On Pell numbers and (k_1A_1, k_2A_2, k_3A_3)-edge colouring in graphs

Lucyna Trojnar-Spelina, Iwona Włoch

Rzeszów University of Technology
Faculty of Mathematics and Applied Physics
al. Powstańców Warszawy 12, 35-959 Rzeszów, Poland
e-mail: lspelina@prz.edu.pl, iwloch@prz.edu.pl

Abstract

In this paper we introduce a special (k_1A_1, k_2A_2, k_3A_3) - edge colouring of a graph. We shall show that for special graphs and special values of k_i, $i = 1, 2, 3$ the number of such colourings generalizes the well-known Pell numbers. Using this graph interpretation we give the direct formula for the generalized Pell numbers. Moreover we show some identities for these numbers.

Keywords: Fibonacci numbers, Pell numbers, Jacobsthal numbers, edge-coloured graphs

MSC 11B37, 11C20, 15B36, 05C69

1 Introduction and preliminary results

In general we use the standard terminology and notation of the combinatorics and the graph theory, see [2], [3]. The nth Fibonacci number F_n is defined by $F_0 = F_1 = 1$ and $F_n = F_{n-1} + F_{n-2}$, for $n \geq 2$. Actually the Fibonacci numbers are studied intensively in a wide sense also in graphs and different combinatorial problems. This interest in mathematical problems is motivated by applications from other branches of science, for example in modern theoretical physics and combinatorial chemistry, see [4], [5].

In general numbers defined recursively by the linear recurrence relation are also named as numbers of Fibonacci type and these recursions appear almost everywhere in mathematics and computer science. Apart the Fibonacci numbers, the well-known are the Pell numbers P_n defined by $P_n = 2P_{n-1} + P_{n-2}$, for $n \geq 2$ with the initial conditions $P_0 = 0$, $P_1 = 1$.

In the mathematical literature there are many generalizations of the Pell numbers with respect to one or more parameters, see for example [6], [7], [8], [9], [10]. In this paper we introduce a one-parameter generalization of the Pell numbers in the distance sense, i.e generalization by the kth order linear recurrence relation.

Let $k \geq 1$, $n \geq 0$ be integers. By a $(2,k)$-distance Pell numbers $P^{(i)}(k,n)$ of the i-th kind, $i = 1,2$, we mean generalized Pell numbers defined recursively by the following relation:

$$P^{(i)}(k,n) = 2P^{(i)}(k,n-k) + P^{(i)}(k,n-2) \quad \text{for} \quad n \geq k \qquad (1)$$

with the following initial conditions $P^{(i)}(k,0) = 0$, $P^{(i)}(k,1) = 1$ and for $2 \leq n \leq k-1$:

$$P^{(1)}(k,n) = \begin{cases} 0 & \text{if } n \text{ is even,} \\ 1 & \text{if } n \text{ is odd,} \end{cases} \qquad \text{and} \quad P^{(2)}(k,n) = 1.$$

It is worth to mention that this type of generalized Pell numbers is motivated by $(2,k)$-distance Fibonacci numbers and $(2,k)$-distance Lucas numbers which were introduced and studied in [1], [11].

The following tables include initial terms of the distance Pell sequences $P^{(i)}(k,n)$, $i = 1,2$, for special values of k and n.

Tab.1. The distance Pell numbers $P^{(1)}(k,n)$ of the first order

n	0	1	2	3	4	5	6	7	8	9	10	11	12
$P^{(1)}(1,n)$	0	1	2	5	12	29	70	169	408	985	2378	5741	13860
$P^{(1)}(2,n)$	0	1	0	3	0	9	0	27	0	81	0	273	0
$P^{(1)}(3,n)$	0	1	0	1	2	1	4	5	6	13	16	25	42
$P^{(1)}(4,n)$	0	1	0	1	0	3	0	5	0	11	0	21	0
$P^{(1)}(5,n)$	0	1	0	1	0	1	2	1	4	1	6	5	8
$P^{(1)}(6,n)$	0	1	0	1	0	1	0	3	0	5	0	7	0
$P^{(1)}(7,n)$	0	1	0	1	0	1	0	1	2	1	4	1	6

Tab.2. The distance Pell numbers $P^{(2)}(k,n)$ of the second order

n	0	1	2	3	4	5	6	7	8	9	10	11	12
$P^{(2)}(1,n)$	0	1	2	5	12	29	70	169	408	985	2378	5741	13860
$P^{(2)}(2,n)$	0	1	1	3	3	9	9	27	27	81	81	273	273
$P^{(2)}(3,n)$	0	1	1	1	3	3	5	9	11	19	29	41	67
$P^{(2)}(4,n)$	0	1	1	1	1	3	3	5	5	11	11	21	21
$P^{(2)}(5,n)$	0	1	1	1	1	1	3	3	5	5	7	11	13
$P^{(2)}(6,n)$	0	1	1	1	1	1	1	3	3	5	5	7	7
$P^{(2)}(7,n)$	0	1	1	1	1	1	1	1	3	3	5	5	7

It is easy to observe that for $i = 1,2$ the number $P^{(i)}(1,n)$ generalizes the classical Pell numbers, i.e for all $n \geq 0$ we have

$$P^{(1)}(1,n) = P^{(2)}(1,n) = P_n.$$

Moreover, for $k = 4$, we get another numbers of the Fibonacci type, namely the Jacobsthal numbers J_n defined by $J_n = J_{n-1} + 2J_{n-2}$ for $n \geq 2$ with $J_0 = J_1 = 1$. For all $n \geq 2$ we have

$$P^{(1)}(4,n) = \begin{cases} 0 & \text{if } n \text{ is even,} \\ J_{\frac{n+3}{2}} & \text{if } n \text{ is odd,} \end{cases}$$

and $P^{(2)}(4,n) = P^{(2)}(4,n+1) = J_{\frac{n+3}{2}}$ for n odd.

The following theorem shows the relation between numbers $P^{(i)}(k,n)$ for $i = 1,2$.

Theorem 1 *Let $k \geq 2$, $n \geq 1$ be integers. Then*

$$P^{(2)}(k,n) = P^{(1)}(k,n) + P^{(1)}(k,n-1).$$

Proof. Let $n = 1, 2, \ldots, k-1$. For $k = 2$ the result is obvious. If $k \geq 3$ then from initial conditions for $P^{(1)}(k,n)$ and $P^{(2)}(k,n)$ we have that $P^{(2)}(k,t) = P^{(1)}(k,t) + P^{(1)}(k,t-1) = 1$. Assume now that $n \geq k-1$ and that the equality

$$P^{(2)}(k,t) = P^{(1)}(k,t) + P^{(1)}(k,t-1)$$

holds for all integers $k \leq t \leq n$. We shall prove that it is true for integer $t = n+1$. Using (1) and induction's assumption we obtain that

$$\begin{aligned}
&P^{(2)}(k,n+1) = 2P^{(2)}(k,n+1-k) + P^{(2)}(k,n-1) = \\
&= 2\left(P^{(1)}(k,n+1-k) + P^{(2)}(k,n-k)\right) + P^{(1)}(k,n-1) + \\
&+ P^{(1)}(k,n-2) = P^{(1)}(k,n+1-k) + P^{(1)}(k,n-1) + P^{(2)}(k,n-k) + \\
&+ P^{(1)}(k,n-2) = P^{(1)}(k,n+1) + P^{(1)}(k,n),
\end{aligned}$$

which ends the proof. □

In this paper we give the graph interpretation of this two types of $(2,k)$-distance Pell numbers with respect to a special edge colouring of a graph. Next using these interpretations we obtain the direct formula for them. We also generalize some results given in [8].

2 (k_1A_1, k_2A_2, k_3A_3) - edge colouring by monochromatic paths in graph

Let G be an edge coloured graph with the set of colours $\{A_1, A_2, A_3\}$. Let $\eta \in \{A_1, A_2, A_3\}$. We say that a subgraph of G is η-monochromatic if all its edges are coloured alike by colour η. By $l(\eta)$ we denote the length of the η-monochromatic path and for $xy \in E(G)$ notation $\eta(xy)$ means the colour η of this edge.

Let $k_i \geq 1$, $i = 1, 2, 3$, be integers. We define a (k_1A_1, k_2A_2, k_3A_3)-*edge colouring by monochromatic paths* in a graph G in such a way that every maximal (with respect to set inclusion) A_i-monochromatic subgraph of G can be partitioned into edge-disjoint paths of the length k_i, $i = 1, 2, 3$. This type of edge-colouring of a graph generalizes edge-colouring introduced and studied by K. Piejko and I. Włoch in [8].

In this paper we consider a special case of a (k_1A_1, k_2A_2, k_3A_3)-edge colouring by monochromatic paths putting $k_1 = k_2 = k$, $k \geq 2$, and $k_3 = 2$. We will derive a formula for the number of all $(kA_1, kA_2, 2A_3)$-edge colourings by monochromatic paths of path $\mathcal{P}_n$. The number of $(kA_1, kA_2, 2A_3)$-edge colourings by monochromatic paths is closely related to $(2, k)$-distance Pell numbers $P^{(i)}(k, n)$, $i = 1, 2$.

Theorem 2 *Let $k \geq 1$, $n \geq 2$ be integers. The number of all $(kA_1, kA_2, 2A_3)$-edge colourings by monochromatic paths of the graph $\mathcal{P}_n$ is equal to $P^{(1)}(k, n)$.*

Proof. Let $V(\mathcal{P}_n)$ be the set of vertices of a graph $\mathcal{P}_n$ with the numbering in the natural fashion. Then the set of edges of this graph is $E(\mathcal{P}_n) = \{x_1x_2, x_2x_3, \ldots, x_{n-1}x_n\}$. Let us denote the number of all $(kA_1, kA_2, 2A_3)$-edge colourings by monochromatic paths of the graph $\mathcal{P}_n$ by $\sigma(k, n)$. By inspection we can verify that $\sigma(k, n) = P^{(1)}(k, n)$ for $n = 2, \ldots, k + 2$. Let now $n \geq k + 3$. We denote by $\sigma_{A_1}(k, n)$, $\sigma_{A_2}(k, n)$, $\sigma_{A_3}(k, n)$, the number of $(kA_1, kA_2, 2A_3)$-edge colourings by monochromatic paths of the graph $\mathcal{P}_n$ for which $\eta(x_{n-1}x_n) = A_1$, $\eta(x_{n-1}x_n) = A_2$, $\eta(x_{n-1}x_n) = A_3$, respectively. It is obvious that $\sigma_{A_1}(k, n)$, $\sigma_{A_2}(k, n)$ are equal to the number of all $(kA_1, kA_2, 2A_3)$-edge colourings by monochromatic paths of the graph $\mathcal{P}_{n-k}$ and $\sigma_{A_3}(k, n)$ is equal to the number of all $(kA_1, kA_2, 2A_3)$-edge colourings by monochromatic paths of the graph $\mathcal{P}_{n-2}$. This means that $\sigma_{A_1}(k, n) = \sigma(k, n-k)$, $\sigma_{A_2}(k, n) = \sigma(k, n-k)$ and $\sigma_{A_3}(k, n) = \sigma(k, n-2)$. Consequently

$$\sigma(k, n) = \sigma_{A_1}(k, n) + \sigma_{A_2}(k, n) + \sigma_{A_3}(k, n)$$

and therefore we get

$$\sigma(k, n) = 2\sigma(k, n - k) + \sigma(k, n - 2).$$

Having regard to the initial conditions we observe that

$$\sigma(k, n) = P^{(1)}(k, n)$$

for all $n \geq 2$. The proof is thus completed. □

Corollary 1 *For all integer $n \geq 2$ the number of all $(A_1, A_2, 2A_3)$-edge colourings of the graph $\mathcal{P}_n$ is equal to P_n.*

In the same manner as Theorem 2 we can prove the following theorem.

Theorem 3 *Let $k \geq 2$, $n \geq 2$ be integers. The number of all $(kA_1, kA_2, 2A_3)$-edge colourings by monochromatic paths of the graph $\mathcal{P}_n$ in which the last edge may be uncoloured is equal to $P^{(2)}(k, n)$.*

This graph interpretation gives a tools to obtain the direct formula for $(2, k)$-distance Pell numbers. Since they are given by the kth order linear recurrence relation, so for an arbitrary $k \geq 2$ the classical methods can not be used.

Firstly we need some preliminary results. Consider an arbitrary edge-colouring by monochromatic paths of the graph $\mathcal{P}_n$ using colours A_1, A_2, A_3 where exactly t edges, $t \geq 0$ are coloured by colours A_1 or A_2 and others edges have the colour A_3. Let $\sigma_{A_3}(n, t)$ denote the number of all such edge-colourings. Then we have

Theorem 4 *Let $n \geq 2$ and $0 \leq t \leq n - 1$ be integers. Then*

$$\sigma_{A_3}(n, t) = \binom{n-1}{t} 2^t.$$

Proof. Let us note that the graph $\mathcal{P}_n$ has $n - 1$ edges. Therefore if $t = 0$, then there is only one possibility of such edge-colouring of the graph $\mathcal{P}_n$, namely all edges of $\mathcal{P}_n$ have the colour A_3. Then we have $\sigma_{A_3}(n, 0) = 1 = \binom{n-1}{0} 2^0$. If $t \geq 1$, then there is $\binom{n-1}{t}$ possibilities of choosing of t edges that will be coloured by colours A_1 or A_2. Let us consider such singular choice of t edges. There is 2^t manners of locations of colours A_1 or A_2 in these t places. Thus the number $\sigma_{A_3}(n, t)$ is equal to $\binom{n-1}{t} 2^t$, what completes the proof. □

Let $k \geq 1$, $n \geq 2$, $0 \leq t \leq \lfloor \frac{n-1}{k} \rfloor$ be integers and let $\sigma_{2A_3}(n, k, t)$ denote the number of $(kA_1, kA_2, 2A_3)$ - edge colourings by monochromatic paths of the graph $\mathcal{P}_n$ such that there are exactly t monochromatic paths coloured by A_i, $i = 1, 2$. Then $t_1 k$ edges of $\mathcal{P}_n$ have colour A_1, $t_2 k$ edges have colour A_2 where $t_1 + t_2 = t$ and other $n - 1 - tk$ edges are coloured by colour A_3. Clearly $\sigma_{2A_3}(n, k, t) > 0$ if $n - 1 - tk$ is even.

Theorem 5 *Let $n \geq 2$, $0 \leq t \leq n - 1$ be integers and $n - 1 - t$ be even. Then*

$$\sigma_{A_3}\left(\frac{n+1+t}{2}, t\right) = \sigma_{2A_3}(n,1,t) = \binom{\frac{n-1+t}{2}}{t}2^t.$$

Proof. Let us consider the graph $\mathcal{P}_n$ with the $(A_1, A_2, 2A_3)$-edge colouring by monochromatic paths. Then we can observe that $\sigma_{2A_3}(n,1,t)$ is equal to the number of (A_1, A_2, A_3)-edge colourings of the graph $\mathcal{P}_{n-\frac{n-1-t}{2}}$ by monochromatic paths, where exactly t edges, $t \geq 0$ have the colour A_1 or A_2 and other edges are coloured by colour A_3. Using Theorem 4 the proof is completed. □

By Theorem 5 we obtain

Corollary 2 *Let $n \geq 2$ and $0 \leq t \leq n-1$ be integers. Then*

$$\sigma_{2A_3}(n,1,t) = \begin{cases} \binom{\frac{n-1+t}{2}}{t}2^t & \text{if} \quad n-1-t \quad \text{is even} \\ 0 & \text{if} \quad n-1-t \quad \text{is odd.} \end{cases}$$

Theorem 6 *Let $k \geq 1$, $n \geq 2$, $0 \leq t \leq \lfloor\frac{n-1}{k}\rfloor$ be integers. For all $s = 0,1,2,\ldots,k-1$ we have $\sigma_{2A_3}(n,k,t) = \sigma_{2A_3}(n-ts, k-s, t)$.*

Proof. Consider a $(kA_1, kA_2, 2A_3)$-edge colouring by monochromatic paths of a graph $\mathcal{P}_n$. Let α be a partition of the set $E(\mathcal{P}_n)$ into edge-disjoint A_i monochromatic paths of length k for $i = 1,2$ or 2 for $i = 3$, respectively. Assume that there are t monochromatic paths of the length k in this partition, then $0 \leq t \leq \lfloor\frac{n-1}{k}\rfloor$.

If every A_i-monochromatic path of the length k, $i = 1,2$, is shorted to a monochromatic path of the length $k-s$ then we obtain the graph $\mathcal{P}_{n-ts}$ such that $t(k-s)$ edges are coloured by A_1 or A_2 and the number of edges coloured by A_3 is the same as in the starting graph. Thus the desired equality follows which ends the proof. □

For the number $\sigma_{2A_3}(n,k,t)$ we can give the recurrence and the direct formula using the previous results.

Corollary 3 *Let $k \geq 1$, $n \geq 2$, $0 \leq t \leq \lfloor\frac{n-1}{k}\rfloor$ be integers. Then*

(i) $\sigma_{2A_3}(n,k,t) = \sigma_{2A_3}(n-t(k-1), 1, t)$,

(ii) $$\sigma_{2A_3}(n,k,t) = \begin{cases} \binom{\frac{1}{2}[n-1-t(k-2)]}{t}2^t & \text{if} \quad n-1-t(k-2) \quad \text{is even,} \\ 0 & \text{if} \quad n-1-t(k-2) \quad \text{is odd.} \end{cases}$$

The next theorem gives the direct formula for the numbers $P^{(1)}(k,n)$.

Theorem 7 *Let $k \geq 1$, $n \geq 2$, $0 \leq t \leq \lfloor\frac{n-1}{k}\rfloor$ be integers. Then*

$$P^{(1)}(k,n) = \begin{cases} 0 & \text{for even } k \text{ and even } n, \\ \sum_{l=0}^{\lfloor\frac{n-1}{k}\rfloor} \binom{\frac{1}{2}[n-1-l(k-2)]}{l} 2^l & \text{for even } k \text{ and odd } n, \\ \sum_{l=0}^{\lfloor\frac{n-1}{2k}\rfloor} \binom{\frac{1}{2}[n-1-2l(k-2)]}{2l} 2^{2l} & \text{for odd } k \text{ and odd } n, \\ \sum_{l=0}^{\lfloor\frac{n-1}{2k}\rfloor} \binom{\frac{1}{2}[n-1-(2l+1)(k-2)]}{2l+1} 2^{2l+1} & \text{for odd } k \text{ and even } n. \end{cases}$$

Proof. Let us recall that $\sum_{t\geq 0} \sigma_{2A_3}(n,k,t)$ is equal to the number of all $(kA_1, kA_2, 2A_3)$-edge colourings of the graph $\mathcal{P}_n$ by monochromatic paths. In view of Theorem 2 we have $\sum_{t\geq 0} \sigma_{2A_3}(n,k,t) = P^{(1)}(k,n)$ where $\sigma_{2A_3}(n,k,t)$ is given in the equality (ii) of Corollary 3.

Note first that t as a number of paths of the length k in the partition of $\mathcal{P}_n$, can be at most $\lfloor\frac{n-1}{k}\rfloor$. The number $\frac{1}{2}[n-1-t(k-2)]$ has to be integer and consequently $n-1-t(k-2)$ has to be even.

Note that for every even n and even k the number $n-1-t(k-2)$ is odd for all values of $t \geq 0$. Therefore by Corollary 3 we have that

$$P^{(1)}(k,n) = \sum_{l=0}^{\lfloor\frac{n-1}{k}\rfloor} \sigma_{2A_3}(n,k,t) = 0$$

for even k and even n. If k is even and n is odd then $n-1-t(k-2)$ is even for all $t \geq 0$. Thus we have

$$P^{(1)}(k,n) = \sum_{l=0}^{\lfloor\frac{n-1}{k}\rfloor} \binom{\frac{1}{2}[n-1-l(k-2)]}{l} 2^l.$$

Now let us consider the case when k and n are both odd. Then the number $n-1-t(k-2)$ is even only if t is even and consequently

$$P^{(1)}(k,n) = \sum_{t=0,\ t \text{ even}}^{\lfloor\frac{n-1}{k}\rfloor} \binom{\frac{1}{2}[n-1-t(k-2)]}{t} 2^t =$$
$$= \sum_{l=0}^{\lfloor\frac{n-1}{2k}\rfloor} \binom{\frac{1}{2}[n-1-2l(k-2)]}{2l} 2^{2l}.$$

Finally, if k is odd and n is even, then the number $n-1-t(k-2)$ is even only if t is odd. Then we have that

$$P^{(1)}(k,n) = \sum_{t=0,\ t \text{ odd}}^{\lfloor \frac{n-1}{k} \rfloor} \binom{\frac{1}{2}[n-1-t(k-2)]}{t} 2^t =$$
$$= \sum_{l=0}^{\lfloor \frac{n-1}{2k} \rfloor} \binom{\frac{1}{2}[n-1-(2l+1)(k-2)]}{2l+1} 2^{2l+1}.$$

The proof is thus completed. □

Note that putting $k = 1$ in Theorem 7 we obtain the following direct formula for nth Pell number i.e

$$P_n = \begin{cases} \sum_{l=0,\ l \text{ even}}^{n-1} \binom{\frac{1}{2}(n-1+l)}{l} 2^l & \text{if } n \text{ is odd,} \\ \sum_{l=1,\ l \text{ odd}}^{n-1} \binom{\frac{1}{2}(n-1+l)}{l} 2^l & \text{if } n \text{ is even.} \end{cases}$$

Using Theorems 1 and 7 we can also give the direct formula for the numbers $P^{(2)}(k,n)$.

Theorem 8 *Let $k \geq 1$, $n \geq 3$, $0 \leq t \leq \lfloor \frac{n-1}{k} \rfloor$ be integers. Then*

$$P^{(2)}(k,n) = \begin{cases} \sum_{l=0}^{\lfloor \frac{n-2}{k} \rfloor} \binom{\frac{1}{2}[n-2-l(k-2)]}{l} 2^l & \textit{for even } k \textit{ and even } n, \\ \sum_{l=0}^{\lfloor \frac{n-1}{k} \rfloor} \binom{\frac{1}{2}[n-1-l(k-2)]}{l} 2^l & \textit{for even } k \textit{ and odd } n, \\ \sum_{l=0}^{\lfloor \frac{n-1}{2k} \rfloor} \binom{\frac{1}{2}[n-1-2l(k-2)]}{2l} 2^{2l} + \\ + \sum_{l=0}^{\lfloor \frac{n-2}{2k} \rfloor} \binom{\frac{1}{2}[n-2-(2l+1)(k-2)]}{2l+1} 2^{2l+1} & \textit{for odd } k \textit{ and odd } n, \\ \sum_{l=0}^{\lfloor \frac{n-2}{2k} \rfloor} \binom{\frac{1}{2}[n-2-2l(k-2)]}{2l} 2^{2l} + \\ + \sum_{l=0}^{\lfloor \frac{n-1}{2k} \rfloor} \binom{\frac{1}{2}[n-1-(2l+1)(k-2)]}{2l+1} 2^{2l+1} & \textit{for odd } k \textit{ and even } n. \end{cases}$$

References

[1] U. Bednarz, D. Bród, M. Wołowiec-Musiał, *On two types of $(2,k)$-distance Lucas numbers,* Ars Combinatoria 115(2014), 467-479.

[2] C. Berge, Principles of Combinatorics, Academic Press New York and London (1971).

[3] R. Diestel, Graph Theory, Springer-Verlag, Heidelberg, New York, Inc., 2005.

[4] I. Gutman, S. Wagner, *Maxima and minima of the Hosaya index and the Merrifield-Simmons index, a survay of result and techniques,* Acta Applicandae Mathematicae 112 3(2010), 323-346.

[5] D. Jakubczyk, *The exact results in the one-dimensional attractive Hubbard model,* Acta Physica Polonica B No 8, 45(2014), 1759-1770.

[6] E. Kiliç and D. Tasci, *The generalized Binet formula, reprezentation and sums of the generalized order-k Pell numbers,* Taiwanese Journal of Mathematics 10(6)(2006), 1661-1670.

[7] E. Kiliç, *The generalized order k-Fibonacci-Pell sequence by matrix methods,* Journal of Computational and Applied Mathemetics 209(2007), 133-145.

[8] K. Piejko, I. Włoch, *On k-distance Pell numbers in 3-edge-coloured graphs,* Journal of Applied Mathematics, (2014), Article ID 428020, 6 pages.

[9] I. Włoch, *On generalized Pell numbers and their graph representations,* Commentationes Mathematicae 48(2)(2008), 169-175.

[10] A. Włoch and M. Wołowiec-Musiał, *Generalized Pell numbers and some relations with Fibonacci numbers,* Ars Combinatoria (2013).

[11] I. Włoch, U. Bednarz, D. Bród, A. Włoch, M. Wołowiec-Musiał, *On a new type of distance Fibonacci numbers,* Discrete Applied Mathematics 161(2013), 2695-2701.