# On the metric dimension of rotationally-symmetric graphs*

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**Abstract.** A family $\mathcal{G}$ of connected graphs is a family with constant metric dimension if $dim(G)$ is finite and does not depend upon the choice of $G$ in $\mathcal{G}$. The metric dimension of some classes of plane graphs has been determined in [2], [3], [4], [9], [10], [14] and [22]. In this paper, we extend this study by considering some classes of plane graphs which are rotationally-symmetric. It is natural to ask for the characterization of classes of rotationally-symmetric plane graphs with constant metric dimension.

Keywords: *Metric dimension, basis, subdivision graph, resolving set, plane graph, rotationally-symmetric*

## 1 Notation and preliminary results

If $G$ is a connected graph, the distance $d(u,v)$ between two vertices $u,v \in V(G)$ is the length of a shortest path between them. Let $W = \{w_1, w_2, ...., w_k\}$ be an ordered set of vertices of $G$ and let $v$ be a vertex of $G$. The representation $r(v|W)$ of $v$ with respect to $W$ is the $k$-tuple $(d(v,w_1), d(v,w_2), ....., d(v,w_k))$. If distinct vertices of $G$ have distinct representations with respect to $W$, then $W$ is called a resolving set or locating set for $G$ [2]. A resolving set of minimum cardinality is called a basis for $G$ and this cardinality is the metric dimension of $G$, denoted by $dim(G)$. The

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concepts of resolving set and metric basis have previously appeared in the literature (see [2-6, 8-11, 14, 16-22]).
For a given ordered set of vertices $W = \{w_1, w_2, ...., w_k\}$ of a graph $G$, the *ith* component of $r(v|W)$ is 0 if and only if $v = w_i$. Thus, to show that $W$ is a resolving set it suffices to verify that $r(x|W) \neq r(y|W)$ for each pair of distinct vertices $x, y \in V(G)\backslash W$.
A useful property in finding $dim(G)$ is the following lemma [21]:

**Lemma 1.** *Let $W$ be a resolving set for a connected graph $G$ and $u, v \in V(G)$. If $d(u,w) = d(v,w)$ for all vertices $w \in V(G) \setminus \{u,v\}$, then $\{u,v\} \cap W \neq \emptyset$.*

Motivated by the problem of uniquely determining the location of an intruder in a network, the concept of metric dimension was introduced by Slater in [19, 20] and studied independently by Harary and Melter in [8]. Applications of this invariant to the navigation of robots in networks are discussed in [16] and applications to chemistry in [5] while applications to problem of pattern recognition and image processing, some of which involve the use of hierarchical data structures are given in [17].
A representation of a graph $G$ is said to be plane if it is drawn on the Euclidean plane such that edges do not cross each other except at vertices of the graph.
By denoting $G + H$ the join of $G$ and $H$ a *wheel* $W_n$ is defined as $W_n = K_1 + C_n$, for $n \geq 3$, a *fan* is $f_n = K_1 + P_n$ for $n \geq 1$ and *Jahangir graph* $J_{2n}, (n \geq 2)$ (also known as *gear graph*) is obtained from the *wheel* $W_{2n}$ by alternately deleting $n$ spokes. Buczkowski *et al.* [2] determined the dimension of the *wheel* $W_n$, Caceres *et al.* [4] the dimension of the *fan* $f_n$ and Tomescu and Javaid [22] the dimension of the *Jahangir graph* $J_{2n}$.

**Theorem 1.** *([2], [4], [22]) Let $W_n$ be a wheel of order $n \geq 3$, $f_n$ be a fan of order $n \geq 1$ and $J_{2n}$ be a Jahangir graph. Then*
*(i) For $n \geq 7$, $dim(W_n) = \lfloor \frac{2n+2}{5} \rfloor$;*
*(ii) For $n \geq 7$, $dim(f_n) = \lfloor \frac{2n+2}{5} \rfloor$;*
*(iii) For $n \geq 4$, $dim(J_{2n}) = \lfloor \frac{2n}{3} \rfloor$.*

The metric dimension of all these plane graphs depends upon the number of vertices in the graph.
On the other hand, we say that a family $\mathcal{G}$ of connected graphs is a family with constant metric dimension if $dim$(G) is finite and does not depend upon the choice of $G$ in $\mathcal{G}$. In [5] it was shown that a graph has metric dimension 1 if and only if it is a *path*, hence paths on $n$ vertices constitute a family of graphs with constant metric dimension. Similarly, *cycles* with $n(\geq 3)$ vertices also constitute such a family of graphs as their metric dimension is 2 and does not depend upon on the number of vertices $n$. In [3]

it was proved that

$$dim(P_m \times C_n)=\begin{cases} 2, & \text{if } n \text{ is odd;} \\ 3, & \text{if } n \text{ is even.} \end{cases}$$

Since *prisms* $D_n$ (also called circular ladders) are the trivalent plane graphs obtained by the cross product of path $P_2$ with a cycle $C_n$, this implies that

$$dim(D_n)=\begin{cases} 2, & \text{if } n \text{ is odd;} \\ 3, & \text{if } n \text{ is even.} \end{cases}$$

So, prisms (circular ladders) constitute a family of 3-*regular graphs* with constant metric dimension. Also Javaid *et al.* proved in [10] that the plane graph *antiprism* $A_n$ constitute a family of regular graphs with constant metric dimension as $dim(A_n) = 3$ for every $n \geq 5$. The prism and the antiprism are *Archimedean* convex polytopes defined e.g. in [13].
A *Cartesian product* of two graphs $G$ and $H$, denoted by $G \times H$, is the graph with vertex set $V(G) \times V(H)$, where two vertices $(x, x')$ and $(y, y')$ are adjacent if and only if $x = y$ and $x'y' \in E(H)$ or $x' = y'$ and $xy \in E(G)$. A *grid* $G_n^m$ is obtained by the cartesian product of two paths $P_n$ by $P_m$. In [14], it was shown that $dim(P_n \times P_m) = 2$, so grids constitute a family of plane graphs with constant metric dimension as their metric dimension is 2 and does not depend upon the number of vertices in the graph. The metric dimension of Cartesian product of graphs has been studied in [3] and [18]. It is shown in [9] that some families of plane graphs generated by convex polytopes constitute the families of plane graphs with constant metric dimension. Note that the problem of determining whether $dim(G) < k$ is an $NP$-complete problem [7]. Some bounds for this invariant, in terms of the diameter of the graph, are given in [16] and it was shown in [5, 16–18] that the metric dimension of trees can be determined efficiently. It appears unlikely that significant progress can be made in determining the dimension of a graph unless it belongs to a class for which the distances between vertices can be described in some systematic manner.
In this paper, we study the metric dimension of some classes of plane graphs which are rotationally-symmetric. In the second section, we study the metric dimension of subdivision graph of a prism (circular ladder). The metric dimension of web graphs defined by Koh et al. [15] and plane graphs $\mathbb{A}_n$ has been studied in third and fourth section. In the fifth section, metric dimension of sun graphs $S_n$ has been determined. It is natural to ask for the characterization of classes of rotationally-symmetric plane graphs with constant metric dimension.

## 2 Subdivision of prism (circular ladder)

The prism $D_n$ (circular ladder), $n \geq 3$ is a cubic graph which can be defined as the cartesian product $P_2 \times C_n$ of a path on two vertices with a cycle on $n$ vertices. Prism $D_n$, $n \geq 3$ consists of an outer $n$-cycle $y_1y_2...y_n$, an inner $n$-cycle $x_1x_2...x_n$, and a set of $n$ spokes $x_iy_i, i = 1, 2, ..., n$. We have $|V(D_n)| = 2n$, $|E(D_n)| = 3n$ and $|F(D_n)| = n+2$, where $|V(D_n)|, |E(D_n)|$ and $|F(D_n)|$ denote the number of vertices, edges and faces of the prism $D_n$, respectively.

The subdivision graph $S(D_n)$ can be obtained by adding a new vertex $u_i$ between $x_i$ and $x_{i+1}$, adding a new vertex $v_i$ between $x_i$ and $y_i$ and adding a new vertex $w_i$ between $y_i$ and $y_{i+1}$. Clearly, $S(D_n)$ has $5n$ vertices and $6n$ edges.

The metric dimension of the generalized prism $P_m \times C_n$ has been deter-

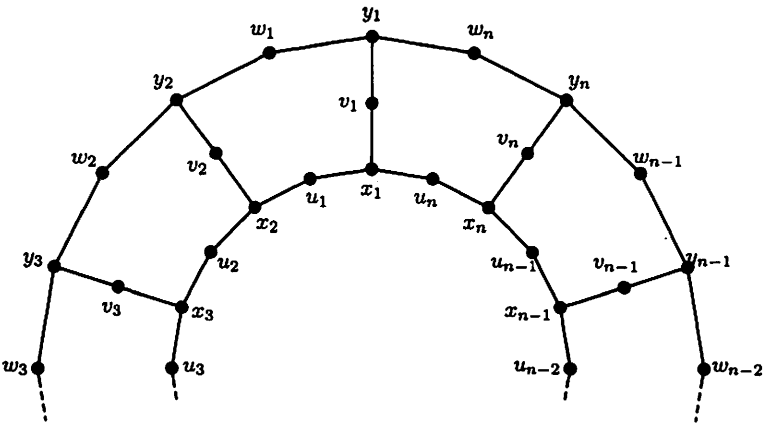


**Fig. 1.** Subdivision of prism

mined in [3] and prism $D_n$ (circular ladder) is actually the cross product of $P_2 \times C_n$. In the next theorem, we prove that the metric dimension of the subdivision graph $S(D_n)$ of the prism is constant and only three vertices appropriately chosen suffice to resolve all the vertices of the subdivision graph $S(D_n)$ of prism.

For our purpose, we call the cycle induced by $\{x_i : 1 \leq i \leq n\} \cup \{u_i : 1 \leq i \leq n\}$, the inner cycle, the cycle induced by $\{y_i : 1 \leq i \leq n\} \cup \{w_i : 1 \leq i \leq n\}$, the outer cycle and set of vertices $\{v_i : 1 \leq i \leq n\}$, the set of interior vertices. Note that the choice of appropriate basis vertices (also refereed to as landmarks in [14]) is the core of the problem.

**Theorem 2.** *Let $S(D_n)$ be the subdivision graph of the circular ladder; then $dim(S(D_n)) = 3$ for every $n \geq 6$.*

*Proof.* We will prove the above equality by double inequalities.
**Case** 1. When $n$ is even.
In this case, we can write $n = 2k$, $k \geq 3$, $k \in \mathbf{Z}^+$. Let $W = \{x_1, x_2, x_{k+1}\} \subset V(S(D_n))$, we show that $W$ is a resolving set for $S(D_n)$ in this case. For this we give representations of any vertex of $V(S(D_n))\backslash W$ with respect to $W$.
Representations for the vertices on the inner cycle are

$$r(x_i|W) = \begin{cases} (2i-2, 2i-4, 2k-2i+2), & 3 \leq i \leq k; \\ (4k-2i+2, 4k-2i+4, 2i-2k-2), & k+2 \leq i \leq 2k. \end{cases}$$

and

$$r(u_i|W) = \begin{cases} (1,1,2k-1), & i = 1; \\ (2i-1, 2i-3, 2k-2i+1), & 2 \leq i \leq k; \\ (2k-1, 2k-1, 1), & i = k+1; \\ (4k-2i+1, 4k-2i+3, 2i-2k-1), & k+2 \leq i \leq 2k. \end{cases}$$

Representations for the set of interior vertices are

$$r(v_i|W) = \begin{cases} (1,3,2k+1), & i = 1; \\ (2i-1, 2i-3, 2k-2i+3), & 2 \leq i \leq k+1; \\ (4k-2i+3, 4k-2i+5, 2i-2k-1), & k+2 \leq i \leq 2k. \end{cases}$$

Representations for the vertices on the outer cycle are

$$r(y_i|W) = \begin{cases} (2,4,2k+2), & i = 1; \\ (2i, 2i-2, 2k-2i+4), & 2 \leq i \leq k+1; \\ (4k-2i+4, 4k-2i+6, 2i-2k), & k+2 \leq i \leq 2k. \end{cases}$$

and

$$r(w_i|W) = \begin{cases} (3,3,2k+1), & i = 1; \\ (2i+1, 2i-1, 2k-2i+3), & 2 \leq i \leq k; \\ (2k+1, 2k+1, 3), & i = k+1; \\ (4k-2i+3, 4k-2i+5, 2i-2k+1), & k+2 \leq i \leq 2k. \end{cases}$$

We note that there are no two vertices having the same representations implying that $dim(S(D_n)) \leq 3$.
On the other hand, we show that $dim(S(D_n)) \geq 3$. Suppose on contrary that $dim(S(D_n)) = 2$, then there are the following possibilities to be discussed.
**(1)** Both vertices are in the inner cycle. Here are the following subcases.

• Both vertices belong to the set $\{x_i : 1 \leq i \leq n\}$. Without loss of generality, we can suppose that one resolving vertex is $x_1$. Suppose that the second resolving vertex is $x_t$ $(2 \leq t \leq k+1)$. Then for $2 \leq t \leq k$, we have $r(u_n|\{x_1, x_t\}) = r(v_1|\{x_1, x_t\}) = (1, 2t-1)$, and for $t = k+1$, we have $r(u_1|\{x_1, x_{k+1}\}) = r(u_n|\{x_1, x_{k+1}\}) = (1, 2k-1)$, a contradiction.
• Both vertices belong to the set $\{u_i : 1 \leq i \leq n\}$. Without loss of generality, we can suppose that one resolving vertex is $u_1$. Suppose that the second resolving vertex is $u_t$ $(2 \leq t \leq k+1)$. Then for $2 \leq t \leq k$, we have $r(u_n|\{u_1, u_t\}) = r(v_1|\{u_1, u_t\}) = (2, 2t)$, and for $t = k+1$, we have $r(x_1|\{u_1, u_{k+1}\}) = r(x_2|\{u_1, u_{k+1}\}) = (1, 2k-1)$, a contradiction.
• One vertex is in the set $\{x_i : 1 \leq i \leq n\}$ and the other one is in the set $\{u_i : 1 \leq i \leq n\}$. Without loss of generality, we can suppose that one resolving vertex is $x_1$. Suppose that the second resolving vertex is $u_t$ $(1 \leq t \leq k+1)$. Then for $1 \leq t \leq k$, we have $r(u_n|\{x_1, u_t\}) = r(v_1|\{x_1, u_t\}) = (1, 2t)$, and for $t = k+1$, we have $r(u_1|\{x_1, u_{k+1}\}) = r(v_1|\{x_1, u_{k+1}\}) = (1, 2k)$, a contradiction.
**(2)** Both vertices are in the set of interior vertices. Without loss of generality, we can suppose that one resolving vertex is $v_1$. Suppose that the second resolving vertex is $v_t$ $(2 \leq t \leq k+1)$. Then for $2 \leq t \leq k+1$, we have $r(x_1|\{v_1, v_t\}) = r(y_1|\{v_1, v_t\}) = (1, 2t-1)$, a contradiction.
**(3)** Both vertices are in the outer cycle. Due to the symmetry of the graph, this case is analogous to case **(1)**.
**(4)** One vertex is in the inner cycle and the other one is in the set of interior vertices. Here are the two subcases.
• One vertex is in the set $\{x_i : 1 \leq i \leq n\}$ and the other one is in the set of interior vertices. Without loss of generality, we can suppose that one resolving vertex is $x_1$. Suppose that the second resolving vertex is $v_t$ $(1 \leq t \leq k+1)$. Then for $t = 1$, we have $r(u_1|\{x_1, v_1\}) = r(u_n|\{x_1, v_1\}) = (1, 2)$. For $2 \leq t \leq k$, $r(u_n|\{x_1, v_t\}) = r(v_1|\{x_1, v_t\}) = (1, 2t)$ and for $t = k+1$, we have $r(u_1|\{x_1, v_{k+1}\}) = r(u_n|\{x_1, v_{k+1}\}) = (1, 2k)$, a contradiction.
• One vertex is in the set $\{u_i : 1 \leq i \leq n\}$ and the other one is in the set of interior vertices. Without loss of generality, we can suppose that one resolving vertex is $u_1$. Suppose that the second resolving vertex is $v_t$ $(1 \leq t \leq k+1)$. Then for $t = 1$, we have $r(w_1|\{u_1, v_1\}) = r(w_n|\{u_1, v_1\}) = (2, 4)$. For $2 \leq t \leq k$, $r(u_n|\{u_1, v_t\}) = r(v_1|\{u_1, v_t\}) = (2, 2t)$ and for $t = k+1$, we have $r(u_n|\{u_1, v_{k+1}\}) = r(v_2|\{u_1, v_{k+1}\}) = (2, 2k)$, a contradiction.
**(5)** One vertex is in the outer cycle and the other one is in the set of interior vertices. Due to the symmetry of the graph, this case is analogous to case **(4)**.
**(6)** One vertex is in the inner cycle and the other one is in the outer cycle. We have the following subcases.
• One vertex is in the set $\{x_i : 1 \leq i \leq n\}$ and the other one is in the set $\{y_i : 1 \leq i \leq n\}$. Without loss of generality, we can suppose that one re-

solving vertex is $x_1$. Suppose that the second resolving vertex is $y_t$ $(1 \leq t \leq k+1)$. Then for $t=1$, we have $r(u_1|\{x_1, y_1\}) = r(u_n|\{x_1, y_1\}) = (1, t+2)$. For $2 \leq t \leq k+1$, we have $r(u_1|\{x_1, y_t\}) = r(v_1|\{x_1, y_t\}) = (1, 2t-1)$, a contradiction.

• One vertex is in the set $\{x_i : 1 \leq i \leq n\}$ and the other one is in the set $\{w_i : 1 \leq i \leq n\}$. Without loss of generality, we can suppose that one resolving vertex is $x_1$. Suppose that the second resolving vertex is $w_t$ $(1 \leq t \leq k+1)$. Then for $t=1$, we have $r(u_1|\{x_1, w_1\}) = r(u_n|\{x_1, w_1\}) = (1, t+3)$. For $2 \leq t \leq k+1$, we have $r(u_1|\{x_1, w_t\}) = r(v_1|\{x_1, w_t\}) = (1, 2t)$, a contradiction.

• One vertex is in the set $\{u_i : 1 \leq i \leq n\}$ and the other is in the set $\{y_i : 1 \leq i \leq n\}$. Due to the symmetry of the graph, this subcase is analogous to above subcase.

• One vertex is in the set $\{u_i : 1 \leq i \leq n\}$ and the other one is in the set $\{w_i : 1 \leq i \leq n\}$. Without loss of generality, we can suppose that one resolving vertex is $u_1$. Suppose that the second resolving vertex is $w_t$ $(1 \leq t \leq k+1)$. Then for $t=1$, we have $r(x_1|\{u_1, w_1\}) = r(x_2|\{u_1, w_1\}) = (1, 3)$. For $t=2$, $r(v_3|\{u_1, w_2\}) = r(w_1|\{u_1, w_2\}) = (4, 2)$ and when $3 \leq t \leq k+1$, we have $r(v_3|\{u_1, w_t\}) = r(w_2|\{u_1, w_t\}) = (4, 2t-4)$, a contradiction.

Hence, from above it follows that there is no resolving set with two vertices for $V(S(D_n))$ implying that $dim(S(D_n)) = 3$ in this case.

**Case 2.** When $n$ is odd.

In this case, we can write $n = 2k+1$, $k \geq 3$, $k \in \mathbf{Z}^+$. Let $W = \{x_1, x_2, u_{k+1}\} \subset V(S(D_n))$, we show that $W$ is a resolving set for $S(D_n)$ in this case. For this we give representations of any vertex of $V(S(D_n))\backslash W$ with respect to $W$.

Representations for the vertices on the inner cycle are

$$r(x_i|W) = \begin{cases} (2i-2, 2i-4, 2k-2i+3), & 3 \leq i \leq k+1; \\ (2k, 2k, 1), & i = k+2; \\ (4k-2i+4, 4k-2i+6, 2i-2k-3), & k+3 \leq i \leq 2k+1. \end{cases}$$

and

$$r(u_i|W) = \begin{cases} (1, 1, 2k), & i = 1; \\ (2i-1, 2i-3, 2k-2i+2), & 2 \leq i \leq k; \\ (4k-2i+3, 4k-2i+5, 2i-2k-2), & k+2 \leq i \leq 2k+1. \end{cases}$$

Representations for the set of interior vertices are

$$r(v_i|W) = \begin{cases} (1, 3, 2k+2), & i = 1; \\ (2i-1, 2i-3, 2k-2i+4), & 2 \leq i \leq k+1; \\ (2k+1, 2k+1, 2), & i = k+2; \\ (4k-2i+5, 4k-2i+7, 2i-2k-2), & k+3 \leq i \leq 2k+1. \end{cases}$$

Representations for the vertices on the outer cycle are

$$r(y_i|W) = \begin{cases} (2,4,2k+3), & i=1; \\ (2i,2i-2,2k-2i+5), & 2 \le i \le k+1; \\ (2k+2,2k+2,3), & i=k+2; \\ (4k-2i+6,4k-2i+8,2i-2k-1), & k+3 \le i \le 2k+1. \end{cases}$$

and

$$r(w_i|W) = \begin{cases} (3,3,2k+2), & i=1; \\ (2i+1,2i-1,2k-2i+4), & 2 \le i \le k; \\ (2k+3,2k+1,4), & i=k+1; \\ (4k-2i+5,4k-2i+7,2i-2k), & k+2 \le i \le 2k+1. \end{cases}$$

Again we see that there are no two vertices having the same representations which implies that $dim(S(D_n)) \le 3$.
On the other hand, suppose that $dim(S(D_n)) = 2$, then there are the same possibilities as in case (i) and contradictions can be deduced analogously. This implies that $dim(S(D_n)) = 3$ in this case, which completes the proof. □

A fundamental question in graph theory concerns how the value of a parameter is affected by making a small change in the graph. If $G'$ is a graph obtained by adding a pendant edge to a nontrivial connected graph $G$, then it is easy to verify that

$$dim(G) \le dim(G') \le dim(G) + 1$$

A *helm* $H_n$, $n \ge 3$ is a graph obtained from a wheel by attaching a pendant vertex to each rim vertex. Javaid [10] proved that $dim(H_n) = dim(W_n)$. In the next two sections, we extend this study by considering the web graph $\mathbb{W}_n$ defined by Koh *et al.* [15] and the plane graph $\mathbb{A}_n$. We prove that by adding a pendant edge at each vertex of the outer cycle of the prism $D_n$ and antiprism $A_n$ (The prsim and antiprism are Archimedian convex polytopes defined in [13]) does not affect their metric dimension.

## 3 The web graph $\mathbb{W}_n$

Koh *et al.* [15] define a web graph as a prism graph $P_3 \times C_n$ with the edges of the outer cycle removed. The web graph $\mathbb{W}_n$ can be obtained from prism $D_n$ by adding a pendant edge at each vertex of the outer cycle of the prism $D_n$. We have

$$V(\mathbb{W}_n) = V(D_n) \cup \{z_i : 1 \le i \le n\}$$

and

$$E(\mathbb{W}_n) = E(D_n) \cup \{y_i z_i : 1 \le i \le n\}$$

In the next theorem, we prove that the metric dimension of the web graph

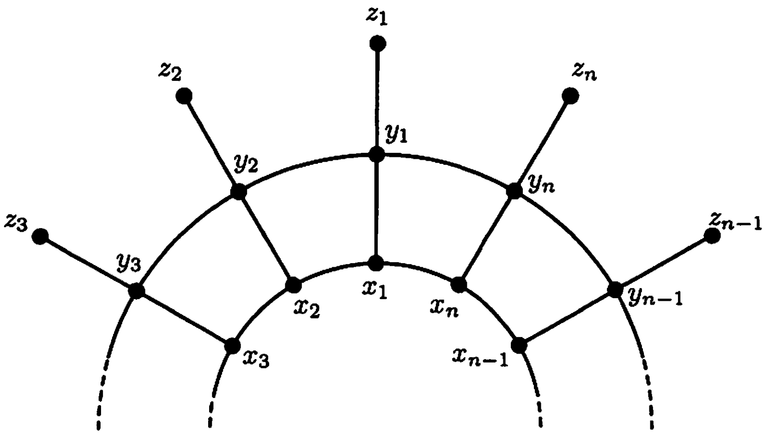


**Fig. 2.** The web graph $\mathbb{W}_n$

$\mathbb{W}_n$ is the same as the metric dimension of the prism (circular ladder) $D_n$. For our purpose, we call the cycle induced by $\{x_i : 1 \le i \le n\}$, the inner cycle, the cycle induced by $\{y_i : 1 \le i \le n\}$, the outer cycle, and set of vertices $\{z_i : 1 \le i \le n\}$, the pendent vertices. Again, the choice of an appropriate basis of vertices (also refereed to as landmarks in [14]) is very important.

**Theorem 3.** *Let* $\mathbb{W}_n$ *be the web graph ; then for every* $n \ge 5$

$$dim(\mathbb{W}_n) = \begin{cases} 2, & \textit{when } n \textit{ is odd;} \\ 3, & \textit{otherwise.} \end{cases}$$

*Proof.* We will prove the above equality by double inequalities.

**Case** 1. When $n$ is odd.

In this case, we can write $n = 2k+1$, $k \ge 2$, $k \in \mathbb{Z}^+$. Let $W = \{x_1, x_{k+1}\} \subset V(\mathbb{W}_n)$, we show that $W$ is a resolving set for $\mathbb{W}_n$ in this case. For this we give the representations of any vertex for $V(\mathbb{W}_n) \backslash W$ with respect to $W$. The representations for the vertices on inner cycle are

$$r(x_i|W) = \begin{cases} (i-1, k-i+1), & 2 \le i \le k; \\ (2k-i+2, i-k-1), & k+2 \le i \le 2k+1. \end{cases}$$

The representations for vertices on the outer cycle are

$$r(y_i|W) = \begin{cases} (i, k-i+2), & 1 \leq i \leq k+1; \\ (2k-i+3, i-k), & k+2 \leq i \leq 2k+1. \end{cases}$$

The representations for pendent vertices are

$$r(z_i|W) = \begin{cases} (i+1, k-i+3), & 1 \leq i \leq k+1; \\ (2k-i+4, i-k+1), & k+2 \leq i \leq 2k+1. \end{cases}$$

We note that there are no two vertices having the same representation implying that $dim(\mathbb{W}_n) \leq 2$. On the other hand, since the plane graph $\mathbb{W}_n$ is not a path, it follows from [5] that $dim(\mathbb{W}_n) = 2$.

**Case** 2. When $n$ is even.

In this case, we can write $n = 2k$, $k \geq 3$, $k \in \mathbb{Z}^+$. We show that $W = \{x_1, x_2, x_{k+1}\} \subset V(\mathbb{W}_n)$ is a resolving set for $W_n$ in this case. For this we give representations for any vertex of $V(\mathbb{W}_n) \backslash W$ with respect to $W$.

The representations for the vertices on inner cycle are

$$r(x_i|W) = \begin{cases} (i-1, i-2, k-i+1), & 3 \leq i \leq k; \\ (2k-i+1, 2k-i+2, i-k-1), & k+2 \leq i \leq 2k. \end{cases}$$

The representations for vertices on the outer cycle are

$$r(y_i|W) = \begin{cases} (1, 2, k+1), & i = 1; \\ (i, i-1, k-i+2), & 2 \leq i \leq k+1; \\ (2k-i+2, 2k-i+3, i-k), & k+2 \leq i \leq 2k. \end{cases}$$

The representations for pendent vertices are

$$r(z_i|W) = \begin{cases} (2, 3, k+2), & i = 1; \\ (i+1, i, k-i+3), & 2 \leq i \leq k+1; \\ (2k-i+3, 2k-i+4, i-k+1), & k+2 \leq i \leq 2k. \end{cases}$$

We note that there are no two vertices having the same representations implying that $dim(\mathbb{W}_n) \leq 3$.

On the other hand, we show that $dim(\mathbb{W}_n) \geq 3$. Suppose on contrary that $dim(\mathbb{W}_n) = 2$, then there are the following possibilities to be discussed.

**(1)** Both vertices are in the inner cycle. Without loss of generality, we can suppose that one resolving vertex is $x_1$. Suppose that the second resolving vertex is $x_t$ $(2 \leq t \leq k+1)$. Then for $2 \leq t \leq k$, we have $r(x_n|\{x_1, x_t\}) = r(y_1|\{x_1, x_t\}) = (1, t)$ and for $t = k+1$, we have $r(x_2|\{x_1, x_{k+1}\}) = r(x_n|\{x_1, x_{k+1}\}) = (1, k-1)$, a contradiction.

**(2)** Both vertices are in the outer cycle. Without loss of generality, we can suppose that one resolving vertex is $y_1$. Suppose that the second resolving vertex is $y_t$ $(2 \leq t \leq k+1)$. Then for $2 \leq t \leq k$, we have $r(y_n|\{y_1, y_t\}) = r(z_1|\{y_1, y_t\}) = (1, t)$ and for $t = k+1$, we have $r(y_2|\{y_1, y_{k+1}\}) = r(y_n|\{y_1, y_{k+1}\}) = (1, k-1)$, a contradiction.

**(3)** Both vertices are in set of pendent vertices. Without loss of generality, we can suppose that one resolving vertex is $z_1$. Suppose that the second resolving vertex is $z_t$ $(2 \leq t \leq k+1)$. Then for $2 \leq t \leq k$, we have $r(x_1|\{z_1, z_t\}) = r(y_n|\{z_1, z_t\}) = (1, t)$ and for $t = k+1$, we have

$r(z_2|\{z_1, z_{k+1}\}) = r(z_n|\{z_1, z_{k+1}\}) = (1, k-1)$, a contradiction.
**(4)** One vertex is in the inner cycle and the other one is in the outer cycle. Without loss of generality, we can suppose that one resolving vertex is $x_1$. Suppose that the second resolving vertex is $y_t$ $(1 \leq t \leq k+1)$. Then for $1 \leq t \leq k$, we have $r(y_n|\{x_1, y_t\}) = r(z_n|\{x_1, y_t\}) = (1, t+1)$ and if $t = k+1$, $r(x_2|\{x_1, y_t\}) = r(x_n|\{x_1, y_t\}) = (1, k)$, a contradiction.
**(5)** One vertex is in the inner cycle and the other one is in the set of pendent vertices. Without loss of generality, we can suppose that one resolving vertex is $x_1$. Suppose that the second resolving vertex is $z_t$ $(1 \leq t \leq k+1)$. Then for $i = 1$, we have $r(y_2|\{x_1, z_t\}) = r(y_n|\{x_1, z_t\}) = (2, 2)$, for $2 \leq t \leq k+1$, $r(x_2|\{x_1, z_t\}) = r(y_1|\{x_1, z_t\}) = (1, t+1)$, a contradiction.
**(6)** One vertex is in the outer cycle and the other one is in the set of pendent vertices. Without loss of generality, we can suppose that one resolving vertex is $y_1$. Suppose that the second resolving vertex is $z_t$ $(1 \leq t \leq k+1)$. Then for $1 \leq t \leq k$, we have $r(x_1|\{y_1, z_t\}) = r(y_n|\{y_1, z_t\}) = (1, t+1)$ and if $t = k+1$, $r(x_2|\{y_1, z_t\}) = r(x_n|\{y_1, z_t\}) = (2, t+1)$, a contradiction.
Hence, from above it follows that there is no resolving set with two vertices for $V(\mathbb{W}_n)$ implying that $dim(\mathbb{W}_n) = 3$ in this case.

□

## 4 The plane graph $\mathbb{A}_n$

The antiprism $A_n$ [1], $n \geq 3$, is a 4-regular graph and, for $n = 3$, it is the octahedron. Antiprism $A_n$, $n \geq 3$, consists of an outer $n$-cycle $y_1 y_2 .... y_n$, an inner $n$-cycle $x_1 x_2 .... x_n$, and a set of $2n$ spokes $x_i y_i$ and $x_{i+1} y_i$, $i = 1, 2, ..., n$ with indices taken modulo $n$. $|V(A_n)| = 2n$, $|E(A_n)| = 4n$ and $|F(A_n)| = 2n+2$. The plane graph $\mathbb{A}_n$ is obtained from the antiprism $A_n$ by adding a pendant edge at each vertex of the outer cycle of the antiprism $A_n$. We have

$$V(\mathbb{A}_n) = V(A_n) \cup \{z_i : 1 \leq i \leq n\}$$

and

$$E(\mathbb{A}_n) = E(A_n) \cup \{y_i z_i : 1 \leq i \leq n\}$$

The metric dimension of the antiprism $A_n$ has been determined in [10]. In the next theorem, we prove that the metric dimension of the plane graph $\mathbb{A}_n$ is the same as the metric dimension of the antiprism $A_n$. For our purpose, we call the cycle induced by $\{x_i : 1 \leq i \leq n\}$, the inner cycle, the cycle induced by $\{y_i : 1 \leq i \leq n\}$, the outer cycle and the set of vertices $\{z_i : 1 \leq i \leq n\}$, the pendent vertices. Once again the choice of an appropriate basis of vertices (also refereed to as landmarks in [14]) is crucial.

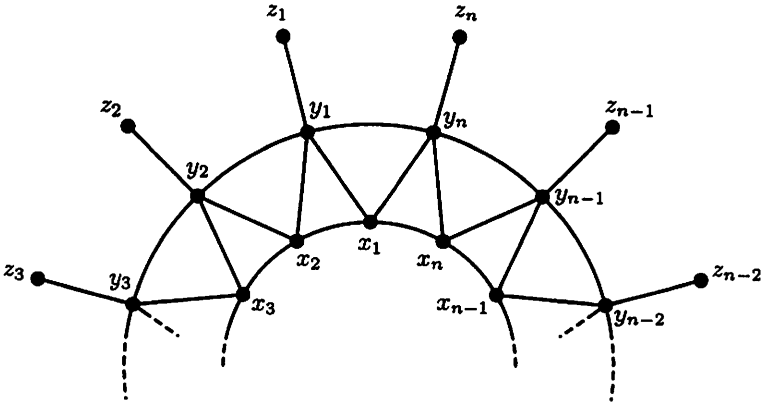


**Fig. 3.** The plane graph $\mathbb{A}_n$

**Theorem 4.** *Let $\mathbb{A}_n$ be the plane graph defined above; then $dim(\mathbb{A}_n) = 3$ for every $n \geq 6$.*

*Proof.* We will prove the above equality by double inequalities.
**Case 1.** When $n$ is even.
In this case, we can write $n = 2k$, $k \geq 3$, $k \in \mathbf{Z}^+$. Let $W = \{x_1, x_3, x_{k+1}\} \subset V(\mathbb{A}_n)$, we show that $W$ is a resolving set for $\mathbb{A}_n$ in this case. For this we give representations for any vertex of $V(\mathbb{A}_n)\backslash W$ with respect to $W$.
The representations for the vertices on inner cycle are

$$r(x_i|W) = \begin{cases} (1,1,k-1), & i = 2; \\ (i-1, i-3, k-i+1), & 4 \leq i \leq k; \\ (k-1, k-1, 1), & i = k+2; \\ (k-2, k, 2), & i = k+3; \\ (2k-i+1, 2k-i+3, i-k-1), & k+4 \leq i \leq 2k. \end{cases}$$

The representations for vertices on the outer cycle are

$$r(y_i|W) = \begin{cases} (1,2,k), & i = 1; \\ (2,1,k-1), & i = 2; \\ (i, i-2, k-i+1), & 3 \leq i \leq k; \\ (k, k-1, 1), & i = k+1; \\ (k-1, k, 2), & i = k+2; \\ (2k-i+1, 2k-i+3, i-k), & k+3 \leq i \leq 2k. \end{cases}$$

The representations for pendant vertices are

$$r(z_i|W) = \begin{cases} (2,3,k+1), & i=1; \\ (3,2,k), & i=2; \\ (i+1,i-1,k-i+2), & 3 \le i \le k; \\ (k+1,k,2), & i=k+1; \\ (k,k+1,3), & i=k+2; \\ (2k-i+2,2k-i+4,i-k+1), & k+3 \le i \le 2k. \end{cases}$$

We note that there are no two vertices having the same representations implying that $dim(\mathbb{A}_n) \le 3$.
On the other hand, we show that $dim(\mathbb{A}_n) \ge 3$. Suppose on contrary that $dim(\mathbb{A}_n) = 2$, then there are the following possibilities to be discussed.
**(1)** Both vertices are in the inner cycle. Without loss of generality, we can suppose that one resolving vertex is $x_1$. Suppose that the second resolving vertex is $x_t$ $(2 \le t \le k+1)$. Then for $2 \le t \le k$, we have $r(x_n|\{x_1, x_t\}) = r(y_n|\{x_1, x_t\}) = (1,t)$ and for $t = k+1$, we have $r(x_2|\{x_1, x_{k+1}\}) = r(x_n|\{x_1, x_{k+1}\}) = (1, k-1)$, a contradiction.
**(2)** Both vertices are in the outer cycle. Without loss of generality, we can suppose that one resolving vertex is $y_1$. Suppose that the second resolving vertex is $y_t$ $(2 \le t \le k+1)$. Then for $2 \le t \le k$, we have $r(x_1|\{y_1, y_t\}) = r(y_n|\{y_1, y_t\}) = (1,t)$ and for $t = k+1$, we have $r(y_2|\{y_1, y_{k+1}\}) = r(y_n|\{y_1, y_{k+1}\}) = (1, k-1)$, a contradiction.
**(3)** Both vertices are in set of pendant vertices. Without loss of generality, we can suppose that one resolving vertex is $z_1$. Suppose that the second resolving vertex is $z_t$ $(2 \le t \le k+1)$. Then for $2 \le t \le k$, we have $r(x_1|\{z_1, z_t\}) = r(y_n|\{z_1, z_t\}) = (2, t+1)$ and for $t = k+1$, we have $r(z_2|\{z_1, z_{k+1}\}) = r(z_n|\{z_1, z_{k+1}\}) = (2, k+1)$, a contradiction.
**(4)** One vertex is in the inner cycle and the other one is in the outer cycle. Without loss of generality, we can suppose that one resolving vertex is $x_1$. Suppose that the second resolving vertex is $y_t$ $(1 \le t \le k+1)$. Then for $1 \le t \le k$, we have $r(y_{n-1}|\{x_1, y_t\}) = r(z_n|\{x_1, y_t\}) = (2, t+1)$ and if $t = k+1$, $r(x_2|\{x_1, y_t\}) = r(y_n|\{x_1, y_t\}) = (1, k)$, a contradiction.
**(5)** One vertex is in the inner cycle and the other one is in the set of pendant vertices. Without loss of generality, we can suppose that one resolving vertex is $x_1$. Suppose that the second resolving vertex is $z_t$ $(1 \le t \le k+1)$. Then for $1 \le t \le k-1$, we have $r(y_{n-1}|\{x_1, z_t\}) = r(z_n|\{x_1, z_t\}) = (1, t+1)$, for $t = k$, $r(x_{n-1}|\{x_1, z_k\}) = r(y_{n-1}|\{x_1, z_k\}) = (2, k+1)$ and if $t = k+1$, $r(x_2|\{x_1, z_{k+1}\}) = r(y_1|\{x_1, z_{k+1}\}) = (1, k+1)$, a contradiction.
**(6)** One vertex is in the outer cycle and the other one is in the set of pendant vertices. Without loss of generality, we can suppose that one resolving vertex is $y_1$. Suppose that the second resolving vertex is $z_t$ $(1 \le t \le k+1)$. Then for $1 \le t \le k-1$, we have $r(x_1|\{y_1, z_t\}) = r(y_n|\{y_1, z_t\}) = (1, t+1)$, for $t = k$, $r(y_n|\{y_1, z_k\}) = r(z_1|\{y_1, z_k\}) = (1, k+1)$ and if $t = k+1$,

$r(z_2|\{y_1, z_{k+1}\}) = r(z_1|\{y_1, z_{k+1}\}) = (2, k+1)$, a contradiction.
Hence, from above it follows that there is no resolving set with two vertices for $V(\mathbb{A}_n)$ implying that $dim(\mathbb{A}_n) = 3$ in this case.

**Case** 2. When $n$ is odd.

In this case, we can write $n = 2k+1$, $k \geq 3$, $k \in \mathbf{Z}^+$. Let $W = \{x_1, x_3, x_{k+1}\} \subset V(\mathbb{A}_n)$, again we show that $W$ is a resolving set for $\mathbb{A}_n$ in this case also. For this we give representations for any vertex of $V(\mathbb{A}_n)\backslash W$ with respect to $W$.

The representations of vertices on the inner cycle are

$$r(x_i|W) = \begin{cases} (1,1,k-1), & i = 2; \\ (i-1, i-3, k-i+1), & 4 \leq i \leq k; \\ (k, k-1, 1), & i = k+2; \\ (k-1, k, 2), & i = k+3; \\ (2k-i+2, 2k-i+4, i-k-1), & k+4 \leq i \leq 2k+1. \end{cases}$$

The representations for vertices on the outer cycle are

$$r(y_i|W) = \begin{cases} (1,2,k), & i = 1; \\ (2,1,k-1), & i = 2; \\ (i, i-2, k-i+1), & 3 \leq i \leq k; \\ (k+1, k-1, 1), & i = k+1; \\ (k,k,2), & i = k+2; \\ (2k-i+2, 2k-i+4, i-k), & k+3 \leq i \leq 2k+1. \end{cases}$$

The representations for pendant vertices are

$$r(z_i|W) = \begin{cases} (2,3,k+1), & i = 1; \\ (3,2,k), & i = 2; \\ (i+1, i-1, k-i+2), & 3 \leq i \leq k; \\ (k+2, k, 2), & i = k+1; \\ (k+1, k+1, 3), & i = k+2; \\ (2k-i+3, 2k-i+5, i-k+1), & k+3 \leq i \leq 2k+1. \end{cases}$$

Again we see that there are no two vertices having the same representations which implies that $dim(\mathbb{A}_n) \leq 3$.

On the other hand, suppose that $dim(\mathbb{A}_n) = 2$, then there are the same possibilities as in case (i) and contradictions can be deduced analogously. This implies that $dim(\mathbb{A}_n) = 3$ in this case, which completes the proof.

□

In the next section, we prove that the metric dimension of a cycle can be affected if we attach an edge terminating in a vertex of degree 1 to each vertex of the cycle.

## 5 The sun graph $S_n$

A sun $S_n$ is a graph with a cycle $C_n$ having an edge terminating in a vertex of degree 1 attached to each vertex of the cycle. The sun $S_n$ consists of the vertex set $V(S_n) = \{x_i : 1 \leq i \leq n\} \cup \{y_i : 1 \leq i \leq n\}$ and edge set $E(S_n) = \{x_i x_{i+1} : 1 \leq i \leq n\} \cup \{x_i y_i : 1 \leq i \leq n\}$, where $i+1$ is taken modulo $n$.

For our purpose, we call the cycle induced by $\{x_i : 1 \leq i \leq n\}$, the cycle

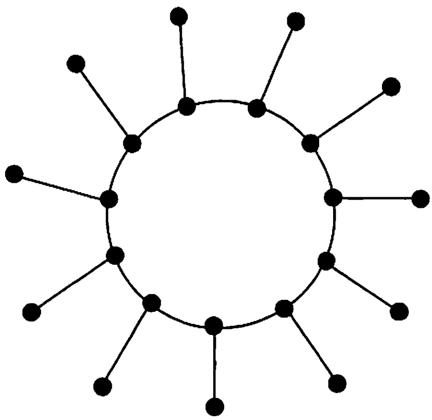

**Fig. 4.** The sun $S_{11}$

and set of vertices $\{y_i : 1 \leq i \leq n\}$, the outer vertices. Once again the choice of an appropriate basis of vertices (also refereed to as landmarks in [14]) is important.

**Theorem 5.** *Let $S_n$ denote the sun graph; then*

$$dim(S_n) = \begin{cases} 2, & \text{when } n \text{ is odd;} \\ 3, & \text{otherwise.} \end{cases}$$

*Proof.* We will prove the above equality by double inequalities.

**Case 1.** When $n$ is odd.

In this case, we can write $n = 2k+1$, $k \geq 2$, $k \in \mathbf{Z}^+$. Let $W = \{y_1, y_k\} \subset V(S_n)$, we show that $W$ is a resolving set for $S_n$ in this case. For this we give representations for any vertex of $V(S_n) \backslash W$ with respect to $W$.

Representations for the vertices on inner cycle are

$$r(x_i|W) = \begin{cases} (i, k-i+1), & 1 \leq i \leq k; \\ (k+1, 2), & i = k+1; \\ (2k-i+3, i-k+1), & k+2 \leq i \leq 2k; \\ (2, k+1), & i = 2k+1. \end{cases}$$

Representations for vertices of the outer vertices are

$$r(y_i|W) = \begin{cases} (i+1, k-i+2), & 2 \leq i \leq k-1; \\ (k+2, 3), & i = k+1; \\ (2k-i+4, i-k+2), & k+2 \leq i \leq 2k; \\ (3, k+2), & i = 2k+1. \end{cases}$$

We note that there are no two vertices having the same representations implying that $dim(S_n) \leq 2$. On the other hand, since the sun graph $S_n$ is not a path, it follows from [5] that $dim(S_n) = 2$ in this case.

**Case** 2. When $n$ is even.

In this case, we can write $n = 2k$, $k \geq 3$, $k \in \mathbf{Z}^+$. Let $W = \{x_1, x_2, x_{k+1}\} \subset V(S_n)$, we show that $W$ is a resolving set for $S_n$ in this case. For this we give representations for any vertex of $V(S_n)\backslash W$ with respect to $W$.

Representations for the vertices on inner cycle are

$$r(x_i|W) = \begin{cases} (i-1, i-2, k-i+1), & 3 \leq i \leq k; \\ (2k-i+1, 2k-i+2, i-k-1), & k+2 \leq i \leq 2k. \end{cases}$$

Representations for vertices of the outer vertices are

$$r(y_i|W) = \begin{cases} (1, 2.k+1), & i = 1; \\ (i, i-1, k-i+2), & 2 \leq i \leq k+1; \\ (2k-i+2, 2k-i+3, i-k), & k+2 \leq i \leq 2k. \end{cases}$$

We note that there are no two vertices having the same representations implying that $dim(S_n) \leq 3$.

On the other hand, we show that $dim(S_n) \geq 3$ by proving that there is no resolving set $W$ such that $|W| = 2$. Suppose on contrary that $dim(S_n) = 2$, then there are the following possibilities to be discussed.

**(1)** Both vertices are in the cycle. Without loss of generality, we can suppose that one resolving vertex is $x_1$. Suppose that the second resolving vertex is $x_t$ $(2 \leq t \leq k+1)$. Then for $2 \leq t \leq k$, we have $r(x_n|\{x_1, x_t\}) = r(y_1|\{x_1, x_t\})$, and for $t = k+1$, we have $r(x_2|\{x_1, x_{k+1}\}) = r(x_n|\{x_1, x_{k+1}\})$, a contradiction.

**(2)** Both vertices belong to the set of outer vertices. Without loss of generality, we can suppose that one resolving vertex is $y_1$. Suppose that the second resolving vertex is $y_t$ $(2 \leq t \leq k+1)$. Then for $1 \leq t \leq k-1$, we have $r(x_{n-1}|\{y_1, y_t\}) = r(y_n|\{y_1, y_t\})$, for $i = k$, $r(x_{n-1}|\{y_1, y_k\}) = r(y_2|\{y_1, y_k\})$ and if $t = k+1$ then $r(x_2|\{y_1, y_{k+1}\}) = r(x_n|\{y_1, y_{k+1}\})$, a contradiction.

**(3)** One vertex is in the cycle and the other one is in the set of outer vertices. Without loss of generality, we can suppose that one resolving vertex is $x_1$. Suppose that the second resolving vertex is $y_t$ $(1 \leq t \leq k+1)$. Then

for $2 \leq t \leq k$, we have $r(x_n|\{x_1, y_t\}) = r(y_1|\{x_1, y_t\})$ and if $t = k+1$, $r(y_2|\{x_1, y_t\}) = r(y_n|\{x_1, y_t\})$, a contradiction.
Hence, from above it follows that there is no resolving set with two vertices for $V(S_n)$ implying that $dim(S_n) = 3$ in this case.

□

# 6 Concluding remarks

In this paper, we have studied the metric dimension of some classes of rotationally-symmetric plane graphs. We prove that the metric dimension of these classes of rotationally-symmetric plane graphs is finite and does not depend upon the number of vertices in these graphs. It is natural to ask for the characterization of rotationally-symmetric plane graphs with constant metric dimension. We have also seen that for prism and antiprism, the metric dimension is not affected if we attach a pendent edge at each vertex of the outer cycle of these graphs. It can be proved in fact that for these graphs if we attach a path $P_t (t \geq 1)$ at each vertex of the outer cycle of these graphs, the metric dimension will not be affected. We also proved that the metric dimension of a cycle is affected if we attach a pendent edge at each vertex of cycle, when the order of the graph is even.
Note that in [17] Melter and Tomescu gave an example of infinite regular graphs (namely the digital plane endowed with city-block and chessboard distances, respectively) having no finite metric basis. We close this section by raising some questions that naturally arise from the text.
**Open Problem 1:** *Characterize the families of rotationally-symmetric graphs $G'$ obtained from rotationally-symmetric graphs $G$ by adding a pendant edge at each vertex of the outer cycle of $G$ such that $dim(G') = dim(G)$.*
**Open Problem 2:** *Characterize the families of rotationally-symmetric graphs $G'$ obtained from rotationally-symmetric graphs $G$ by adding a pendant edge at each vertex of outer cycle of $G$ such that $dim(G') = 1+dim(G)$.*
**Open Problem 3:** *Is it the case that the subdivision graph of every convex polytope (having constant metric dimension) will have constant metric dimension?*

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