 4	1 4 3 2	6 1 5 3	5 3 2 4	1 5 6 2
3 4 5 1	4 1 2 6	2 3 6 5	3 5 4 1	4 2 1 6	2 6 3 5
5 6 2 4	2 5 3 1	6 1 4 3	4 6 2 5	1 5 3 2	3 1 4 6
1 3 6 5	3 4 5 2	5 2 1 6	2 3 6 4	6 4 5 1	4 2 1 3

Fig. 19: 6-star-coloring of $C_4 \square C_4 \square C_6$

234265	416153	652342	534161	426523	615341
616354	325421	546163	213254	635461	542132
452413	261635	134524	352616	241345	163526
161532	543246	321615	465432	153216	324654

Fig. 20: 6-star-coloring of $C_4 \square C_6 \square C_6$

□

Similar to the proof of Proposition 1, we have

Corollary 3
(1) Let $i, j, k \geq 1$. *Then* $\chi_s(C_{4i} \square C_{4j} \square C_{4k}) = 6$.
(2) Let $i, j, k \geq 1$. *Then* $\chi_s(C_{6i} \square C_{6j} \square C_{6k}) = 6$.
(3) Let $i, j, k \geq 1$. *Then* $\chi_s(C_{4i} \square C_{4j} \square C_{6k}) = 6$.
(4) Let $i, j, k \geq 1$. *Then* $\chi_s(C_{4i} \square C_{6j} \square C_{6k}) = 6$.

3.4 Cartesian product of four or more graphs

By Theorem 2, we have $\chi_s(C_4 \square C_4 \square C_4 \square C_4) \leq 37$. We improve this upper bound by the following result:

Theorem 10 *Let* $i, j, k, \ell \geq 1$. $\chi_s(C_{4i} \square C_{4j} \square C_{4k} \square C_{4\ell}) \leq 9$.

Proof. Let M_i be the i-th $(C_4\Box C_4\Box C_4)$-layer of $C_4\Box C_4\Box C_4\Box C_4$. The pattern depicted in Fig. 21 is a 9-star-coloring of $C_4\Box C_4\Box C_4\Box C_4$. Therefore, $\chi_s(C_4\Box C_4\Box C_4\Box C_4) \leq 9$. By Lemma 3, we have $\chi_s(C_{4i}\Box C_{4j}\Box C_{4k}\Box C_{4\ell}) \leq 9$.

	8617	2131	1716	3121		4151	1819	5141	1918
M1	1898	8415	9181	8514	M2	8213	6171	1312	7161
	8716	3121	1617	2131		5141	1918	4151	1819
	9181	1514	8191	1415		1312	7161	1213	6171
	1716	3121	1617	2131		5141	1918	4151	1819
M3	9181	1514	8191	1415	M4	8312	7161	1213	6171
	1617	2131	1716	3121		4151	1819	5141	1918
	8191	1415	9181	1514		1213	6171	1312	7161

Fig. 21: A 9-star-coloring of $C_4\Box C_4\Box C_4\Box C_4$

□

The following result improves the upper bound of $\chi_s(Q_6)$:

Theorem 11 $\chi_s(Q_2) = 3$, $\chi_s(Q_3) = 4$, $\chi_s(Q_4) = 5$, $\chi_s(Q_5) = 6$, $\chi_s(Q_6) = 6$.

Proof. By Theorem 1, we have $\chi_s(Q_d) \leq d+1$ for $d \geq 2$. Now for the upper bound, we only need to consider Q_6. Let the sequence 6, 2, 2, 3, 2, 4, 5, 6, 3, 5, 4, 1, 1, 3, 3, 2, 4, 5, 1, 4, 3, 1, 4, 2, 2, 6, 5, 3, 5, 4, 6, 5, 5, 1, 4, 5, 3, 5, 1, 2, 2, 4, 6, 3, 4, 6, 5, 4, 2, 3, 3, 6, 6, 4, 5, 3, 1, 5, 4, 2, 3, 2, 2, 1 be a coloring $f : V(Q_6) \to \{1,2,3,4,5,6\}$ of Q_6 by the lexicographic order, i.e., $f(000000) = 6$, $f(000001) = 2$, $f(000010) = 2$, $f(000011) = 3$, etc. Then it can be seen that f is a 6-star-coloring of Q_6. Therefore, the upper bounds are established.

Since $Q_2 \cong C_4$, we have $\chi_s(Q_2) \geq 3$. By Lemma 4, we have $\chi_s(Q_4) \geq 5$ and $\chi_s(Q_6) \geq \chi_s(Q_5) \geq 6$. The lower bound for $\chi_s(Q_3)$ follows from Theorem 7. Therefore, the lower bounds are established. □

It is likely that the star chromatic number of $C_i\Box C_j\Box C_k)$ for even i, j, k is 6 and the star chromatic number of $C_i\Box C_j\Box C_k)$ is 7 if one of $\{i, j, k\}$ is even. We therefore propose the following:

Conjecture 1 *Let $i, j, k \geq 3$. Then*
(1) $\chi_s(C_i\Box C_j\Box C_k) = 6$ if each of $\{i, j, k\}$ is even,
(2) $\chi_s(C_i\Box C_j\Box C_k) = 7$ if at least one of $\{i, j, k\}$ is odd.

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