Hamiltonian properties of almost locally connected claw-free graphs *

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Abstract: G is almost locally connected if $B(G)$ is an independent set and for any $x \in B(G)$, there is a vertex y in $V(G) \setminus \{x\}$ such that $N(x) \cup \{y\}$ induces a connected subgraph of G, where $B(G)$ denotes the set of vertices of G that are not locally connected. In this paper, we prove that an almost locally connected claw-free graph on at least 4 vertices is Hamilton-connected if and only if it is 3-connected. This generalizes a result by Asratian that a locally connected claw-free graph on at least 4 vertices is Hamilton-connected if and only if it is 3-connected [Journal of Graph Theory 23 (1996) 191–201].

Keywords: almost locally connected, claw-free graph, hamiltonian, Hamilton-connected

1 Introduction

All graphs considered in this paper are simple and finite. We use [2] for notation and terminology not defined here. A graph G is *claw-free* if it

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*This research is supported by National Natural Science Foundation of China, Tian Yuan Special Foundation (Grant No. 11426125), National Science Foundation of China (Grant No. 51405214), Educational Commission of Liaoning Province (Grant No. L2014239).

does not contain the *claw* $K_{1,3}$ as an induced subgraph. For two distinct vertices x and y of a graph G, an *(x, y)-path* in G is a path of G with end vertices x and y. If an (x, y)-path of G contains all the vertices of a graph G, then this (x, y)-path is a *Hamilton path* (from x to y or between x and y) of G, and the graph G is called *traceable.* A graph G is *hamiltonian* if there exists a pair of adjacent vertices $\{x, y\}$ in G and a Hamilton path between x and y in $G-\{x, y\}$, i.e., if there exists a (Hamilton) cycle in G containing all the vertices of G. A graph G is *Hamilton-connected* (this is sometimes called hamiltonian-connected, but we adopt the terminology of [2]) if for every pair of vertices $\{x, y\}$ of G there exists a Hamilton path between x and y. A graph G is *panconnected* if for each k and for each pair of distinct vertices u and v with $d(u, v) \leq k \leq |V(G)| - 1$, there exists a (u, v)-path of length k, where $d(u, v)$ is the *distance* between u and v in G, i.e., the length (number of edges) of a shortest (u, v)-path of G. For a vertex v of G, the *neighborhood* $N(v)$ is the set of all the vertices that are adjacent to v in G; the *closed neighborhood* of v is the set $N[v] = N(v) \cup \{v\}$. For a nonempty subset S of $V(G)$, $G[S]$ denotes the subgraph induced by S in G. A vertex v is *locally connected* if $G[N(v)]$ is connected. A graph G is *locally connected* if every vertex v of G is locally connected.

Local connectivity conditions have been a popular subject because they play certain roles for hamiltonian properties of claw-free graphs. While arbitrarily high levels of connectivity cannot guarantee hamiltonian properties of general graphs, even mild local connectivity conditions can do so for claw-free graphs, as we can see from the earliest result in this area.

Theorem 1 (Oberly and Sumner [7]). *Every connected, locally connected claw-free graph on at least three vertices is hamiltonian.*

Similar results on hamiltonian properties of graphs were obtained in, e.g., [4], [5] and [6], under this and stronger local connectivity conditions.

Notice that we can easily prove Theorem 1 if we use the well-known Ryjáček closure in [9]. The key elements of Ryjáček closure are as follows. Let G be a claw-free graph and let v be a locally connected vertex of G. If $G[N(v)]$ is not a complete subgraph of G, add all the missing edges to $G[N(v)]$ to turn it into a complete subgraph, and denote the newly obtained graph by G_v. Then G_v is again a claw-free graph, and G_v is hamiltonian if and only if G is hamiltonian. Moreover, a locally connected vertex in G remains locally connected in G_v, so repeatedly applying this procedure to a connected, locally connected graph turns the graph into a complete graph. Since a complete graph on at least three vertices is trivially hamiltonian, this shows that a connected, locally connected claw-free graph on at least three vertices is hamiltonian.

Asratian proved the following theorem in [1].

Theorem 2 (Asratian [1]). *Let G be a locally connected claw-free graph on at least four vertices. Then G is Hamilton-connected if and only if G is 3-connected.*

A natural question is if we can weaken the local connectivity condition in Theorem 2 and still maintain the same result as the one in Theorem 1. This motivates the following concept introduced by Teng and You in [11]. Let $B(G)$ denote the set of vertices of a graph G that are *locally disconnected*, i.e., not locally connected. A subset S of the vertices of a graph G is called *independent* if no pair of vertices of S is adjacent in G. A graph G is called *almost locally connected* if $B(G)$ is independent and for any $x \in B(G)$, there is a vertex y in $V(G) \setminus \{x\}$ such that $G[N(x) \cup \{y\}]$ is connected. From the definition it is straightforward to see that any locally connected graph G is also almost locally connected, since for such a graph G, $B(G) = \emptyset$. On the other hand, it is easy to give examples of almost locally connected claw-free graphs that are not locally connected (see Figure 1).

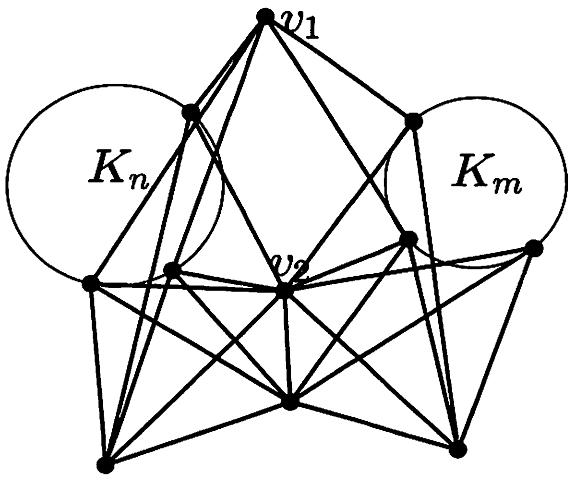

Figure 1: An almost locally connected but not locally connected claw-free graph

Supposing that G is a connected, almost locally connected claw-free graph with $B(G) \neq \emptyset$, consider a vertex x of $B(G)$. First of all, note that the neighborhood of x induces two disjoint complete graphs in G (if G has at least three vertices). By the definition, for some vertex y in $V(G) \setminus \{x\}$, the subgraph $G[N(x) \cup \{y\}]$ is connected, hence y has at least two neighbors z_1 and z_2 in two different (complete) components of $G[N(x)]$, respectively. Then clearly $xy \notin E(G)$; otherwise x would be locally connected. Since

$B(G)$ is an independent set, both z_1 and z_2 are locally connected. Applying the Ryjáček closure with respect to z_1, in the resulting graph G_{z_1} the vertices x and y are adjacent. In fact, it is not difficult to check that both x and y are locally connected in G_{z_1}. Continuing in the same way for other vertices of $B(G)$, one can deduce that the closure of a connected, almost locally connected, claw-free graph is complete. Hence a result which is similar to the one in Theorem 1 for the almost locally connected graphs can be easily obtained.

In this paper, we prove the following analogue of Theorem 2 for almost locally connected claw-free graphs.

Theorem 3. *Let G be an almost locally connected claw-free graph on at least four vertices. Then G is Hamilton-connected if and only if G is 3-connected.*

Another natural question is whether the conclusion of Theorems 2 and 3 can be strengthened from Hamilton-connected to panconnected claw-free graphs. This question was answered affirmatively in case of local connectivity by Sheng, Tian and Wei [10]. They proved the conjecture by Broersma and Veldman [3] that every locally connected claw-free graph of order at least 4 is panconnected if and only if it is 3-connected. We were not able to prove a counterpart of this result for almost locally connected graphs and leave it as an open problem.

2 Preliminaries

Before we prove Theorem 3, we introduce some additional terminology and auxiliary results.

For two nonempty vertex sets A and B of a graph G, we define $E(A, B) = \{xy \in E(G) : x \in A, y \in B\}$.

We let P denote a path by $x_1x_2 \dots x_k$, and we will use x_i^+ to denote the successor x_{i+1} of x_i on P for $1 \leq i \leq k-1$, in the direction specified by the ordering of the vertices of P. Similarly, we use x_i^- to denote x_{i-1} for $2 \leq i \leq k$. We denote by P^- the path P with the reverse orientation, so $P^- = x_kx_{k-1} \dots x_1$. With x_iPx_j $(i < j)$ we denote the consecutive vertices on P from x_i to x_j (inclusive), and with $x_jP^-x_i$ $(j > i)$ we denote the consecutive vertices on P from x_j to x_i in the reverse order.

We now present some useful lemmas that are identical or follow implicitly from the proofs in [1]. Most of these statements are exactly the same statements as in [1], but here we deal with locally connected vertices in an almost locally connected graph, while in [1] all vertices are assumed to be locally connected. The proof of Lemma 1 is implicit in the proof of Proposition 2.2 in [1].

Lemma 1. *Let G be a connected claw-free graph and let u be a locally connected vertex of G. Furthermore, let w be a cut vertex of $H = G[N(u)]$. Then the following properties hold:*

(1) the graph $H - w$ has two components and each of them is a complete graph;

(2) the graph H has at most two cut vertices. Moreover, if H has two cut vertices v_1 and v_2, then $v_1v_2 \in E(G)$.

The proof of Lemma 2 is implicit in the proof of Proposition 2.3 in [1].

Lemma 2. *Let G be a connected claw-free graph and let u be a locally connected vertex of G. If $v \in N(u)$ and v is not a cut vertex of $G[N(u)]$, then there is a Hamilton (u, v)-path in $G[N[u]]$.*

The proof of Lemma 3 is implicit in the proofs of Theorems 3.1-3.3 in [1].

Lemma 3. *If G is a 3-connected claw-free graph, and u and v are both locally connected vertices of G, then there exists a (u, v)-path P in G such that $N(u) \cup N(v) \subseteq V(P)$.*

The following observation follows immediately from the definition of an almost locally connected graph.

Observation 1. *If G is a connected, almost locally connected graph and u is a locally disconnected vertex of G, then any vertex $v \in N(u)$ is locally connected.*

Lemma 4. *Let G be a connected, almost locally connected graph and u be a locally disconnected vertex of G. Then for any vertex $v \in N(u)$, $G[N(v) \setminus N[u]]$ is a complete graph and there is an edge xy with $x \in N(u) \cap N(v)$ and $y \in N(v) \setminus N[u]$.*

Proof. Notice that u is locally disconnected and G is claw-free. Let $N(u) = A \cup B$, where $G[A]$ and $G[B]$ are two disjoint complete graphs. Suppose $v \in A$. Then for any two distinct vertices $y_1, y_2 \in N(v) \setminus N[u]$, $y_1y_2 \in E(G)$ since $G[v, u, y_1, y_2] \neq K_{1,3}$. By Observation 1, v is locally connected. Thus $A \setminus \{v\} \neq \emptyset$, and there is an edge xy with $x \in A \setminus \{v\}$ and $y \in N(v) \setminus A$. Since $G[A]$ is a complete graph and $v \in A$, $x \in N(v) \cap N(u)$. This completes the proof of Lemma 4. □

Let z be an internal vertex of a (u, v)-path P of a graph G with $u \neq v$. We say that P has a *local detour* at z if there exists a path in $G[N(z) \setminus \{u, v\}]$ with origin outside P and terminal which is a neighbor of z on P. The following result was obtained in [5].

Lemma 5. *Let G be a claw-free graph with order $|V(G)| \geq 3$ and let P be a (u, v)-path of length k with $u \neq v$ and $3 \leq k \leq |V(G)| - 2$. If P has a local detour, then G contains a (u, v)-path Q of length $k + 1$ with $V(P) \subset V(Q)$.*

Our final result of this section shows that in a connected, almost locally connected claw-free graph, a Hamilton (u, v)-path is guaranteed by the existence of a (u, v)-path (of length at least 3) containing all neighbors of u and v.

Theorem 4. *If G is a connected, almost locally connected claw-free graph, and there is a (u, v)-path P of length at least 3 such that $N(u) \cup N(v) \subseteq V(P)$, then G contains a Hamilton (u, v)-path.*

Proof. Suppose, to the contrary, that G does not contain a Hamilton (u, v)-path. Let P be a longest (u, v)-path of length k of G such that $N(u) \cup N(v) \subseteq V(P)$. Then $k < |V(G)| - 1$ and $V(G) \setminus V(P) \neq \emptyset$. Suppose $x \in V(P)$, $y \in V(G) \setminus V(P)$ and $xy \in E(G)$. Since $N(u) \cup N(v) \subseteq V(P)$, $x \notin \{u, v\}$. Then we obtain $x^-x^+ \in E(G)$ since G is claw-free. By Lemma 5, x is not locally connected; otherwise we can get a (u, v)-path P' of length $k + 1$ such that $V(P) \subset V(P')$, a contradiction with the choice of P. Thus assume that $N(x) = A \cup B$, where $G[A]$ and $G[B]$ are two disjoint complete graphs. Without loss of generality, let $y \in B$. Then $x^-, x^+ \in A$.

We first prove a number of claims, followed by short proofs, before we reach our final contradiction.

Claim 1. $A \subseteq V(P)$.
Proof. Suppose that there is a vertex $z \in A$ such that $z \notin V(P)$. Since $G[A]$ is a complete graph, $zx^+, zx^- \in E(G)$. It is easy to get a (u, v)-path P' of length $k + 1$ such that $V(P) \subset V(P')$, a contradiction. □

Claim 2. $B \cap V(P) = \emptyset$.
Proof. If $B = \{y\}$, then the claim is obviously true. Suppose that $|B| \geq 2$ and there is a vertex $y' \in B \cap V(P)$. Since $G[B]$ is a complete graph, $yy' \in E(G)$. Since $N(u) \cup N(v) \subseteq V(P)$, $y' \notin \{u, v\}$. Then by Lemma 5, y' is not locally connected; otherwise we can get a (u, v)-path P' of length $k+1$ such that $V(P) \subset V(P')$, a contradiction. Since x and y' are not locally connected and $y'x \in E(G)$, we obtain a contradiction with the definition of almost local connectedness. This completes the proof of Claim 2. □

Since G is almost locally connected, there is a vertex w connecting A and B. By Claim 2, $B \cap V(P) = \emptyset$. Without loss of generality, we may assume that $wy, wz \in E(G)$ for some $z \in A$.

Claim 3. $w \in V(P)$, $w \notin \{x^-, x^+\}$ and $w^-w^+ \in E(G)$.
Proof. Suppose that $w \in V(G) \setminus V(P)$. By Claim 1, $z \in V(P)$. Since $wz \in E(G)$ and $N(u) \cup N(v) \subseteq V(P)$, $z \notin \{u, v\}$. Obviously, $z^-z^+ \in E(G)$. If $z = x^-$, then we can easily obtain a (u,v)-path $P' = uPx^-wyxPv$ of length $k+2$ such that $V(P) \subset V(P')$, a contradiction. Similarly, $z \neq x^+$. Thus, without loss of generality, assume that z is on $x^{++}Pv$. Since $G[A]$ is a complete graph and $\{x^-, x^+, z\} \subseteq A$, we get that $zx^-, zx^+ \in E(G)$. Then we can obtain a (u,v)-path $P' = uPxywzx^+Pz^-z^+Pv$ of length $k+2$ such that $V(P) \subset V(P')$, a contradiction. We conclude that $w \in V(P)$.

It is clear that $w \notin \{x^-, x^+\}$. Since $G[w, w^-, w^+, y] \neq K_{1,3}$ and $yw^-, yw^+ \notin E(G)$, $w^-w^+ \in E(G)$. □

Claim 4. w is not locally connected.
Proof. Since $wy \in E(G)$ and $N(u) \cup N(v) \subseteq V(P)$, $w \notin \{u, v\}$. Thus by Lemma 5, w is not locally connected. □

By Claim 3, without loss of generality, we may assume that $w \in x^{++}Pv$. By the definition of almost local connectedness, $G[N(x) \cup \{w\}]$ is connected. Then there is a path Q in $G[N(x) \cup \{w\}]$ of length at most 3 connecting y and x^+, since $G[A]$ and $G[B]$ are two disjoint complete subgraphs of $G[N(x)]$. If $Q = ywx^+$, then we can obtain a (u,v)-path $P' = uPxywx^+Pw^-w^+Pv$ of length $k+1$ such that $V(P) \subset V(P')$, a contradiction. Let $Q = ywzx^+$. Since G is claw-free and using Claim 4, we assume that $N(w) = A_1 \cup B_1$, where $G[A_1], G[B_1]$ are two disjoint complete graphs. Then, without loss of generality, we may assume that $y \in A_1, z \in B_1$. Obviously, $w^-, w^+ \in B_1$ and $w^-z, w^+z \in E(G)$, since $w^-y, w^+y \notin E(G)$ and $G[B_1]$ is a complete graph. If $z = w^-$, then we can easily obtain a (u,v)-path $P' = uPx^-x^+PzxywPv$ of length $k+1$ such that $V(P) \subset V(P')$, a contradiction. Similarly, $z \neq w^+$. Without loss of generality, we may assume that z is on $w^{++}Pv$. Since $\{x^-, x^+, z\} \subseteq A$ and $G[A]$ is a complete graph, $x^-z, x^+z \in E(G)$. If $z^-z^+ \in E(G)$, then we can get a (u,v)-path $P' = uPxywP^-x^+zw^+Pz^-z^+Pv$ of length $k+1$ such that $V(P) \subset V(P')$, a contradiction. Since $G[z, z^-, z^+, x^+] \neq K_{1,3}$ and $z^-z^+ \notin E(G)$, $z^-x^+ \in E(G)$ or $z^+x^+ \in E(G)$. If $z^-x^+ \in E(G)$, then we can obtain a (u,v)-path $P' = uPxywP^-x^+z^-P^-w^+zPv$ of length $k+1$ such that $V(P) \subset V(P')$, a contradiction. If $z^+x^+ \in E(G)$, then we can get a (u,v)-path $P' = uPxywzP^-w^+w^-P^-x^+z^+Pv$ of length $k+1$ such that $V(P) \subset V(P')$, a contradiction. This completes the proof of Theorem 4. □

By Theorem 4, the existence of a (u, v)-path P with $N(u) \cup N(v) \subseteq V(P)$ guarantees the existence of a Hamilton (u, v)-path in a 3-connected, almost locally connected claw-free graph. Using Lemma 3, this shows that any two distinct locally connected vertices are connected by a Hamilton path in a 3-connected, almost locally connected graph. Hence, in order to prove Theorem 3, it suffices to prove the existence of a Hamilton (u, v)-path in a 3-connected, almost locally connected claw-free graph on at least 4 vertices, in case at least one of u and v is a locally disconnected vertex. By Theorem 4, it suffices to prove the existence of a (u, v)-path P with $N(u) \cup N(v) \subseteq V(P)$ in these cases. We give a separate proof for the three remaining cases in the next section. In fact, we follow a slightly different case distinction involving the distance $d(u, v)$ between u and v.

3 The remaining cases

Throughout this section we assume that G is a 3-connected, almost locally connected claw-free graph, and that u is a locally disconnected vertex of G. We complete the proof of Theorem 4 by distinguishing the following cases and proving the existence of a (u, v)-path P such that $N(u) \cup N(v) \subseteq V(P)$ in all these cases:

- v is a vertex of G such that $d(u, v) = 1$, i.e., $v \in N(u)$;
- v is a vertex of G such that $d(u, v) = 2$ and $N(u) \cup \{v\}$ is connected;
- v is a vertex of G such that $d(u, v) = 2$ and $N(u) \cup \{v\}$ is disconnected;
- v is a vertex of G such that $d(u, v) \geq 3$.

We use the following notation. Suppose H is a graph with $V(H) = A \cup \{u, v\}$, where $H[A]$ is a complete graph. Then we let $u[A]v$ denote a Hamilton (u, v)-path of H.

Case 1. *v is a vertex of G with $d(u, v) = 1$.*

Proof. Notice that G is claw-free and that u is locally disconnected. Then $N(u) = A \cup B$, where $G[A]$ and $G[B]$ are two disjoint complete graphs. Since $d(u, v) = 1$, we may assume $v \in A$. Then by Observation 1, v is locally connected. Suppose first that u is a cut vertex of $G[N(v)]$. Then there are two vertices $y_1, y_2 \in N(v) \cap N(u)$ such that y_1 and y_2 belong to two distinct components of $G[N(v) \setminus \{u\}]$, respectively. Obviously $y_1, y_2 \in A$ and $y_1 y_2 \notin E(G)$, which contradicts that $G[A]$ is a complete graph. Thus u is not a cut vertex of $G[N(v)]$, and $G[N(v) \setminus \{u\}]$ is connected. Notice that G is almost locally connected, assume that w connects A and B such that $wa, bw \in E(G)$ for some $a \in A$ and $b \in B$. Using Lemma 1, suppose first that there are two distinct cut vertices v_1 and v_2 of $G[N(v) \setminus \{u\}]$ in A. Then $N(v) \cap A = \{v_1, v_2\}$. Using Lemma 1, let $G[H_1]$ and $G[H_2]$ be two distinct complete subgraphs of $G[N(v) \setminus \{u, v_1, v_2\}]$. By Lemma 4, $G[(H_1 \cup H_2) \setminus A]$ is a complete graph since $(H_1 \cup H_2) \setminus A \subseteq N(v) \setminus A$, a contradiction with $H_1 \cap H_2 = \emptyset$. Therefore, A contains at most one cut vertex of $G[N(v) \setminus \{u\}]$.

Now first assume that $w \notin N(v)$. We first deal with the case that $|N(w) \cap A| \geq 2$ and, without loss of generality, we may assume that a is not a cut vertex of $G[N(v) \setminus \{u\}]$. Then by Lemma 2, we can get a Hamilton (a, v)-path Q_0 of $G[N[v] \setminus \{u\}]$. Obviously, $A \subseteq V(Q_0)$ and $B \cap V(Q_0) = \emptyset$. Thus we can obtain a (u, v)-path $P = u[B \setminus \{b\}]bwaQ_0v$ such that $N(u) \cup N(v) \subseteq V(P)$. Similarly, if $N(w) \cap A = \{a\}$ and a is not a cut vertex of $G[N(v) \setminus \{u\}]$, then we can obtain a (u, v)-path P such that $N(u) \cup N(v) \subseteq V(P)$. We next deal with the case that $N(w) \cap A = \{a\}$ and a is a cut vertex of $G[N(v) \setminus \{u\}]$. Let $H = N(v) \setminus N[u]$. Then $H \neq \emptyset$ and by Lemma 4, $G[H]$ is a complete graph. Obviously, there is a vertex $c \in H$ such that $ca \in E(G)$. By Lemma 4, $cw \in E(G)$, since $c, w \in N(a) \setminus A$. Suppose that c is not a cut vertex of $G[N(v) \setminus \{u\}]$. Then by Lemma 2, there is a Hamilton (c, v)-path Q_1 of $G[N[v] \setminus \{u\}]$. It follows that we can obtain a (u, v)-path $P = u[B \setminus \{b\}]bwcQ_1v$ such that $N(u) \cup N(v) \subseteq V(P)$. Suppose that c is a cut vertex of $G[N(v) \setminus \{u\}]$. Then $ca \in E(G)$ by Lemma 1. $G - \{v, c\}$ contains at least one (x, y)-path Q_2 with an orientation from x to y connecting $H \setminus \{c\}$ and $N(u)$ such that $x \in H \setminus \{c\}, y \in N(u)$ and $(V(Q_2) \setminus \{x, y\}) \cap (N(u) \cup H) = \emptyset$ since G is 3-connected. Suppose $y \in A$. If $w \notin V(Q_2)$, then we can obtain a (u, v)-path $P = u[B \setminus \{b\}]bwc[H \setminus \{x, c\}]xQ_2y[A \setminus \{y, v\}]v$ such that $N(u) \cup N(v) \subseteq V(P)$. If $w \in V(Q_2)$, then we can obtain a (u, v)-path $P = u[B \setminus \{b\}]bwQ_2^- x[H \setminus \{x, c\}]ca[A \setminus \{a, v\}]v$ such that $N(u) \cup N(v) \subseteq V(P)$. Suppose $y \in B$. Then we can obtain a (u, v)-path $P = u[B \setminus \{y\}]yQ_2^- x[H \setminus \{x, c\}]ca[A \setminus \{a, v\}]v$ such that $N(u) \cup N(v) \subseteq V(P)$. This completes all the subcases when $w \notin N(v)$.

Suppose next that $w \in N(v)$. Then, using similar arguments as above, we can obtain a (u, v)-path P such that $N(u) \cup N(v) \subseteq V(P)$. This completes the proof of Case 1. □

Case 2. *v is a vertex of G with $d(u, v) = 2$ and $N(u) \cup \{v\}$ is connected.*

Proof. As before, let $N(u) = A \cup B$, where $G[A]$ and $G[B]$ are two disjoint complete graphs. Moreover, assume that w connects A and B such that $aw, bw \in E(G)$ for some $a \in A$ and $b \in B$. Since v connects A and B, without loss of generality, assume that $v = w$. We complete the proof by distinguishing the following two subcases.

Case 2.1. *v is locally connected.*

For any vertex $x \in N(v) \setminus N[u]$, if there is a vertex $y_1 \in N(v) \cap A$ such that $xy_1 \notin E(G)$, then for any vertex $z_1 \in N(v) \cap B$, $xz_1 \in E(G)$ since $G[v, y_1, z_1, x] \neq K_{1,3}$ and $y_1 z_1 \notin E(G)$. Similarly, if there is a vertex $y_2 \in N(v) \cap B$ such that $xy_2 \notin E(G)$, then for any vertex $z_2 \in N(v) \cap A$, $xz_2 \in E(G)$. Thus $G[\{x\} \cup (N(v) \cap A)]$ or $G[\{x\} \cup (N(v) \cap B)]$ is a complete graph. Let $H_1 = \{z \in N(v) \setminus N[u] : G[\{z\} \cup (N(v) \cap A)]$ is a complete graph$\}$. Then for any two distinct vertices $h_1, h_2 \in H_1$ and any vertex $y \in N(v) \cap A$, $h_1 y, h_2 y \in E(G)$ and $h_1 h_2 \in E(G)$ by Lemma 4. Thus $G[H_1 \cup (N(v) \cap A)]$ is a complete graph. Let $H_2 = N(v) \setminus (N[u] \cup H_1)$. Obviously, $H_2 = \emptyset$ or $H_2 = \{h \in N(v) \setminus N[u] : h \notin H_1$ and $G[\{h\} \cup (N(v) \cap B)]$ is a complete graph$\}$. Similarly as for $G[H_1 \cup (N(v) \cap A)]$, we get that $G[H_2 \cup (N(v) \cap B)]$ is also a complete graph. Since v is locally connected, $E(H_1, N(v) \cap B) \neq \emptyset$, $E(H_2, N(v) \cap A) \neq \emptyset$ or $E(H_1, H_2) \neq \emptyset$. Without loss of generality, assume that $E(H_1, N(v) \cap B) \neq \emptyset$ and $x'y' \in E(H_1, N(v) \cap B)$ ($x' \in H_1$, $y' \in N(v) \cap B$). Without loss of generality, we only consider the case that $N(v) \cap B = \{b\}$ (i.e., $y' = b$); the other cases are similarly dealt. Then by Observation 1 and Lemma 4, b is locally connected and there is an edge $x_1' y_1'$ such that $x_1' \in B \setminus \{b\}$ and $y_1' \in N(b) \setminus B$. Without loss of generality, assume that $y_1' \in H_2$. Then we can obtain a (u, v)-path $P = u[A \setminus \{a\}]a[H_1 \setminus \{x'\}]x'b[B \setminus \{b, x_1'\}]x_1' y_1'[H_2 \setminus \{y_1'\}]v$ such that $N(u) \cup N(v) \subseteq V(P)$.

Case 2.2. *v is not locally connected.*

As before, let $N(v) = A_1 \cup B_1$, where $G[A_1]$ and $G[B_1]$ are two disjoint complete graphs. Since v connects A and B, without loss of generality, assume that $A_1 \cap A \neq \emptyset, B_1 \cap B \neq \emptyset$. Then $A_1 \cap B = B_1 \cap A = \emptyset$.

We first prove the following claim.

Claim. For any vertex $x \in A \cup A_1$ and $y \in B \cup B_1$, there is a Hamilton (u, x)-path Q_0 of $G[A \cup A_1 \cup \{u\}]$ and a Hamilton (y, v)-path Q_1 of $G[B \cup B_1 \cup \{v\}]$, respectively.

Proof. Without loss of generality, assume that $x \in A \setminus A_1$ and $A_1 \cap A = \{a\}$. Then by Observation 1 and Lemma 4, a is locally connected and there is an edge $z_1 z_2$ such that $z_1 \in A \setminus \{a\}$ and $z_2 \in N(a) \setminus A$. Since $\{z_2, v\} \subseteq N(a) \setminus A$, by Lemma 4, $z_2 v \in E(G)$ (i.e., $z_2 \in A_1 \setminus A$). Without loss of generality, assume that $z_1 \neq x$. Then we can obtain a Hamilton (u, x)-path $Q_0 = u[A \setminus \{x, z_1, a\}]a[A_1 \setminus \{a, z_2\}]z_2 z_1 x$ of $G[A \cup A_1]$. Symmetrically, we can also obtain a Hamilton (y, v)-path Q_1 of $G[B \cup B_1]$ for any vertex $y \in B \cup B_1$. This completes the proof of the claim. $\square$

Since G is 3-connected, $G - \{u, v\}$ contains at least one (x, y)-path Q_2 connecting $A \cup A_1$ and $B \cup B_1$, with an orientation from x to y, such that $x \in A \cup A_1$, $y \in B \cup B_1$ and $(V(Q_2) \setminus \{x, y\}) \cap (A \cup A_1) = (V(Q_2) \setminus \{x, y\}) \cap (B \cup B_1) = \emptyset$. By the above Claim, there is a Hamilton (u, x)-path Q_0 of $G[A \cup A_1]$ with an orientation from u to x, and a Hamilton (y, v)-path Q_1 of $G[B \cup B_1]$ with an orientation from y to v. Then we can obtain a (u, v)-path $P = Q_0 Q_2 Q_1$ such that $N(u) \cup N(v) \subseteq V(P)$. This completes the proof for Case 2. $\square$

Case 3. *v is a vertex of G with $d(u, v) = 2$ and $N(u) \cup \{v\}$ is disconnected;*

Proof. As before, let $N(u) = A \cup B$ and let w connect the two disjoint complete subgraphs $G[A]$ and $G[B]$ of $G[N(u)]$ such that $aw, bw \in E(G)$ for some $a \in A$ and $b \in B$. Since $d(u, v) = 2$ and v does not connect A and B, suppose $N(v) \cap A \neq \emptyset$ and $N(v) \cap B = \emptyset$.

We first prove the following claim.

Claim. $w \in N(v)$ if and only if $N(w) \cap A \cap N(v) \neq \emptyset$ and $N(w) \cap A \subseteq N(v)$.

Proof of Claim. If $w \in N(v)$, then for any vertex $x \in N(w) \cap A$, $xv \in E(G)$, since $G[w, x, b, v] \neq K_{1,3}$ and $xb, bv \notin E(G)$. Suppose $w \notin N(v)$. If $x' \in N(w) \cap A \cap N(v)$, then $G[x', u, w, v] = K_{1,3}$, a contradiction. This completes the proof of the claim.

We complete the proof by distinguishing two subcases.

Case 3.1. *v is locally connected.*

We first deal with the subcase that $w \notin N(v)$. Then by the above Claim, $a \notin N(v)$. If there is a vertex $y \in N(v) \cap A$ such that y is not a cut vertex of $G[N(v)]$, then by Lemma 2 there is a Hamilton (y, v)-path Q_0 of $G[N(v)]$. Thus we can obtain a (u, v)-path $P = u[B \setminus \{b\}]bwa[A \setminus N(v) \cup \{a\}]yQ_0v$ such that $N(u) \cup N(v) \subseteq V(P)$. If every vertex in $N(v) \cap A$ is a cut vertex of $G[N(v)]$, then by Lemma 1, $|N(v) \cap A| \leq 2$. Suppose $N(v) \cap A = \{z\}$ and z is a cut vertex of $G[N(v)]$. Let H_1 and H_2 be two distinct components of $G[N(v) \setminus \{z\}]$. Then by Lemma 1, $N(v) \setminus \{z\} = H_1 \cup H_2$ and $G[H_1], G[H_2]$ are two disjoint complete graphs. By Observation 1 and Lemma 4, z is locally connected and $G[H_1 \cup H_2]$ is a complete graph since $H_1 \cup H_2 \subseteq N(z) \setminus A$, a contradiction. Thus let $N(v) \cap A = \{v_1, v_2\}$ and v_1, v_2 be cut vertices of $G[N(v)]$. Assume that H_1' and H_2' are two distinct components of $G[N(v) \setminus \{v_1\}]$ and $v_2 \in H_2'$. By Observation 1 and Lemma 4, v_1 is locally connected and there is an edge y_1y_2 such that $y_1 \in A \setminus \{v_1\}, y_2 \in N(v_1) \setminus A$. Since $\{y_2, v\} \subseteq N(v_1) \setminus A$, $y_2v \in E(G)$ by Lemma 4. Obviously, $y_2 \in H_1'$. Then we can obtain a (u,v)-path $P = u[B \setminus \{b\}]bwa[A \setminus \{a, y_1, v_1, v_2\}]y_1y_2[H_1' \setminus \{y_2\}]v_1v_2[H_2' \setminus \{v_2\}]v$ such that $N(u) \cup N(v) \subseteq V(P)$. This completes the proof for the subcase that $w \notin N(v)$.

Suppose next that $w \in N(v)$. By the Claim of Case 3, $a \in N(v)$, and then by Lemma 4, $N(a) - A \subseteq N(v)$. Then for any two distinct vertices $x_1, x_2 \in N(v) \setminus N(w)$, $x_1x_2 \in E(G)$ since $G[v, x_1, x_2, w] \neq K_{1,3}$. It follows that $G[T_1]$ is a complete graph, where $T_1 = \{y : y \in N(v) \setminus N(w)\}$. Let $T_2 = N(v) \cap N(w)$. Obviously, $N(v) = T_1 \cup T_2$. For any vertex $z \in N(v) \cap N(w)$, $za \in E(G)$ or $zb \in E(G)$, since $G[w, z, a, b] \neq K_{1,3}$ and $N(v) \cap N(u) = \emptyset$. By Lemma 4, a and b are locally connected vertices, and $N(a) - A$ and $N(b) - B$ are complete graphs. It follows that $G[T_3]$ and $G[T_4]$ are complete graphs, where $T_3 = \{y : y \in T_2 \cap (N(a) - A)\}$, $T_4 = \{y : y \notin T_3, y \in T_2 \cap (N(b) - B)\}$. Obviously, $T_2 = T_3 \cup T_4$ and then $N(u) = T_1 \cup T_2 = T_1 \cup T_3 \cup T_4$. Since $N(u)$ is connected, $E(T_1, T_3) \neq \emptyset$ or $E(T_1, T_4) \neq \emptyset$. Without loss of generality, $E(T_1, T_3) \neq \emptyset$ and $y_1y_3 \in E(T_1, T_3) (y_1 \in T_1, y_3 \in T_3)$. Since $N(a) - A \subseteq N(v)$, $N[a] = A \cup T_3$. By Lemma 4, there is an edge $y'y_3'$ such that $y' \in A - \{a\}$, $y_3' \in T_3$. If $y_3' \neq y_3$, then we can obtain a (u,v)-path $P = u[B \setminus \{b\}]b[T_4]wa[A \setminus \{a, y'\}]y'y_3'[T_3 \setminus \{y_3, y_3'\}]y_3y_1[T_1 \setminus \{y_1\}]v$ such that $N(u) \cup N(v) \subseteq V(P)$. Similarly, if $y_3 = y_3'$ and $|E(A - \{a\}, T_3)| \geq 2$, then we can obtain a (u,v)-path P such that $N(u) \cup N(v) \subseteq V(P)$. Suppose $E(A - \{a\}, T_3) = \{y'y_3\}$. Since $G[y_3, w, y', y_1] \neq K_{1,3}$ and $N(w) \cap T_1 \neq \emptyset$, $wy' \in E(G)$ or $y'y_1 \in E(G)$. If $wy' \in E(G)$, then we can obtain a (u,v)-path $P = u[B \setminus \{b\}]b[T_4]wy'[A \setminus \{y', a\}]a[T_3 \setminus \{y_3\}]y_3y_1[T_1 \setminus \{y_1\}]v$ such that $N(u) \cup N(v) \subseteq V(P)$. If $y'y_1 \in E(G)$, then we can obtain a (u,v)-path $P = u[B \setminus \{b\}]b[T_4]w[T_3]a[A \setminus \{a, y'\}]y'y_1[T_1 \setminus \{y_1\}]v$ such that $N(u) \cup N(v) \subseteq V(P)$.

Case 3.2. *v is not locally connected.*

As before, let $N(v) = A_1 \cup B_1$, where $G[A_1]$ and $G[B_1]$ are two disjoint complete graphs. Since v does not connect A and B and $d(u,v) = 2$, without loss of generality, let $A_1 \cap A \neq \emptyset$. Then we obtain $B_1 \cap A = B_1 \cap B = \emptyset$. Since G is 3-connected, $G - \{w, v\}$ contains an (x,y)-path Q_0, with an orientation from x to y, connecting $N(u) \cup A_1$ and B_1 such that $x \in N(u) \cup A_1$, $y \in B_1$ and $(V(Q_0) \setminus \{x, y\}) \cap (N(u) \cup N(v)) = \emptyset$. We get that $x \notin A \cap A_1$; otherwise $G[x, u, x^+, v] = K_{1,3}$, where x^+ is the successor of x in the orientation of Q_0.

Suppose first that $w \notin N(v)$. Without loss of generality, we only consider the case that $N(w) \cap A = \{a\}$, $A \cap A_1 = \{z\}$ and $x = a$; the other cases are similarly dealt. Then by Observation 1 and Lemma 4, a is locally connected and there is an edge x_1x_2 such that $x_1 \in A \setminus \{a\}$ and $x_2 \in N(a) \setminus A$. Moreover, z is locally connected and there is an edge y_1y_2 such that $y_1 \in A \setminus \{z\}$ and $y_2 \in N(z) \setminus A$. Without loss of generality, assume that $x_1 = y_1$, $x_1 \neq z$ and $x_2 = a^+$, where a^+ is the successor of a in the orientation of Q_0. Since $\{x_2, w\} \subseteq N(a) \setminus A$, we get that $x_2w \in E(G)$ by Lemma 4. Similarly, $y_2v \in E(G)$. Obviously, $y_2 \in A_1$. We also have $x_2 \neq y_2$; otherwise $G[y_2, w, v, x_1] = K_{1,3}$, a contradiction. Now we can obtain a (u, v)-path $P = u[B \setminus \{b\}]wa[A \setminus \{a, z, x_1\}]z[A_1 \setminus \{z, y_2\}]y_2x_1Q_0y[B_1 \setminus \{y\}]v$ such that $N(u) \cup N(v) \subseteq V(P)$, where y^- is the successor of y in the orientation of Q_0^-.

Suppose next that $w \in N(v)$. Then $w \in A_1$ and by the above Claim, $a \in A \cap A_1$. Without loss of generality, assume that $x \in A \setminus A_1$. Then we can obtain a (u, v)-path $P = u[B \setminus \{b\}]bw[A_1 \setminus A]a[A \setminus \{a, x\}]xQ_0y[B_1 \setminus \{y\}]v$ such that $N(u) \cup N(v) \subseteq V(P)$. This completes the proof for Case 3. □

Case 4. *v is a vertex of G with $d(u, v) \geq 3$.*

Proof. As before, let $N(u) = A \cup B$ and let w connect the two disjoint complete graphs $G[A]$ and $G[B]$ such that aw and $bw \in E(G)$ for some $a \in A$ and $b \in B$. We complete the proof by distinguishing two subcases.

Case 4.1. *v is locally connected.*

Using Lemma 1, without loss of generality, assume that $G[N(v)]$ contains two distinct cut vertices v_1 and v_2. Since G is 3-connected, $G \setminus \{v_1, v_2\}$ contains at least one (x, y)-path Q_0, with an orientation from x to y, connecting $N(u)$ and $N(v)$ such that $x \in N(u)$, $y \in N(v)$ and $(V(Q_0) \setminus \{x, y\}) \cap (N[u] \cup N[v]) = \emptyset$. Suppose $w \in V(Q_0)$ and w^+ is the successor of w in the orientation of Q_0. Since $(V(Q_0) \setminus \{x, y\}) \cap (N[u] \cup N[v]) = \emptyset$, $w^+ \notin \{a, b\}$. Then $w^+a \in E(G)$ or $w^+b \in E(G)$ since $G[w, w^+, a, b] \neq K_{1,3}$ and $ab \notin E(G)$. It follows that we can replace Q_0 by the path aw^+Q_0y or bw^+Q_0y. Thus without loss of generality, we assume that $w \notin V(Q_0)$ and we only consider the case that $N(w) \cap A = \{a\}$ and $x = a$. Then by Observation 1 and Lemma 4, a is locally connected and there is an edge x_1x_2 such that $x_1 \in A \setminus \{a\}$ and $x_2 \in N(a) \setminus A$. Since $\{x_2, w\} \subseteq N(a) \setminus A$, $x_2w \in E(G)$ by Lemma 4. Without loss of generality, assume that $x_2 \notin V(Q_0)$. By Lemma 2, we can get a Hamilton (y, v)-path Q_1 of $G[N(v)]$. Then we can obtain a (u, v)-path $P = u[B \setminus \{b\}]bwx_2x_1[A \setminus \{a, x_1\}]aQ_0yQ_1v$ such that $N(u) \cup N(v) \subseteq V(P)$.

Case 4.2. *v is not locally connected.*

As before, let $N(v) = A_1 \cup B_1$ and let w_1 connect the two disjoint complete graphs $G[A_1]$ and $G[B_1]$ such that a_1w_1 and $b_1w_1 \in E(G)$ for some $a_1 \in A_1$ and $b_1 \in B_1$.

We first suppose that $w_1 \neq w$. Since G is connected, there is an (x', y')-path Q_2 with an orientation from x' to y' connecting $N(u)$ and $N(v)$ such that $x' \in N(u)$, $y' \in N(v)$ and $(V(Q_2) \setminus \{x', y'\}) \cap (N[u] \cup N[v]) = \emptyset$. As in Case 4.1, without loss of generality, assume that $w_1, w \notin V(Q_2)$ and we only consider the case that $N(w) \cap A = \{a\}$, $N(w_1) \cap A_1 = \{a_1\}$, $x' = a$ and $y' = a_1$; the other cases are similarly dealt. Then by Observation 1 and Lemma 4, a is locally connected and there is an edge $x'_1x'_2$ such that $x'_1 \in A \setminus \{a\}$ and $x'_2 \in N(a) \setminus A$. Similarly, a_1 is locally connected and there is an edge $y'_1y'_2$ such that $y'_1 \in A_1 \setminus \{a_1\}$ and $y'_2 \in N(a_1) \setminus A_1$. Moreover, by Lemma 4, $G[N(a) \setminus A]$ and $G[N(a_1) \setminus A_1]$ are complete graphs. Thus without loss of generality, assume that $x'_2 = a^+$ and $y'_2 = a_1^-$, where a^+ is the successor of a in the orientation of Q_2, and a_1^- is the successor of a_1 in the orientation of Q_2^-. Then we can obtain a (u, v)-path $P = u[B \setminus \{b\}]bwa[A \setminus \{a, x'_1\}]x'_1x'_2Q_2y'_2y'_1[A_1 \setminus \{y'_1, a_1\}]a_1w_1b_1[B_1 \setminus \{b_1\}]v$ such that $N(u) \cup N(v) \subseteq V(P)$.

Next we suppose that $w = w_1$. Since $G[w, a, b, a_1] \neq K_{1,3}$ and $ab \notin E(G)$, $aa_1 \in E(G)$ or $ba_1 \in E(G)$. Similarly, $ab_1 \in E(G)$ or $b_1b \in E(G)$. If aa_1 and $b_1a \in E(G)$, then $G[a, u, a_1, b_1] = K_{1,3}$, a contradiction. Thus $aa_1, bb_1 \in E(G)$ or $ab_1, ba_1 \in E(G)$. Without loss of generality, assume that aa_1, $bb_1 \in E(G)$, and we only consider the case that $N(w) \cap B_1 = \{b_1\}$ and $N(w) \cap A = \{a\}$; the other cases are similar. By Observation 1 and Lemma 4, a is locally connected and there is an edge y_1y_2 such that $y_1 \in A \setminus \{a\}$ and $y_2 \in N(a) \setminus A$. Similarly, b_1 is locally connected and there is an edge z_1z_2 such that $z_1 \in B_1 \setminus \{b_1\}$ and $z_2 \in N(b_1) \setminus B_1$. Without loss of generality, assume that $b \neq z_2$ and $a_1 \neq y_2$. Then by Lemma 4, $G[N(a) \setminus A]$ and $G[N(b_1) \setminus B_1]$ are two complete graphs. Thus bz_2, wz_2, wy_2 and $a_1y_2 \in E(G)$. Without loss of generality, assume that $y_2, z_2 \notin N[u] \cup N[v]$. First suppose $y_2 = z_2$. Then we can obtain a (u, v)-path $P = u[B \setminus \{b\}]bb_1[B_1 \setminus \{b_1, z_1\}]z_1y_2y_1[A \setminus \{y_1, a\}]aa_1[A_1 \setminus \{a_1\}]v$ such that $N(u) \cup N(v) \subseteq V(P)$. Next suppose $y_2 \neq z_2$. Then we can get a Hamilton (u, w)-path $Q_3 = u[B \setminus \{b\}]bz_2z_1[B_1 \setminus \{z_1, b_1\}]b_1w$ of $G[B \cup B_1 \cup \{u, z_2, w\}]$. Similarly, we can get a Hamilton (a, v)-path Q_4 of $G[A \cup A_1 \cup \{y_2\}]$. Thus we can obtain a (u, v)-path $P = Q_3Q_4$ such that $N(u) \cup N(v) \subseteq V(P)$. This completes the proof for Case 4.

$\square$

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