Sufficient Conditions for Burst Error Identification and Correction in LRTJ-Spaces

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Abstract. In [8], the author introduced the notion of burst errors for 2-dimensional array coding systems. Also, in [10], the author introduced a series of metrics called Lee-RT-Jain-Metric(LRTJ-metric) [3] for array codes which is a generalization of both classical Lee metric [12] and array RT metric [14]. In this paper, we obtain sufficient conditions on the parameters of array codes equipped with LRTJ-metric for the identification and correction of burst array errors.

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1. Introduction

In a classical coding setting, codes are subsets/subspaces of ambient space F_q^n, the space of all n-tuples with entries from a finite field F_q, and are investigated with respect to the Hamming metric [13] and Lee metric [12]. In [14], array codes which are subsets/subspaces of the linear space of all m by s matrices $\mathrm{Mat}_{m\times s}(F_q)$ with entries from a finite field F_q endowed with a generalized Hamming metric known as RT-metric (or m-metric) were introduced. Motivated by the idea to have a generalized Lee metric for array code, the author introduced a new series of metrics on the space $\mathrm{Mat}_{m\times s}(\mathbb{Z}_q)$ which is a generalization of both Lee metric for classical coding and RT-metric for array coding and named this metric as Lee-RT-Jain-metric (LRTJ-metric).

Here is a model of an information transmission for which array coding is useful. Suppose that a sender transmits messages, each being an s-tuple of m-tuples of q-ary symbols, transmitted over m parallel channels. There is an interfering noise in the channels which create errors in the transmitted message. An important and practical situation is when errors are not

*Address for Communication: 32, Uttranchal 5, I.P. Extension, Delhi 110092, India scattered randomly in the code matrix (or code array) but are in cluster form and are confined to a submatrix (or subarray) part of the code array. Motivated by this idea, the author introduced the notion of burst errors in array coding [8] and obtained some lower and construction upper bounds [8, 9] on the parameters of m-metric array codes for the identification and correction of burst errors. However, the choice of a metric for a given parallel communication system plays an important role as the channel model should match the metric d to be employed for developing a suitable array code, and hence for a communication system to operate reliably. Thus, given a modulation scheme, one metric may be better than another. The LRTJ-metric is useful over non-binary communication channels than RT-metric as this metric takes into account magnitude of change rather than only position of change. The author has already obtained lower bounds for burst error correction in LRTJ-metric array codes [11]. In this paper, we obtain construction upper bounds or equivalently sufficient conditions for the burst error identification and correction in LRTJ-metric array codes.

2. Definitions and Notations

Let $\mathbf{Z}_q$ be the ring of integers modulo q. Let $Mat_{m\times s}(\mathbf{Z}_q)$ be the set of all $m \times s$ matrices with entries from $\mathbf{Z}_q$. Then $Mat_{m\times s}(\mathbf{Z}_q)$ is a module over $\mathbf{Z}_q$. Let V be a $\mathbf{Z}_q$-submodule of the module $Mat_{m\times s}(\mathbf{Z}_q)$. Then V is called an array code (In fact, linear array code). For q prime, $\mathbf{Z}_q$ becomes a field and correspondingly $Mat_{m\times s}(\mathbf{Z}_q)$ and V become the vector space and a sub space respectively over the field $\mathbf{Z}_q$. We note that the space $Mat_{m\times s}(\mathbf{Z}_q)$ is identifiable with the space $\mathbf{Z}_q^{ms}$. Every matrix in $Mat_{m\times s}(\mathbf{Z}_q)$ can be represented as an $1\times ms$ vector by writing the first row of matrix followed by second row and so on. Similarly, every vector in $\mathbf{Z}_q^{ms}$ can be represented as an $m \times s$ matrix in $Mat_{m\times s}(\mathbf{Z}_q)$ by separating the co-ordinates of the vector into m groups of s-coordinates. Also, we define the modular value $|a|$ of an element $a \in \mathbf{Z}_q$ by

$$|a| = \begin{cases} a & \text{if} \quad 0 \leq a \leq q/2 \\ q-a & \text{if} \quad q/2 < a \leq q-1. \end{cases}$$

We note that the non-zero modular value $|a|$ can be obtained by two different elements a and $q-a$ of $\mathbf{Z}_q$ provided $\{q$ is odd$\}$ or $\{q$ is even and $a \neq [q/2]\}$, i.e.

$$|a| = |q-a| \quad \text{if} \quad \begin{cases} q \quad \text{is odd} \\ \text{or} \\ q \text{ is even and} \quad a \neq q/2. \end{cases}$$

If q is even and $a = [q/2]$ or if $a = 0$, then $|a|$ is obtained in only one way viz., $|a| = a$.

Thus, there may be one or two equivalent values of $|a|$ which we shall refer to as repetitive equivalent values of a. The number of repetitive equivalent values of a will be denoted by e_a, where

$$e_a = \begin{cases} 1 & \text{if } \{\, q \text{ is even and } a = q/2\} \text{ or } \{a = 0\} \\ 2 & \text{if } \{\, q \text{ is odd and } a \neq 0\} \text{ or } \{q \text{ is even, } a \neq 0 \text{ and } a \neq q/2\}. \end{cases}$$

We now define the LRTJ-metric in the space $Mat_{m\times s}(\mathbf{Z}_q)$ as follows [10] :

Let $Y \in Mat_{1\times s}(\mathbf{Z}_q)$ with $Y = (y_1, y_2, \cdots, y_s)$.

Define the row-weight of Y as

$$wt_\rho(Y) = \begin{cases} \max_{j=1}^{s}|y_j| + \max_{j=1}^{s}\{j-1 \mid y_j \neq 0\} & \text{if } Y \neq 0 \\ 0 & \text{if } Y = 0. \end{cases}$$

Then $0 \leq wt_\rho(Y) \leq [q/2]+s-1$. Extending the definition of the row-weight to the class of all $m \times s$ matrices as

$$wt\rho(A) = \sum_{i=1}^{m} wt\rho(R_i)$$

where $A = \begin{bmatrix} R_1 \\ R_2 \\ \cdots \\ R_m \end{bmatrix} \in Mat_{m\times s}(\mathbf{Z}_q)$ and R_i denotes the i^{th} row of A.

Then wt_ρ satisfies $0 \leq wt_\rho(A) \leq m([q/2]+s-1)\ \forall\ A \in Mat_{m\times s}(\mathbf{Z}_q)$ and determines a metric on $Mat_{m\times s}(\mathbf{Z}_q)$ if we set $d(A, A') = wt_\rho(A - A')\ \forall\ A, A' \in Mat_{m\times s}(\mathbf{Z}_q)$. We name this metric as Lee-RT-Jain-metric (or LRTJ-metric) because of the following observations:

1. For $s = 1$, it is just the classical Lee metric [12].

2. For $q = 2, 3$, this metric reduces to the RT-metric [14].

Remarks.

1. For $q > 3$,

$$\text{LRTJ-wt } (A) > \text{RT-wt } (A)\ \forall\ A \in Mat_{m\times s}(\mathbf{Z}_q)$$

2. For $s = 1$ and $q = 2, 3$, LRTJ-metric reduces to the Hamming metric [13].

Notations. We shall use the following notations:

1. $[x]$ = The largest integer less than or equal to x.
2. Q_i will denote the sum of repetitive equivalent values up to i i.e.,

$$Q_i = e_0 + e_1 + \cdots + e_i$$

where e_i denotes the repetitive equivalent value of i.

3. $< x, y >= \min \{x, y\}$.

Also, the definition of bursts for array coding [8] runs as follows:

Definition 2.1. A burst of order pr(or $p \times r$)$(1 \leq p \leq m, 1 \leq r \leq s)$ in the space $Mat_{m \times s}(\mathbf{Z}_q)$ is an $m \times s$ matrix in which all the nonzero entries are confined to some $p \times r$ submatrix which has non-zero first and last rows as well as non-zero first and last columns.

Note. For $p = 1$, Definition 2.1 reduces to the definition of burst for classical codes [6].

Definition 2.2. A burst of order pr or less $(1 \leq p \leq m, 1 \leq r \leq s)$ in the space $Mat_{m \times s}(\mathbf{Z}_q)$ is a burst of order cd(or $c \times d$) where $1 \leq c \leq p \leq m$ and $1 \leq d \leq r \leq s$.

3. Sufficient Condition for Burst Error Identification in LRTJ-Metric Array Codes

To derive the results in this and subsequent section, we shall identify the space $Mat_{m \times s}(\mathbf{Z}_q)$ with the space $\mathbf{Z}_q^{ms}$ i.e an $m \times s$ matrix over $\mathbf{Z}_q$ is considered as an ms-tuple over $\mathbf{Z}_q$ arranged into m groups of s elements each. Each group of s elements in an ms-tuple is called a block. Also, s is called the block length or block size and m is called the block value. Each block of an ms-tuple has a LRTJ-weight and sum of LRTJ-weights of all the m blocks of an ms-tuple is the LRTJ-weight of that ms-tuple. Also, columns of generator matrix G and parity check matrix H of a linear array code V are grouped into m blocks of s columns each. Therefore, generator matrix G and parity check matrix H of a linear array code V are represented as $G = [G_1, G_2, \cdots, G_m], H = [H_1, H_2, \cdots, H_m]$ where G_i and H_i are the i^{th} block $(1 \leq i \leq m)$ of generator and parity check matrix respectively of the code V and are given by

$$G_i = [G_{i1}, G_{i2}, \cdots, G_{is}],$$

and

$$H_i = [H_{i1}, H_{i2}, \cdots, H_{is}],$$

where each $G_{ij}(1 \leq i \leq m, 1 \leq j \leq s)$ is a $k \times 1$ column vector and each $H_{ij}(1 \leq i \leq m, 1 \leq j \leq s)$ is an $(ms - k) \times 1$ column vector.

Also, we give the following definition:

Definition 3.1. A linear combination of $m \times s$ vectors $u_{11}, \cdots, u_{1s}, u_{21}, \cdots, u_{2s}, \cdots, u_{m1}, \cdots u_{ms}$ given by

$$\alpha_{11}u_{11} + \cdots + \alpha_{1s}u_{1s} + \alpha_{21}u_{21} + \cdots + \alpha_{2s}u_{2s} + \cdots + \alpha_{m1}u_{m1} + \cdots + \alpha_{ms}u_{ms},$$

where $\alpha_{ij} \in \mathbf{Z}_q, u_{ij} \in \mathbf{Z}_q^{ms-k}(1 \leq i \leq m, 1 \leq j \leq s)$ is called a linear combination of LRTJ-weight w if

$$\begin{aligned}&\text{LRTJ-}wt(\alpha_{11}, \cdots, \alpha 1s) + \text{GLRTP-}wt(\alpha_{21}, \cdots, \alpha_{2s} + \cdots + \\ &+ \cdots \text{LRTJ-}wt(\alpha_{m1}, \cdots, \alpha_{ms}) = w,\end{aligned}$$

where $\forall\, i(1 \leq i \leq m)$,

$$\text{LRTJ-}wt(\alpha_{i1}, \cdots, \alpha_{is}) = \begin{cases} \max\limits_{j=1}^{s}|\alpha_{ij}| + \max\limits_{j=1}^{s}\{j-1\}|\alpha_{ij} \neq 0\} \; if \; (\alpha_{i1} \cdots \alpha_{is}) \neq 0 \\ \\ 0 \qquad\qquad if \; (\alpha_{i1}, \cdots, \alpha_{is}) = 0. \end{cases}$$

Now we obtain the sufficient condition for burst error identification with LRTJ-weight constraint in linear array codes.

Theorem 3.1. *Let q be prime and m, s, p, r, k, w be positive integers satisfying $1 \leq p \leq m, 1 \leq r \leq s$, $1 \leq w \leq p([q/2] + s - 1)$ and $1 \leq k \leq ms$, then there exists an $[m \times s, k]$ linear LRTJ-metric array code over $\mathbf{Z}_q$, i.e. a linear array code with m as block value and s as the block size, that has no burst of order pr or less with LRTJ-weight w or less as a code array provided*

$$\begin{aligned} q^{ms-k} > 1 \quad &+ \quad \sum_{\substack{j=1: \\ j \leq w}}^{s} q^{<j-1,r-1>} V^{p-1,q}_{j,r,w-(j-1)-[q/2]} \\ &+ \quad \sum_{\substack{j=1: \\ j > w}}^{s} \Big((m-1)s + (j-1) \Big) \end{aligned} \tag{1}$$

where

$$V^{p-1,q}_{j,r,w-(j-1)-[q/2]} \;=\; \sum_{r_{lf}} \frac{(p-1)!}{\prod\limits_{l=1}^{[q/2]} \prod\limits_{f=1}^{<j,r>} r_{lf}! \left((p-1) - \sum\limits_{l=1}^{[q/2]} \prod\limits_{f=1}^{<j,r>} r_{lf} \right)!} \times$$

$$\times \prod_{l=1}^{[q/2]} \prod_{f=1}^{<j,r>} \left(e_l \left(Q_l \right)^{f-2} \left(Q_l + (f-1)(Q_{l-1} - 1) \right) \right)^{r_{lf}} \tag{2}$$

and $r_{lf}(1 \leq l \leq [q/2], 1 \leq f \leq \ < j, r >)$ being nonnegative integers satisfying

$$\sum_{l=1}^{[q/2]} \sum_{f=1}^{<j,r>} r_{lf} \leq p - 1,$$

$$\sum_{l=1}^{[q/2]} \sum_{f=1}^{<j,r>} (l + (f-1)) r_{lf} \leq w - (j-1) - [q/2]. \tag{3}$$

Proof. The existence of such a code will be proved by constructing a suitable $(ms - k) \times ms$ parity check matrix H for the desired code. To detect any burst of order pr or less with LRTJ-weight w or less, it is necessary and sufficient that no linear combination of LRTJ-weight w or less involving r(or fewer) consecutive columns in p (or fewer) consecutive blocks should be zero. Suppose that $i - 1 (1 \leq i \leq m)$ blocks $H_1, H_2, \cdots, H_{i-1}$ have been chosen suitably. To add the j^{th} column $(1 \leq j \leq s)$ in the i^{th} block, we can have either of two mutually exclusive cases:

Case (i): When $j \leq w$.

In this case, j^{th} column in the i^{th} block can be added, provided it is not a linear combination of $l_j^{th}, (l_j + 1)^{th}, \cdots, j^{th}$ columns from the immediately preceding $< i-1, p-1 >$ blocks having LRTJ-weight $w - (j-1) - [q/2]$ or less (where $l_j =< 1, j - r + 1 >$) together with any linear combination of $l_j^{th}, (l_j + 1)^{th}, \cdots, (j-1)^{th}$ columns in the i^{th} block. Therefore, column $H_{ij}(1 \leq j \leq s)$ can be added to H provided

$$\begin{aligned} H_{ij} \ \neq \ & \sum_{g=i-<i,p>+1}^{i-1} (\alpha_{g,l_j} H_{g,l_j} + \alpha_{g,l_j+1} H_{g,l_j+1} + \cdots + \alpha_{g,j} H_{g,j}) \\ & + \alpha_{i,l_j} H_{i,l_j} + \alpha_{i,l_j+1} H_{i,l_j+1} + \cdots + \alpha_{i,j-1} H_{i,j-1} \end{aligned} \tag{4}$$

where

$$\sum_{g=i-<i,p>+1}^{i-1} \text{LRTJ-}wt(\alpha_{g,l_j}, \alpha_{g,l_j+1}, \cdots, \alpha_{g,j}) \leq w - (j-1) - [q/2]. \tag{5}$$

The number of linear combinations occuring in the R.H.S. of (4) subject to constraint (5) is given by

$$q^{<j-1,r-1>} V^{<i-1,p-1>,q}_{j,r,w-(j-1)-[q/2]} \tag{6}$$

where $V^{<i-1,p-1>,q}_{j,r,w-(j-1)-[q/2]}$ is given by

$$V^{<i-1,p-1>}_{j,r,w-(j-1)-[q/2]} = \sum_{r_{lf}} \frac{(<i-1,p-1>)!}{\prod_{l=1}^{[q/2]}\prod_{f=1}^{<j,r>} r_{lf}! \left(<i-1,p-1>) - \sum_{l=1}^{[q/2]}\sum_{f=1}^{<j,r>} r_{lf} \right)!} \times$$

$$\times \prod_{l=1}^{[q/2]}\prod_{f=1}^{<j,r>} \left(e_l \left(Q_l \right)^{f-2} \left(Q_l + (f-1)(Q_{l-1}-1) \right) \right)^{r_{lf}} \qquad (7)$$

and $r_{lf}(1 \leq l \leq [q/2], 1 \leq f \leq< j, r >)$ being nonnegative integers satisfying

$$\sum_{l=1}^{[q/2]}\sum_{f=1}^{<j,r>} r_{lf} \leq< i-1, p-1 >$$

$$\sum_{l=1}^{[q/2]}\sum_{f=1}^{<j,r>} (l+(f-1))r_{lf} \leq w-(j-1)-[q/2]. \qquad (8)$$

Case (ii). When $j > w$.

In this case, the j^{th} column in the i^{th} block can be selected from the set of all $(ms-k)$-tuples provided it is not selected previously. The number of $(ms-k)$-tuples selected in the construction of H so far is given by

$$(i-1)s+(j-1). \qquad (9)$$

Thus, i^{th} block can be added to H provided the summation of the number enumerated in (6) ($j \leq w$) and in (9) ($j > w$) for $j = 1$ to s including the array of all zeros is less than the total number of $(ms-k)$-tuples. Therefore, i^{th} block H_i can be added to H provided that

$$q^{ms-k} > 1 + \sum_{\substack{j=1: \\ j\leq w}}^{s} q^{<j-1,r-1>} V^{<i-1,p-1>,q}_{j,r,w-(j-1)-[q/2]} + \sum_{\substack{j=1: \\ j>w}}^{s} ((i-1)s+(j-1)) \qquad (10)$$

where $V^{<i-1,p-1>,q}_{j,r,w-(j-1)-[q/2]}$ is given by (7) staisfying (8).

For the existence of an $[m \times s, k]$ linear array code over $\mathbf{Z}_q$, inequality (10) should hold for $i = m$ so that it is possible to add upto m^{th} block to form an $(ms-k)\times ms$ parity check matrix H and we get (1) by noting that $<m,p>=p$.

Hence the theorem. $\square$

Example 3.1. Take $m=s=p=r=2, w=3, k=2$ and $q=5$. Then we have

$$\text{R.H.S. of (1)} = 1+\sum_{j=1}^{2} 5^{<j-1,1>} V^{1,5}_{j,2,2-j} = L(\text{say}),$$

where $V^{1,5}_{j,2,2-j}$ is given by (2) on taking $p=r=2, w=3$ and $q=5$.

Therefore,

$$L = 1+5^0 V^{1,5}_{1,2,1} + 5^1 V^{1,5}_{2,2,0}. \tag{11}$$

Now,

$$\begin{aligned} V^{1,5}_{1,2,1} &= \sum_{r_{11},r_{21}} \frac{1!}{r_{11}!r_{21}!(1-(r_{11}+r_{21}))!} \times \\ &\quad \times (e_1(Q_1)^{-1}(Q_1))^{r_{11}} \times (e_2(Q_2)^{-1}(Q_2))^{r_{21}} \\ &= \sum_{r_{11},r_{21}} \frac{1!}{r_{11}!r_{21}!(1-(r_{11}+r_{21}))!} \times e_1^{r_{11}} \times e_2^{r_{21}}, \end{aligned}$$

where r_{11}, r_{21} are nonnegative integers satisfying the following constraints:

$$\begin{aligned} r_{11}+r_{21} &\le 1, \\ r_{11}+2r_{21} &\le 1. \end{aligned} \tag{12}$$

The feasible solutions for (r_{11}, r_{21}) satisfying the constraint (12) are given by

$$(r_{11}, r_{21}) = (0,0), (1,0).$$

Therefore,

$$V^{1,5}_{1,2,1} = 1+\frac{1!}{1!0!1!}e_1^1 = 1+e_1 = 1+2 = 3. \quad (\text{Note that } e_1 = 2 \text{ over } Z_5).$$

Again,

$$\begin{aligned} V^{1,5}_{2,2,0} &= \sum_{r_{11},r_{21},r_{12},r_{22}} \frac{1!}{r_{11}!r_{21}!r_{12}!r_{22}!(1-(r_{11}+r_{21}+r_{12}+r_{22}))!} \times \\ &\quad \times (e_1(Q_1)^{-1}(Q_1))^{r_{11}} \times (e_2(Q_2)^{-1}(Q_2))^{r_{21}} \times \\ &\quad \times (e_1(Q_1)^{0}(Q_1+(Q_0-1)))^{r_{12}} \times (e_2(Q_2)^{0}(Q_2+(Q_1-1)))^{r_{22}} \\ &= \sum_{r_{11},r_{21},r_{12},r_{22}} \frac{1!}{r_{11}!r_{21}!r_{12}!r_{22}!(1-(r_{11}+r_{21}+r_{12}+r_{22}))!} \times \\ &\quad \times e_1^{r_{11}} \times e_2^{r_{21}} \times (e_1(Q_1+(Q_0-1)))^{r_{12}} \times (e_2(Q_2+(Q_1-1)))^{r_{22}} \end{aligned}$$

subject to

$$
\begin{aligned}
&r_{11}, r_{21}, r_{12}, r_{22} \geq 0, \\
&r_{11} + r_{21} + r_{12} + r_{22} \leq 1 \\
&r_{11} + 2r_{21} + 2r_{12} + 3r_{22} \leq 0
\end{aligned} \tag{13}
$$

The only feasible solutions for $(r_{11}, r_{21}, r_{12}, r_{22})$ satisfying (13) is the null solution.

Therefore $V_{2,2,0}^{1,2} = 1$.

Thus from (11)

$$L = 1 + 3 + 5 = 9$$

Also, $q^{ms-k} = 5^{4-2} = 5^2 = 25$.

Therefore, L.H.S. of (1) $= 25 > 9 =$ R.H.S. of (1) and hence there exists a $[2 \times 2, 2]$ linear array code V over $\mathbf{Z}_5$ that detects all bursts of order 2×2 or less having LRTJ-weight 3 or less. Consider the following $(2 \times 2 - 2) \times (2 \times 2) = 2 \times 4$ parity check matrix of a $[2 \times 2, 2]$ linear array code over $\mathbf{Z}_5$ constructed by the algorithm discussed in Theorem 3.1.

$$H = \begin{bmatrix} 1 & 0 & \vdots & 4 & 2 \\ 0 & 1 & \vdots & 2 & 3 \end{bmatrix}_{2\times 4}$$

The generator matrix G corresponding to the parity check matrix H is given by

$$G = \begin{bmatrix} 1 & 3 & \vdots & 1 & 0 \\ 3 & 2 & \vdots & 0 & 1 \end{bmatrix}_{2\times 4}$$

The 25 code arrays of the code $V \subseteq \mathrm{Mat}_{2\times 2}(\mathbf{Z}_5)$ with G as generator matrix and H as parity check matrix are given by

$v_0 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$, LRTJ-$wt(v_0) = 0$; $\quad v_1 = \begin{pmatrix} 3 & 2 \\ 0 & 1 \end{pmatrix}$, LRTJ-$wt(v_1) = 5$;

$v_2 = \begin{pmatrix} 1 & 4 \\ 0 & 2 \end{pmatrix}$, LRTJ-$wt(v_2) = 5$; $\quad v_3 = \begin{pmatrix} 4 & 1 \\ 0 & 3 \end{pmatrix}$, LRTJ-$wt(v_3) = 5$;

$v_4 = \begin{pmatrix} 2 & 3 \\ 0 & 4 \end{pmatrix}$, LRTJ-$wt(v_4) = 5$; $\quad v_5 = \begin{pmatrix} 1 & 3 \\ 1 & 0 \end{pmatrix}$, LRTJ-$wt(v_5) = 4$;

$v_6 = \begin{pmatrix} 4 & 0 \\ 1 & 1 \end{pmatrix}$, LRTJ-$wt(v_6) = 3$; $\quad v_7 = \begin{pmatrix} 2 & 2 \\ 1 & 2 \end{pmatrix}$, LRTJ-$wt(v_7) = 6$;

$v_8 = \begin{pmatrix} 0 & 4 \\ 1 & 3 \end{pmatrix}$, LRTJ-$wt(v_8) = 5$; $v_9 = \begin{pmatrix} 3 & 1 \\ 1 & 4 \end{pmatrix}$, LRTJ-$wt(v_9) = 5$;

$v_{10} = \begin{pmatrix} 2 & 1 \\ 2 & 0 \end{pmatrix}$, LRTJ-$wt(v_{10}) = 5$; $v_{11} = \begin{pmatrix} 0 & 3 \\ 2 & 1 \end{pmatrix}$, LRTJ-$wt(v_{11}) = 6$;

$v_{12} = \begin{pmatrix} 3 & 0 \\ 2 & 2 \end{pmatrix}$, LRTJ-$wt(v_{12}) = 5$; $v_{13} = \begin{pmatrix} 1 & 2 \\ 2 & 3 \end{pmatrix}$, LRTJ-$wt(v_{13}) = 6$;

$v_{14} = \begin{pmatrix} 4 & 4 \\ 2 & 4 \end{pmatrix}$, LRTJ-$wt(v_{14}) = 5$; $v_{15} = \begin{pmatrix} 3 & 4 \\ 3 & 0 \end{pmatrix}$, LRTJ-$wt(v_{15}) = 5$;

$v_{16} = \begin{pmatrix} 1 & 1 \\ 3 & 1 \end{pmatrix}$, LRTJ-$wt(v_{16}) = 5$; $v_{17} = \begin{pmatrix} 4 & 3 \\ 3 & 2 \end{pmatrix}$, LRTJ-$wt(v_{17}) = 6$;

$v_{18} = \begin{pmatrix} 2 & 0 \\ 3 & 3 \end{pmatrix}$, LRTJ-$wt(v_{18}) = 5$; $v_{19} = \begin{pmatrix} 0 & 2 \\ 3 & 4 \end{pmatrix}$, LRTJ-$wt(v_{19}) = 6$;

$v_{20} = \begin{pmatrix} 4 & 2 \\ 4 & 0 \end{pmatrix}$, LRTJ-$wt(v_{20}) = 4$; $v_{21} = \begin{pmatrix} 2 & 4 \\ 4 & 1 \end{pmatrix}$, LRTJ-$wt(v_{21}) = 5$;

$v_{22} = \begin{pmatrix} 0 & 1 \\ 4 & 2 \end{pmatrix}$, LRTJ-$wt(v_{22}) = 5$; $v_{23} = \begin{pmatrix} 3 & 3 \\ 4 & 3 \end{pmatrix}$, LRTJ-$wt(v_{23}) = 6$;

$v_{24} = \begin{pmatrix} 1 & 0 \\ 4 & 4 \end{pmatrix}$, LRTJ-$wt(v_{24}) = 3$.

We observe that none of the code array is a burst of order 2×2 or less over $\mathbf{Z}_5$ having LRTJ-weight 3 or less. Therefore, sufficient condition (1) is justified and hence the code V detects these type of burst errors.

Note that in Example 3.1, Case (ii) of Theorem 3.1 does not occur as $j \leq w$ always. The following example illustrates both the cases of Theorem 3.1.

Example 3.2. Take $m = s = 3, p = r = 2, w = 2, q = 2$ and $k = 5$. Then

$$\begin{aligned}
\text{R.H.S. of}(1) &= 1 + \sum_{\substack{j=1:\\ j\leq 2}}^{3} 2^{<j-1,1>} V^{1,2}_{j,2,2-j} + \sum_{\substack{j=1:\\ j>2}}^{3} (6 + (j-1)) \\
&= 1 + 2^{<0,1>} V^{1,2}_{1,2,1} + 2^{<1,1>} V^{1,2}_{2,2,0} + 8 \\
&= 1 + (2^0 \times 2) + (2^1 \times 1) + 7 \quad (\text{since } V^{1,2}_{1,2,1} = 2, V^{1,2}_{2,2,0} = 1) \\
&= 1 + 2 + 2 + 8 = 13.
\end{aligned}$$

Also $q^{ms-k} = 2^{9-5} = 2^4 = 16$.

Therefore, L.H.S. of (1) $= 16 > 13 =$ R.H.S. of (1) and hence there exists a $[3 \times 3, 5]$ linear array code V over $\mathbf{Z}_2$ that detects/identifies all bursts of order 2 × 2 or less having LRTJ-weight 2 or less. Consider the following $(3\times 3-5)\times(3\times 3)=4\times 9$ parity check matrix of a $[3\times 3, 3]$ linear array code over $\mathbf{Z}_2$.

$$H=\left[\begin{array}{ccc:ccc:ccc} 1&0&0&0&1&1&1&0&1\\ 0&1&0&0&0&1&0&1&1\\ 0&0&1&0&1&0&1&1&1\\ 0&0&0&1&0&1&1&0&1 \end{array}\right]_{4\times 9}$$

The generator matrix G corresponding to the parity check matrix H is given by

$$G=\left[\begin{array}{ccc:ccc:ccc} 1&0&1&0&1&0&0&0&0\\ 1&0&0&1&0&1&0&0&0\\ 1&0&1&1&0&0&1&0&0\\ 0&1&1&0&0&0&0&1&0\\ 1&1&1&1&0&0&0&0&1 \end{array}\right]_{5\times 9}$$

The 32 code arrays of the code $V\subseteq \mathrm{Mat}_{3\times 3}\mathbf{Z}_2$ with G as generator matrix and H as parity check matrix are given by

$$v_0=\begin{pmatrix}0&0&0\\0&0&0\\0&0&0\end{pmatrix},v_1=\begin{pmatrix}1&1&1\\1&0&0\\0&0&1\end{pmatrix},v_2=\begin{pmatrix}0&1&1\\0&0&0\\0&1&0\end{pmatrix},$$

$$v_3=\begin{pmatrix}1&0&1\\1&0&0\\1&0&0\end{pmatrix},v_4=\begin{pmatrix}1&1&0\\1&0&1\\0&0&0\end{pmatrix},v_5=\begin{pmatrix}1&0&1\\0&1&0\\0&0&0\end{pmatrix},$$

$$v_6=\begin{pmatrix}1&0&0\\1&0&0\\0&1&1\end{pmatrix},v_7=\begin{pmatrix}0&1&0\\0&0&0\\1&0&1\end{pmatrix},v_8=\begin{pmatrix}1&1&0\\1&0&0\\1&1&0\end{pmatrix},$$

$$v_9=\begin{pmatrix}0&0&1\\0&0&0\\1&1&1\end{pmatrix},v_{10}=\begin{pmatrix}0&0&1\\0&0&1\\0&0&1\end{pmatrix},v_{11}=\begin{pmatrix}1&0&1\\1&0&1\\0&1&0\end{pmatrix},$$

$$v_{12}=\begin{pmatrix}0&1&0\\0&0&1\\0&1&1\end{pmatrix},v_{13}=\begin{pmatrix}0&1&1\\0&0&1\\1&0&0\end{pmatrix},v_{14}=\begin{pmatrix}1&0&0\\1&0&1\\1&0&1\end{pmatrix},$$

$$v_{15} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}, v_{16} = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 1 \end{pmatrix}, v_{17} = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

$$v_{18} = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 0 \end{pmatrix}, v_{19} = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix}, v_{20} = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix},$$

$$v_{21} = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}, v_{22} = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 1 & 0 \end{pmatrix}, v_{23} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 1 & 1 \end{pmatrix},$$

$$v_{24} = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix}, v_{25} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}, v_{26} = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 0 \end{pmatrix},$$

$$v_{27} = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{pmatrix}, v_{28} = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 0 \end{pmatrix}, v_{29} = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 1 & 1 \\ 1 & 0 & 1 \end{pmatrix},$$

$$v_{30} = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 0 \end{pmatrix}, v_{31} = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}.$$

We observe that none of the code array is a burst of order 2×2 or less having LRTJ-weight 2 or less and hence the code V detects these type of burst errors.

4. Sufficient Condition for Burst Error Correction in LRTJ-Metric Array Codes

In this section, we obtain sufficient condition for burst error correction with LRTJ-weight constraint in linear array codes. To prove the desired bound, we need the following lemma [11]:

Lemma 4.1. *The number of bursts of order* $pr(1 \leq p \leq m, 1 \leq r \leq s)$ *in* $Mat_{m\times s}(\mathbb{Z}_q)$ *having LRTJ-weight* w *or less* $(1 \leq w \leq m([q/2] + s - 1))$ *is given by*

$$B^{p\times r}_{m\times s}(\mathbf{Z}_q,w)=\begin{cases} m\displaystyle\sum_{j=1}^{min(w,s)}(Q_{w-(j-1)}-1) & \text{if } p=r=1,\\ m\displaystyle\sum_{j=1}^{min(w-r+1,s-r+1)}(Q_{w-(j+r-2)})^{r-2}\times \\ \times(Q_{w-(j+r-2)}-1)^2 & \text{if } p=1, r\geq 2,\\ (m-p+1)\displaystyle\sum_{j=1}^{min(w-r+1,s-r+1)}(L^p_{j,r}-2L^{p-1}_{j,r}+L^{p-2}_{j,r}) \\ & \text{if } p\geq 2, r\geq 1, \end{cases} \tag{14}$$

where

$$\begin{aligned} L^p_{j,r} &= \sum_{r_{lf}} \frac{p!}{\prod_{l=1}^{[q/2]}\prod_{f=j}^{j+r-1} r_{lf}!\left(p-\sum_{l=1}^{[q/2]}\sum_{f=j}^{j+r-1} r_{lf}\right)!}\times \\ &\quad\times\prod_{l=1}^{[q/2]}\prod_{f=j}^{j+r-1}\left(e_l(Q_l)^{f-j-1}(Q_l+(f-j)(Q_{l-1}-1))\right)^{r_{lf}}, \end{aligned} \tag{15}$$

and r_{lf} $(1\leq l\leq [q/2], j\leq f\leq j+r-1$ *in the expression for* $L^p_{j,r}$ *are non-negative integers satisfying the following constraints:*

$$\begin{aligned} &\textit{at least one of } r_{lj}>0\ (1\leq l\leq [q/2],\ j \textit{ fixed occuring in} \\ &\textit{the expression for } L^p_{j,r}), \\ &\textit{at least one of } r_{l,j+r-1}>0\ (1\leq l\leq [q/2],\ j+r-1 \textit{ fixed}), \\ &\sum_{l=1}^{[q/2]}\sum_{f=j}^{j+r-1}(l+(f-1))r_{lf}\leq w, \\ &\sum_{l=1}^{[q/2]}\sum_{f=j}^{j+r-1} r_{lf}\leq p. \end{aligned} \tag{16}$$

Theorem 4.1. *Let q be prime and m, s, p, r, k, w be positive integers satisfying $1\leq p\leq [m/2], 1\leq r\leq s, 1\leq k\leq ms$, and $1\leq w\leq p([q/2]+s-1)$, then a sufficient condition for the existence of an $[m\times s, k]$ linear array code over $\mathbf{Z}_q$ that corrects all bursts of order pr or less having LRTJ-weight* *w or less is given by:*

$$q^{ms-k} > 1+\left(\sum_{c=1}^{p}\sum_{d=1}^{r}B^{c\times d}_{(m-p)\times s}(\mathbf{Z}_q,w)\right)\times \quad (17)$$

$$\times\left(\sum_{\substack{j=1:\\ j\le w}}^{s} q^{<j-1,r-1>}V^{p-1,q}_{j,r,w-(j-1)-\lceil q/2\rceil}\right)+$$

$$+\sum_{\substack{j=1:\\ j>w}}^{s}((m-1)s+(j-1)) \quad (18)$$

where $B^{c\times d}_{(m-p)\times s}(\mathbf{Z}_q,w)$ is given by (14) in Lemma 4.1 and $V^{p-1,q}_{j,r,w-(j-1)-\lceil q/2\rceil}$ is given by (2).

Proof. We construct the parity check matrix of the desired code by using the fact that to correct all bursts of order pr or less with LRTJ-weight w or less, it is necessary and sufficient that no code array consists of the sum of two bursts of order pr or less having LRTJ-weight w or less. Thus no linear combination involving two sets of r (or fewer) consecutive columns in p (or fewer) consecutive blocks having LRTJ-weight w or less should be zero. Suppose that $(m-1)$ blocks $H_1, H_2, \cdots, H_{m-1}$ of the parity check matrix H have been chosen suitably. To add the j^{th} cloumn $(1 \le j \le s)$ in the m^{th} block, we can have either of the two mutually exclusive cases:

Case (i): When $j \le w$.

In this case, j^{th} column in the m^{th} block can be added, provided it is not a linear combination of $l_j^{th}, (l_j+1)^{th}, \cdots, j^{th}$ columns from the immediately preceding $(p-1)$ blocks having LRTJ-weight $w-(j-1)-\lceil q/2\rceil$ or less (where $l_j =< 1, j-r+1 >$) together with any linear combination of $l_j^{th}, (l_j+1)^{th}, \cdots, (j-1)^{th}$ columns in the m^{th} block and any linear combination of r (or fewer) consecutive columns in p (or fewer) consecutive blocks among the first $(m-p)$ blocks having LRTJ-weight w or less. In other words, j^{th} column $(j \le w)$ in the m^{th} block can be added to H provided that

$$H_{mj} \neq \sum_{g=m-p-1}^{m-1} (\alpha_{g,l_j}H_{g,l_j} + \alpha_{g,l_j+1}H_{g,l_j+1} + \cdots + \alpha_{g,j}H_{g,j})$$
$$+\alpha_{m,l_j}H_{m,l_j} + \alpha_{m,l_j+1}H_{m,l_j+1} + \cdots + \alpha_{m,j-1}H_{m,j-1} \quad (19)$$

+linear combination which form a burst of order pr or less having LRTJ-weight w or less in the first $(m-p)$ blocks.

subject to

$$\sum_{g=m-p+1}^{m-1} \text{LRTJ-}wt(\alpha_{g,l_j}, \alpha_{g,l_j+1}, \cdots, \alpha_{g,j}) \leq w - (j-1) - [q/2]. \quad (20)$$

Now, the number of linear combinations occuring in the R.H.S. of (18) satisfying constraint (19) is given by

$$\left(q^{<j-1,r-1>} V^{p-1,q}_{j,r,w-(j-1)-[q/2]}\right) \left(\sum_{c=1}^{p} \sum_{d=1}^{r} B^{c\times d}_{(m-p)\times s}(\mathbf{Z}_q, w)\right). \quad (21)$$

Case ii. When $j > w$.

In this case, the j^{th} column in the m^{th} block can be selected from the set of all $(ms-k)$-tuples provided it is not selected previously. The number of $(ms-k)$-tuples selected in the construction of H so far is given by

$$(m-1)s + (j-1). \quad (22)$$

Now, m^{th} block can be added to H provided we can add all the s columns of the m^{th} block. Therefore, m^{th} block can be added to H provided the sum of the numbers for each $j = 1$ to s enumerated in (20) (for $j \leq w$) and in (21) (for $j > w$) including the pattern of all zeros is less than the total number of available $(ms-k)$-tuples. Thus, H_m can be added to H provided that

$$\begin{aligned} q^{ms-k} \quad > \quad & 1 + \left(\sum_{c=1}^{p} \sum_{d=1}^{r} B^{c\times d}_{(m-p)\times s}(\mathbf{Z}_q, w))\right) \times \\ & \times \left(\sum_{\substack{j=1:\\ j\leq w}}^{s} q^{<j-1,r-1>} V^{p-1,q}_{j,r,w-(j-1)-[q/2]}\right) + \\ & + \sum_{\substack{j=1:\\ j>w}}^{s} ((m-1)s + (j-1)). \end{aligned}$$

Hence the theroem. □

Example 4.1. Take $m = 4, s = 3, p = 2, r = 1, w = 2, k = 9$ and $q = 5$. Then

$$\text{R.H.S. of(17)} \quad = \quad 1 + \left(\sum_{c=1}^{2} \sum_{d=1}^{1} B^{c\times d}_{2\times 3}(\mathbf{Z}_5, 2)\right) \left(\sum_{\substack{j=1:\\ j\leq 2}}^{3} 2^{<j-1,0>} V^{1,5}_{j,1,1-j}\right) +$$

$$\sum_{\substack{j=1:\\ j>2}}^{3}(9+(j-1))$$

$$= 1+\left(B^{1\times 1}_{2\times 3}(\mathbf{Z}_5,2)+B^{2\times 1}_{2\times 3}(\mathbf{Z}_5,2)\right)\left(V^{1,5}_{1,1,0} \tag{23}\right.$$

$$\left. +V^{1,5}_{2,1,-1}\right)+(9+2) \tag{24}$$

Now

$$V^{1,5}_{1,1,0}=1 \quad \text{and } V^{1,5}_{2,1,-1}=0. \tag{25}$$

Also,

$$\begin{aligned} B^{1\times 1}_{2\times 3}(\mathbf{Z}_5,2) &= 2\sum_{j=1}^{min(2,3)}(Q_{3-j}-1)=2\sum_{j=1}^{2}(Q_{3-j}-1)\\ &= 2((Q_2-1)+(Q_1-1))=2(4+2)=12. \qquad (26)\\ &\quad \text{(Note that over } \mathbf{Z}_5, Q_1=3, Q_2=5). \end{aligned}$$

Again,

$$\begin{aligned} B^{2\times 1}_{2\times 3}(\mathbf{Z}_5,2) &= \sum_{j=1}^{min(2,3)}(L^2_{j,1}+L^0_{j,1}-2L^1_{j,1})\\ &= (L^2_{1,1}+L^0_{1,1}-2L^1_{1,1})+(L^2_{2,1}+L^0_{2,1}-2L^1_{2,1})\\ &= (12+0-8)+(12+0-12)=4. \qquad (27) \end{aligned}$$

(Note that from (15) and (16) in Lemma 4.1, we have $L^2_{1,1}=12, L^1_{1,1}=4, L^0_{1,1}=0, L^2_{2,1}=12, L^1_{2,1}=6$ and $L^0_{2,1}=0$).

Using (23), (24) and (25) in (22), we get

$$\text{R.H.S. of } (17)=1+(12+4)(1+0)+11=28.$$

Also, L.H.S. of (17) $=5^{ms-k}=5^{4\times 3-9}=5^3=125$.

Therefore, L.H.S. of (17) $=125>28=$ R.H.S. of (17) and hence by Theorem 4.1, there exists a $[4\times 3,9]$ linear array code over $\mathbf{Z}_5$ that corrects all bursts of order 2×1 or less having LRTJ-weight 2 or less. Consider the following $(4\times 3-9)\times(4\times 3)=3\times 12$ parity check matrix of a $[4\times 3,9]$ linear array code over $\mathbf{Z}_5$ constructed by the procedure discussed in Theorem 4.1.

$$H=\begin{bmatrix} 1&0&0&\vdots&0&1&1&\vdots&1&0&0&\vdots&0&1&2\\ 0&1&0&\vdots&1&2&4&\vdots&2&1&4&\vdots&1&3&4\\ 0&0&1&\vdots&2&1&0&\vdots&3&4&3&\vdots&3&3&2 \end{bmatrix}_{3\times 12}$$

The code $V \subseteq Mat_{4\times3}(Z_5)$ which is the null subspace of H corrects all bursts of order 2×1 or less having LRTJ-weight 2 or less since syndromes of these error patterns are all distinct as seen from Table 4.1.

Table 4.1.

Burst Errors of order 2×1 or less having LRTJ-weight 2 or less	Syndromes
$\begin{pmatrix} 1 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = (100\ 100\ 000\ 000)$	(112)
$\begin{pmatrix} 1 & 0 & 0 \\ 4 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = (100\ 400\ 000\ 000)$	(143)
$\begin{pmatrix} 4 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = (400\ 100\ 000\ 000)$	(412)
$\begin{pmatrix} 4 & 0 & 0 \\ 4 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = (400\ 400\ 000\ 000)$	(443)
$\begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = (000\ 100\ 100\ 000)$	(130)
$\begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 4 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = (000\ 100\ 400\ 000)$	(444)
$\begin{pmatrix} 0 & 0 & 0 \\ 4 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = (000\ 400\ 100\ 000)$	(111)
$\begin{pmatrix} 0 & 0 & 0 \\ 4 & 0 & 0 \\ 4 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = (000\ 400\ 400\ 000)$	(420)

Table contd.

Burst Errors of order 2×1 or less having LRTJ-weight 2 or less	Syndromes
$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} = (000\ 000\ 100\ 100)$	(131)
$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \\ 4 & 0 & 0 \end{pmatrix} = (000\ 000\ 100\ 400)$	(110)
$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 4 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} = (000\ 000\ 400\ 100)$	(440)
$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 4 & 0 & 0 \\ 4 & 0 & 0 \end{pmatrix} = (000\ 000\ 400\ 400)$	(424)
$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = (100\ 000\ 000\ 000)$	(100)
$\begin{pmatrix} 2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = (200\ 000\ 000\ 000)$	(200)
$\begin{pmatrix} 3 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = (300\ 000\ 000\ 000)$	(300)
$\begin{pmatrix} 4 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = (400\ 000\ 000\ 000)$	(400)
$\begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = (000\ 100\ 000\ 000)$	(012)

Table contd.

Burst Errors of order 2×1 or less having LRTJ-weight 2 or less	Syndromes
$\begin{pmatrix} 0 & 0 & 0 \\ 2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ = (000 200 000 000)	(024)
$\begin{pmatrix} 0 & 0 & 0 \\ 3 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ = (000 300 000 000)	(031)
$\begin{pmatrix} 0 & 0 & 0 \\ 4 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ = (000 400 000 000)	(043)
$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ = (000 000 100 000)	(123)
$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 2 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ = (000 000 200 000)	(241)
$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 3 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ = (000 000 300 000)	(314)
$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 4 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ = (000 000 400 000)	(432)
$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}$ = (000 000 000 100)	(013)
$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 2 & 0 & 0 \end{pmatrix}$ = (000 000 000 200)	(021)

Table contd.

Burst Errors of order 2×1 or less having LRTJ-weight 2 or less	Syndromes
$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 3 & 0 & 0 \end{pmatrix} = (000\ 000\ 000\ 300)$	(034)
$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 4 & 0 & 0 \end{pmatrix} = (000\ 000\ 000\ 400)$	(042)
$\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = (010\ 000\ 000\ 000)$	(010)
$\begin{pmatrix} 0 & 4 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = (040\ 000\ 000\ 000)$	(040)
$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = (000\ 010\ 000\ 000)$	(121)
$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = (000\ 040\ 000\ 000)$	(434)
$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} = (000\ 000\ 010\ 000)$	(014)
$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 0 \end{pmatrix} = (000\ 000\ 040\ 000)$	(041)
$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} = (000\ 000\ 000\ 010)$	(133)
$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 4 & 0 \end{pmatrix} = (000\ 000\ 000\ 040)$	(422)

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