SOME RESULTS ON THE EXTERIOR DEGREE OF EXTRA-SPECIAL GROUPS

PEYMAN NIROOMAND

ABSTRACT. The concept of exterior degree of a finite group G is introduced by the author in a joint paper [13] which is the probability of randomly two elements g and h in G such that $g \wedge h = 1$. In the present paper, a necessary and sufficient condition for a non cyclic group is given when its exterior degree achieves the upper bound $(p^2+p-1)/p^3$ in which p is the smallest prime number dividing the order of G. We also compute the exterior degree of all extra-special p-groups. Finally, for an extra-special p-group H and a group G when $G/Z^\wedge(G)$ is p-group, we will show that $d^\wedge(G) = d^\wedge(H)$ if and only if $G/Z^\wedge(G) \cong H/Z^\wedge(H)$ provided that $d^\wedge(G) \neq 11/32$.

1. Introduction and some known results

Commutativity degree. For a finite group G the commutativity degree $d(G)$ of G is defined as the ratio

$$d(G) = \frac{|\{(x,y) \in G \times G \mid xyx^{-1}y^{-1} = 1\}|}{|G|^2}.$$

Clearly, abelian groups are characterized by the property $d(G) = 1$. By one of the known results $d(G)$ is bounded by $5/8$ for any finite non abelian group G, and it achieves the bound if and only if $G/Z(G)$ is isomorphic to the elementary abelian 2-group of order 4.

The table I in [13] shows that groups with the same commutativity degree do not need to be isomorphic. However, in [8, Lemma 2.4] is shown that two groups have the same commutativity degree when they are isoclinic. Recall that the notion of isoclinic was introduced by P. Hall in [11], which is a more general equivalence relation in the class of all groups. There is a wide literature about the commutativity degree of a group. To get some more details about this concept the references [7, 8, 9, 10] could be useful.

2010 *Mathematics Subject Classification.* Primary 20F99; Secondary 20P05.

Key words and phrases. Exterior degree; commutativity degree; exterior centre; capable group; Schur multiplier; extra-special group.

Exterior degree. The exterior square $G \wedge G$ of G is a group generated by the symbols $g \wedge h$ subject to the following relations

$$gg' \wedge h = ({}^{g}g' \wedge {}^{g}h)(g \wedge h), \; g \wedge hh' = (g \wedge h)({}^{h}g \wedge {}^{h}h') \text{ and } g \wedge g,$$

for all $g, g', h, h' \in G$ where ${}^{g}g' = gg'g^{-1}$ (see [4] for more details). The existence of an epimorphism $\kappa : G \wedge G \to G'$ (sending $g \wedge h$ to $ghg^{-1}h^{-1}$) follows from the defining relation of $G \wedge G$. We know from [4] that the kernel of κ is isomorphic to the Schur multiplier $\mathcal{M}(G)$ of G.

The exterior degree has been modeled as the following ratio

$$d^{\wedge}(G) = |\{(x, y) \in G \times G \mid g \wedge h = 1_{\wedge}\}|/|G|^2$$

by the author in a joint paper [13]. Obviously, $d^{\wedge}(G) \leq d(G)$ and when the equality holds G is called *unidegree*. From the definition of κ, a group is unidegree if its Schur multiplier is trivial. Since every unidegree group is unicentral by [13, Corollary 2.5], $d^{\wedge}(G) < d(G)$ for all non unicentral groups. Recall from [13, Corollary 2.4 (iii)] that the cyclic groups are characterized by $d^{\wedge}(G) = 1$ as well as $d(G) = 1$ characterized abelian groups.

At least we may conclude from [13, Theorems 2.3, 2.8 and Corollary 2.5] that knowing the mount of exterior degree not only gives a lower bound for the commutativity degree but also helps us to illustrate when a group G is capable or unicentral. Similar to the commutativity degree of groups, Proposition 1.2 emphasizes that the groups with the same exterior degree do not need to be isomorphic (see also Example 2). Moreover, Corollary 1.1 shows the exterior degree of non cyclic group is bounded above by $(p^2 + p - 1)/p^3$ where p is the smallest prime number dividing the order of G.

In the present manuscript, we give a necessary and sufficient condition when the exterior degree of a non abelian group achieves the upper bound.

As we mentioned about the commutativity degree, two isoclinic groups G and H have the same commutativity degree but not vice versa. Analogously, two groups have the same exterior degree if $G/Z^{\wedge}(G) \cong H/Z^{\wedge}(H)$ by Proposition 1.2. Of course, the converse is not true in general. It seems $Z^{\wedge}(G)$ plays the same role relative to $d^{\wedge}(G)$ as $Z(G)$ plays relative to $d(G)$ (see [8]). Here, in the case H is an extra-special and $G/Z^{\wedge}(G)$ is a p-group, we will show that $d^{\wedge}(G) = d^{\wedge}(H)$ implies that $G/Z^{\wedge}(G) \cong H/Z^{\wedge}(H)$ provided that $d^{\wedge}(G) \neq 11/32$. To do this we need to compute the exterior degree of all extra-special p-groups.

Recall that [13] describes the concept of exterior centralizer of an element x of G which is equal to $C_G^{\wedge}(g) = \{a \in G \mid a \wedge h = 1_{\wedge}\}$. The exterior center, the intersection of exterior centralizer of all elements G, is denoted by $Z^{\wedge}(G)$. It follows from [5, Proposition 16 (i)] that the exterior centre is a subgroup of $Z(G)$ which allows us to decide whether G is a capable group. More precisely, G is capable if and only if $Z^{\wedge}(G) = 1$. Furthermore, it is known that $Z^{\wedge}(G)$ is the smallest subgroup of G such that $G/Z^{\wedge}(G)$ is capable.

Throughout this paper we assume that G is a finite group, D_{2n} and $E_1 = \langle a, b, c \mid a^p = b^p = c^p = 1, [a, c] = [b, c] = 1, [a, b] = c\rangle$ denote the dihedral group of order $2n$ and the extra-special p-group of order p^3 and exponent p. Finally, $C_p^{(n)}$ denotes the direct product of n-copies of the cyclic group of order p.

Known results. We summarize some known results without proof which will be used throughout this paper without any further references.

Corollary 1.1. (*See* [13, Corollary 2.4(i)]) *Let p be the smallest prime number dividing the order of a non cyclic group G. Then*

$$d^{\wedge}(G) \leq (p^2 + p - 1)/p^3.$$

Proposition 1.2. (*See* [13, Proposition 2.6]) *Let N be a normal subgroup of a finite group G. Then*

$$d^{\wedge}(G) \leq d^{\wedge}(\frac{G}{N}),$$

and the equality holds if N is contained in $Z^{\wedge}(G)$.

Lemma 1.3. (*See* [13, Examples 3.1 and 3.3]) *For all $n \geq 2$, we have*

(i) $d^{\wedge}(D_{2n}) = \dfrac{n+3}{4n}$,

(ii) $d^{\wedge}(C_p^{(n)}) = \dfrac{p^n + p^{n-1} - 1}{p^{2n-1}}$.

2. Main results

Corollary 1.1 shows that a non cyclic group has

$$d^{\wedge}(G) \leq (p^2 + p - 1)/p^3,$$

where p is the smallest prime number dividing the order of G. The following lemma gives a necessary and sufficient condition when $d^{\wedge}(G)$ is attained the upper bound. In particular, when $p = 2$ the analogues result was obtained for the commutativity degree, see [6, Theorem 2.2].

Lemma 2.1. *Let G be a group and p be the smallest prime divisor of the order of G. Then*

$$d^{\wedge}(G) = \frac{p^2 + p - 1}{p^3} \text{ if and only if } \frac{G}{Z^{\wedge}(G)} \cong C_p \times C_p.$$

Proof. First assume that $\frac{G}{Z^{\wedge}(G)} \cong C_p \times C_p$. Proposition 1.2 follows that $d^{\wedge}(G) = d^{\wedge}(\frac{G}{Z^{\wedge}(G)})$ and hence $d^{\wedge}(G) = \frac{p^2+p-1}{p^3}$ due to Lemma 1.3.

Conversely, since for all $g \notin Z^{\wedge}(G)$ we have $[G : C_G^{\wedge}(g)] \geq p$. We should have

$$\begin{aligned}
\frac{p^2+p-1}{p^3} &= d^{\wedge}(G) = \frac{1}{|G|^2}\sum_{g\in G}|C_G^{\wedge}(g)| \\
&\leq \frac{1}{|G|^2}(|Z^{\wedge}(G)||G| + (|G| - |Z^{\wedge}(G)|)\frac{|G|}{p}) \\
&= \frac{1}{p} + \frac{|Z^{\wedge}(G)|}{|G|}(1 - \frac{1}{p}),
\end{aligned}$$

thus $|G|/|Z^{\wedge}(G)| \leq p^2$. In the case that G is non abelian, the results holds. Otherwise since G is not cyclic ($d^{\wedge}(G) \neq 1$), [5, Proposition 18 (ii)] implies that $\frac{G}{Z^{\wedge}(G)} \cong C_p \times C_p$. □

In the following lemma we compute the exterior degree of all extra-special p-groups. We note that the exterior degree of non capable non abelian group achieves the upper bound.

Theorem 2.2. *Let G be an extra-special p-group. Then the exterior degree of G is equal to one of the following cases.*

(i). $d^{\wedge}(E_1) = \frac{p^3+p^2-1}{p^5}$ and $d^{\wedge}(D_8) = \frac{7}{16}$.

(ii). $d^{\wedge}(G) = \frac{p^{2n}+p^{2n-1}-1}{p^{4n-1}}$, when G is non capable of order p^{2n+1};

Proof. (*i*). First assume that $G \cong E_1$. By a consequence of [1, Theorem 4.3], we should have

$$E_1 \wedge E_1 \cong \langle a \wedge b\rangle \times \langle a \wedge c\rangle \times \langle b \wedge c\rangle,$$

which is an elementary abelian of order p^3. It can be easily seen that the order of all centralizer of non central elements of E_1 is equal to p^2. Let $x \in \{a, b, c\}$, then the exterior centralizer of x is equal to $\langle x\rangle$. Since for instance let $x = a$ and $g = a^m b^n c^l$ be an arbitrary element of E_1, by using [1, Proposition 3.5] we should have

$$g \wedge a = (a \wedge b)^{-n}(a \wedge c)^{-l}(b \wedge c)^{\frac{1}{2}n(n-1)}.$$

Hence $g = a^m$, which means that $C_G^{\wedge}(a) = \langle a\rangle$. By a similar way, it can be shown that $C_G^{\wedge}(b) = \langle b\rangle$ and $C_G^{\wedge}(c) = \langle c\rangle$.

We claim that the order of all exterior centralizer of non identity elements of E_1 is equal to p, and so

$$\begin{aligned} d^{\wedge}(E_1) &= \frac{1}{|E_1|^2}|\{(g,h) \in E_1 \times E_1 : g \wedge h = 1\}| \\ &= \frac{1}{|E_1|^2}\sum_{g\in E_1}|C^{\wedge}_{E_1}(g)| = \frac{p^3+(p^3-1)p}{p^6} = \frac{p^3+p^2-1}{p^5}. \end{aligned}$$

By contrary, suppose that $|C^{\wedge}_G(g)| \neq p$ for a given non trivial element $g = a^m b^n c^l$. Since $Z^{\wedge}(G)$ is trivial, we should have $|C^{\wedge}_G(g)| = p^2$ and hence $Z(E_1) \subseteq C^{\wedge}_G(g)$. Thus $c \in C^{\wedge}_G(g)$ hence $c \wedge a^m b^n c^l = (a \wedge c)^m (b \wedge c)^n = 1$, and so $g = c^l$ which is a contradiction.

The exterior degree of D_8 is obtained directly by using Lemma 1.3.

(*ii*). Since $G/Z^{\wedge}(G)$ is elementary abelian of order p^{2n}, by invoking Lemma 1.3 we have $d^{\wedge}(G) = \dfrac{p^{2n}+p^{2n-1}-1}{p^{4n-1}}$. □

For the group of order p^3, the following lemma emphasizes that the groups can be classified by the mentioned amount of $d^{\wedge}(G)$ in Theorem 2.2.

Lemma 2.3. *Let G be group of order p^3. Then*

(i) $d^{\wedge}(G) = \dfrac{p^3+p^2-1}{p^5}$ *if and only if* $G \cong C_p^{(3)}$ *or* E_1;

(ii) $d^{\wedge}(G) = \dfrac{7}{16}$ *if and only if* $G \cong D_8$.

(iii) $d^{\wedge}(G) = \dfrac{p^2+p-1}{p^3}$ *if and only if G is not capable.*

Proof. (*i*) Assume that $G \cong C_p^{(3)}$, Lemma 1.3 follows that $d^{\wedge}(G) = \dfrac{p^3+p^2-1}{p^5}$. In the case $G \cong E_1$, Theorem 2.2 (*i*) follows the result.

Conversely, suppose that there exists an element g of G such that $|C^{\wedge}_G(g)| \geq p^2$, hence we have

$$\begin{aligned} \frac{p^3+p^2-1}{p^5} &= d^{\wedge}(G) \\ &= \frac{1}{|G|^2}\sum_{x\in G}|C^{\wedge}_G(g)| = \frac{1}{|G|^2}\Big(\sum_{g\in Z^{\wedge}(G)}|C^{\wedge}_G(g)| + \sum_{g\notin Z^{\wedge}(G)}|C^{\wedge}_G(g)|\Big) \\ &\geq \frac{1}{p^6}\big(|G||Z^{\wedge}(G)| + (|G| - |Z^{\wedge}(G)| - 1)p + p^2\big), \end{aligned}$$

and so $(p+1)|Z^{\wedge}(G)| \leq p$ which is a contradiction. Hence for all non identity element $g \in G$, we should have $|C^{\wedge}_G(g)| = p$. By a similar technique, one may see that $|Z^{\wedge}(G)| = 1$. Thus G is capable and $G \cong C_p^{(3)}$ or $G \cong E_1$ by using [2], [3, Corollary 8.2] and the assumption.

(*ii*) It follows from Theorem 2.2 (*i*) that $d^{\wedge}(D_8) = \frac{7}{16}$. Conversely, if G is non capable, then Lemma 2.1 implies that $d^{\wedge}(G) = \frac{p^2+p-1}{p^3}$ which is a contradiction. Hence [3, Corollary 8.2], [2] and the pervious part imply that $G \cong D_8$.

(*iii*) It is obtained by a similar argument in (*i*) and (*ii*). □

The following results generalize Lemma 2.3.

Theorem 2.4. *Let G be a non abelian group such that $[G : Z^{\wedge}(G)] = p^n$ $(p \neq 2)$. Then*

(i) $d^{\wedge}(G) = \frac{p^n + p^{n-1} - 1}{p^{2n-1}}$ $(n \neq 3)$ *if and only if* $\frac{G}{Z^{\wedge}(G)} \cong \frac{H}{Z^{\wedge}(H)}$, *where H is a non capable extra-special p-group of order p^{2n+1};*

(ii) $d^{\wedge}(G) = \frac{p^3+p^2-1}{p^5}$ *if and only if* $\frac{G}{Z^{\wedge}(G)} \cong H$, *where H is the capable extra-special p-group of order p^3.*

Proof. (*i*) We shall prove the result by using induction on the order of G. For $|G| = p^3$, we have $n = 2$ and so $G/Z^{\wedge}(G) \cong C_p \times C_p$. Thus Lemmas 2.1 and 2.3 imply that $d^{\wedge}(G) = \frac{p^2+p-1}{p^3}$ if and only if $\frac{G}{Z^{\wedge}(G)} \cong C_p \times C_p$. Hence $\frac{G}{Z^{\wedge}(G)} \cong \frac{H}{Z^{\wedge}(H)}$, where H is an non capable extra-special p-group of order p^3.

Now assume that $|G| > p^3$ and let $d^{\wedge}(G) = \frac{p^n + p^{n-1} - 1}{p^{2n-1}}$ $(n \neq 3)$. If G is not capable, then $|G/Z^{\wedge}(G)| < |G|$ and also $[G/Z^{\wedge}(G) : Z^{\wedge}(G/Z^{\wedge}(G))] = [G : Z^{\wedge}(G)] = p^n$. On the other hand, $d^{\wedge}(G) = d^{\wedge}(G/Z^{\wedge}(G))$ due to Proposition 1.2. Hence by induction hypothesis $\frac{G/Z^{\wedge}(G)}{Z^{\wedge}(G/Z^{\wedge}(G))} \cong \frac{H}{Z^{\wedge}(H)}$, where H is non capable extra-special p-group. Thus the result follows since $Z^{\wedge}(G/Z^{\wedge}(G)) = 1$. When G is capable, since $[G : Z^{\wedge}(G)] = p^n$, G is p-group and by using [3, Corollary 8.2], [2] and assumption $G \cong E_1$, and so $d^{\wedge}(G) = \frac{p^3+p^2-1}{p^5}$ due to Lemma 2.3 (*i*) which is a contradiction. The converse is obtained by using Lemma 1.3, Proposition 1.2 and Lemma 2.3.

(*ii*) Suppose that $d^{\wedge}(G) = \frac{p^3+p^2-1}{p^5}$, the result follows when G is capable. In the case G is non capable $G/Z^{\wedge}(G)$ satisfies the induction hypothesis and so the result follows. The converse is obtained directly by using Lemma 2.3 (i), as required. □

Corollary 2.5. *Let G be a non abelian group such that $[G : Z^{\wedge}(G)] = p^n$ $(p \neq 2)$. If $d^{\wedge}(G) = \dfrac{p^n + p^{n-1} - 1}{p^{2n-1}}$ $(n \neq 3)$. Then G is non capable.*

Theorem 2.6. *Let G be a non abelian group such that $[G : Z^{\wedge}(G)] = 2^n$ $(n \geq 3)$. Then*

$$d^{\wedge}(G) = \frac{7}{16} \text{ if and only if } \frac{G}{Z^{\wedge}(G)} \cong D_8.$$

Proof. Let $|G| = p^3$, since $d^{\wedge}(G) = d^{\wedge}(G/Z^{\wedge}(G))$ according to Proposition 1.2, the result is obtained by using Lemma 2.3 (ii). Otherwise, it is obtained by invoking induction and Lemma 2.3 (ii). □

Some examples. The following examples show that there exists a group which is not a p-group satisfying in Theorem 2.6 (see [13, Example 2.4]).

Example 2.7. Let $G = C_3 \times D_8$. Then $d^{\wedge}(G) = 7/16$ and $Z^{\wedge}(G) = C_3$.

The next two examples emphasize that $d^{\wedge}(G) = d^{\wedge}(H)$ does not imply that $G/Z^{\wedge}(G) \cong H/Z^{\wedge}(H)$ in general (see Theorem 2.4 (ii)).

Example 2.8. It is known that $d^{\wedge}(D_{16}) = d^{\wedge}(C_2 \times Q_2) = 11/32$ by [13, Examples 2.4]. On the other hand, by using the same reference $Z^{\wedge}(D_{16}) = 1$ and $Z^{\wedge}(C_2 \times Q_2) = C_2$.

Example 2.9. Table I in [13] shows that $d^{\wedge}(C_2 \times D_8) = 1/4$ and $Z^{\wedge}(C_2 \times D_8) = 1$. On the other hand, if $G \cong \langle a, b \mid a^4 = b^6 = (ab)^2 = (a^{-1}b)^2 = 1\rangle$, then $d^{\wedge}(G) = 1/4$ and $Z^{\wedge}(G) = 1$.

References

[1] M. Bacon and L.-C. Kappe, The nonabelian tensor square of a 2-generator p-group of class 2, Arch. Math.(Basel). **61** (1993), 508–516.

[2] R. Baer, Groups with preassigned central and central quotient group, Trans. Amer. Math.Soc. **44** (1938), 387-412.

[3] F.R. Beyl, U. Felgner and P. Schmid, On groups occurring as center factor groups, J. Algebra **61**(1979), 161-177.

[4] R. Brown and J.-L. Loday, Van Kampen theorems for diagrams of spaces, Topology **26** (1987), 311-335.

[5] G. Ellis, Tensor products and q-crossed modules, J. London Math. Soc. **2** (1995), 241–258.

[6] A. Erfanian, R. Rezaei and P. Lescot, On the relative commutativity degree of a subgroup of a finite groups, Comm. Algebra **35** (2007), 4183-4197.

[7] W.H. Gustafson, What is the probability that two groups elements commute? Amer. Math. Monthly **80** (1973), 1031-1304.

[8] P. Lescot, Isoclinism classes and commutativity degrees of finite groups, J. Algebra **177** (1995), 847-869.
[9] P. Lescot, Central extensions and commutativity degree, Comm. Algebra 29 (**10**) (2001), 4451-4460.
[10] G.A. Miller, Relative number of noninvariant operators in a group, Proc. Nat. Acad. Sci. USA. 30 (**2**) (1944), 25-28 .
[11] P. Hall, The classification of prime-power groups, J. Reine Ang. Math. 182 (1940), 130141.
[12] P. Niroomand and F. Russo, A note on the exterior centralizer, Arch. Math. (Basel). 93(**6**) (2009), 505-512.
[13] P. Niroomand, and R. Rezaei, On the exterior degree of finite groups, Comm. Algebra 39 (**1**) (2011), 335-343.

SCHOOL OF MATHEMATICS AND COMPUTER SCIENCE, DAMGHAN UNIVERSITY, DAMGHAN, IRAN
E-mail address: `niroomand@du.ac.ir`, `p_niroomand@yahoo.com`