SOME NEW FINITE SUMS INVOLVING GENERALIZED FIBONACCI AND LUCAS NUMBERS

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ABSTRACT. In this paper, we compute various finite sums that alternate according to $(-1)^{\binom{n}{k}}$ involving the generalized Fibonacci and Lucas numbers for $k = 3, 4, 5$ and even k of the form 2^m with $m > 1$.

1. INTRODUCTION

Define the second order linear sequences $\{U_n\}$ and $\{V_n\}$ for $n \geq 2$ by

$$U_n = pU_{n-1} + U_{n-2}, \quad U_0 = 0, \ U_1 = 1,$$
$$V_n = pV_{n-1} + V_{n-2}, \quad V_0 = 2, \ V_1 = p.$$

When $p = 1$, then $U_n = F_n$ (nth Fibonacci number) and $V_n = L_n$ (nth Lucas number). Also when $p = 2$, then $U_n = P_n$ (nth Pell number) and $V_n = Q_n$ (nth Pell-Lucas number).

Also the Binet formulas of the sequences $\{U_n\}$ and $\{V_n\}$ are

$$U_n = \frac{\alpha^n - \beta^n}{\alpha - \beta} \quad \text{and} \quad V_n = \alpha^n + \beta^n,$$

where $\alpha, \beta = (p \pm \sqrt{\Delta})/2$.

Many author computed sums and alternating sums involving generalized Fibonacci and Lucas numbers as well as their certain products. For example, Melham [7] computed various finite non-alternating sums, alternating sums and sums that alternate according to $(-1)^{\binom{n}{2}}$ for certain generalized Fibonacci and Lucas sequences. In his result, the sums that alternate according to $(-1)^{\binom{n}{2}}$ are interesting because this sign function was firstly used in such sums. Here we recall one example for the readers:

$$\sum_{n=0}^{4j+3} (-1)^{\binom{n+1}{2}} F_{2n} = \frac{1}{3} F_{4j+4} L_{4j+3}.$$

The authors [3] presented generalized results on non-alternating sums, alternating sums and sums that alternate according to $(-1)^{\binom{n}{2}}$ by considering the results of [7]. We may also refer [2, 4, 5, 6, 8, 9] for some other known results.

In this paper, by considering early used sign function $(-1)^{\binom{n}{2}}$, we will derive new sums that alternate according to $(-1)^{\binom{n}{3}}$, $(-1)^{\binom{n}{4}}$, $(-1)^{\binom{n}{5}}$ and $(-1)^{\binom{n}{2^m}}$ for the generalized Fibonacci sequence $\{U_n\}$ and Lucas sequence $\{V_n\}$.

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2. Sums of the generalized Fibonacci and Lucas numbers

We are interested to evaluate

$$\sum_{k=0}^{n} X_{kr}\,(-1)^{\binom{k}{3}}$$

in closed form where r and n are integers, and X_n either U_n or V_n.

Now we will give our first result:

Theorem 1. *For any integer r,*

$$\sum_{k=0}^{n}(-1)^{\binom{k}{3}}\,U_{kr} = \frac{U_{nr}-(-1)^r\left(U_{(n+1)r}-U_r\right)}{2\,[r \text{ is even}]-(-1)^r\,V_r} -2\frac{U_{nr}-U_{(n+4)r}+U_{3r}+(-1)^r\,U_r}{2-V_{4r}},$$

where [ ] *is the Iverson's notation defined as in* [1].

Proof. Consider

$$\begin{aligned}
&\sum_{k=0}^{n}(-1)^{\binom{k}{3}}\,U_{kr}\\
&= \sum_{k=0}^{n}U_{kr}-2\sum_{k=0}^{(n-3)/4}U_{(4k+3)r}\\
&= \frac{1}{\alpha-\beta}\left(\sum_{k=0}^{n}\left(\alpha^{kr}-\beta^{kr}\right)-2\sum_{k=0}^{(n-3)/4}\alpha^{(4k+3)r}+2\sum_{k=0}^{(n-3)/4}\beta^{(4k+3)r}\right)\\
&= \frac{1}{\alpha-\beta}\left(\frac{1-\alpha^{(n+1)r}}{1-\alpha^r}-\frac{1-\beta^{(n+1)r}}{1-\beta^r}-2\frac{\alpha^{3r}-\alpha^{(n+4)r}}{1-\alpha^{4r}}+2\frac{\beta^{3r}-\beta^{(n+4)r}}{1-\beta^{4r}}\right)\\
&= \frac{1}{\alpha-\beta}\frac{\left(-\alpha^{(n+1)r}+\beta^{(n+1)r}+\alpha^{(n+1)r}\beta^r-\alpha^r\beta^{(n+1)r}+\alpha^r-\beta^r\right)}{(1-\alpha^r)(1-\beta^r)}\\
&\quad -\frac{2}{\alpha-\beta}\left(\frac{-\alpha^{(n+4)r}+\beta^{(n+4)r}+\alpha^{(n+4)r}\beta^{4r}-\alpha^{4r}\beta^{(n+4)r}}{(1-\alpha^{4r})(1-\beta^{4r})}\right.\\
&\quad \left.+\frac{\alpha^{3r}-\beta^{3r}-\alpha^{3r}\beta^{4r}+\alpha^{4r}\beta^{3r}}{(1-\alpha^{4r})(1-\beta^{4r})}\right),
\end{aligned}$$

which, by $\alpha\beta=-1$, equals

$$\begin{aligned}
&= \frac{1}{\alpha-\beta}\frac{-\alpha^{(n+1)r}+\beta^{(n+1)r}+(\alpha\beta)^r(\alpha^{nr}-\beta^{nr})+\alpha^r-\beta^r}{(\alpha\beta)^r-(\alpha^r+\beta^r)+1} \\
&\quad -\frac{2}{\alpha-\beta}\frac{-\alpha^{(n+4)r}+\beta^{(n+4)r}+(\alpha\beta)^{4r}(\alpha^{nr}-\beta^{nr})}{(\alpha\beta)^{4r}-\left(\alpha^{4r}+\beta^{4r}\right)+1} \\
&\quad -\frac{2}{\alpha-\beta}\frac{\alpha^{3r}-\beta^{3r}+(\alpha\beta)^{3r}(\alpha^r-\beta^r)}{(\alpha\beta)^{4r}-\left(\alpha^{4r}+\beta^{4r}\right)+1} \\
&= \frac{-U_{(n+1)r}+(-1)^r U_{nr}+U_r}{(-1)^r-V_r+1}-2\frac{-U_{(n+4)r}+U_{nr}+U_{3r}+(-1)^r U_r}{2-V_{4r}} \\
&= \begin{cases} \dfrac{U_{nr}-U_{(n+1)r}+U_r}{2-V_r}-2\dfrac{U_{nr}-U_{(n+4)r}+U_{3r}+U_r}{2-V_{4r}} & \text{if } r \text{ is even}, \\[2ex] \dfrac{U_{nr}+U_{(n+1)r}-U_r}{V_r}-2\dfrac{U_{nr}-U_{(n+4)r}+U_{3r}-U_r}{2-V_4} & \text{if } r \text{ is odd}, \end{cases}
\end{aligned}$$

as claimed.

□

For example, as consequences of our result, when $U_n=F_n$ and $V_n=L_n$, we get for $r=1,2$ and 3

$$\sum_{k=0}^{4n+3}(-1)^{\binom{k}{3}}F_k=F_{2n+2}F_{2n},$$

$$\sum_{k=0}^{4n+3}(-1)^{\binom{k}{3}}F_{2k}=-\frac{5F_{8n+6}+L_{8n+5}+9}{15}$$

and

$$\sum_{k=0}^{4n+3}(-1)^{\binom{k}{3}}F_{3k}=\frac{-25F_{12n+9}+8L_{12n+8}-6}{20},$$

respectively.

When $U_n=P_n$, we get for $r=1$,

$$\sum_{k=0}^{4n+3}(-1)^{\binom{k}{3}}P_k=-P_{2n+2}P_{2n+1}.$$

Theorem 2. *For any integer r,*

$$\begin{aligned}
\sum_{k=0}^{n}(-1)^{\binom{k}{3}}V_{kr} &= \frac{V_{nr}-(-1)^r\left(V_{(n+1)r}+V_r-2\right)}{2\,[r\text{ is even}]-(-1)^r V_r} \\
&\quad -2\frac{V_{nr}-V_{(n+4)r}+V_{3r}-(-1)^r V_r}{2-V_{4r}},
\end{aligned}$$

where [ ] is the Iverson's notation defined as in [1].

Proof. Consider

$$\sum_{k=0}^{n}(-1)^{\binom{k}{3}}V_{kr}=\sum_{k=0}^{n}V_{kr}-2\sum_{k=0}^{(n-3)/4}V_{(4k+3)r}$$

$$\begin{aligned}
&= \sum_{k=0}^{n}\left(\alpha^{kr}+\beta^{kr}\right)-2\alpha^{3r}\sum_{k=0}^{(n-3)/4}\alpha^{4kr}-2\beta^{3r}\sum_{k=0}^{(n-3)/4}\beta^{4kr}\\
&= \frac{1-\alpha^{(n+1)r}}{1-\alpha^{r}}+\frac{1-\beta^{(n+1)r}}{1-\beta^{r}}-2\frac{\alpha^{3r}-\alpha^{(n+4)r}}{1-\alpha^{4r}}-2\frac{\beta^{3r}-\beta^{(n+4)r}}{1-\beta^{4r}}\\
&= \frac{\alpha^{(n+1)r}\beta^{r}+\alpha^{r}\beta^{(n+1)r}-\alpha^{(n+1)r}-\beta^{(n+1)r}-\alpha^{r}-\beta^{r}+2}{(1-\alpha^{r})(1-\beta^{r})}\\
&\quad -2\frac{\alpha^{(n+4)r}\beta^{4r}+\alpha^{4r}\beta^{(n+4)r}-\alpha^{(n+4)r}-\beta^{(n+4)r}}{(1-\alpha^{4r})(1-\beta^{4r})}\\
&\quad -2\frac{\alpha^{3r}+\beta^{3r}-\alpha^{3r}\beta^{4r}-\alpha^{4r}\beta^{3r}}{(1-\alpha^{4r})(1-\beta^{4r})},
\end{aligned}$$

which, by $\alpha\beta=-1$, equals

$$\begin{aligned}
&= \frac{-\alpha^{(n+1)r}-\beta^{(n+1)r}+(\alpha\beta)^{r}(\alpha^{nr}+\beta^{nr})-(\alpha^{r}+\beta^{r})+2}{(\alpha\beta)^{r}-(\alpha^{r}+\beta^{r})+1}\\
&\quad -2\frac{(\alpha\beta)^{4r}(\alpha^{nr}+\beta^{nr})-\alpha^{(n+4)r}-\beta^{(n+4)r}}{(\alpha\beta)^{4r}-\left(\alpha^{4r}+\beta^{4r}\right)+1}\\
&\quad -2\frac{\alpha^{3r}+\beta^{3r}-(\alpha\beta)^{3r}(\alpha^{r}+\beta^{r})}{(\alpha\beta)^{4r}-\left(\alpha^{4r}+\beta^{4r}\right)+1}\\
&= \frac{-V_{(n+1)r}+(-1)^{r}V_{nr}-V_{r}+2}{(-1)^{r}-V_{r}+1}-2\frac{V_{nr}-V_{(n+4)r}+V_{3r}-(-1)^{r}V_{r}}{2-V_{4r}},
\end{aligned}$$

which, by the cases of r gives

$$\sum_{k=0}^{n}(-1)^{\binom{k}{3}}V_{kr}
=\begin{cases}
\dfrac{-V_{(n+1)r}+V_{nr}-V_{r}+2}{2-V_{r}}-2\dfrac{V_{nr}-V_{(n+4)r}+V_{3r}-V_{r}}{2-V_{4r}} & \text{if } r \text{ is even},\\[2ex]
\dfrac{V_{(n+1)r}+V_{nr}+V_{r}-2}{V_{r}}-2\dfrac{V_{nr}-V_{(n+4)r}+V_{3r}+V_{r}}{2-V_{4r}} & \text{if } r \text{ is odd},
\end{cases}$$

as claimed. □

For example, when $U_n=F_n$ and $V_n=L_n$, we get for $r=1,2$ and 3

$$\sum_{k=0}^{4n+3}L_k(-1)^{\binom{k}{3}}=F_{2n+2}L_{2n},$$

$$\sum_{k=0}^{4n+3}L_{2k}(-1)^{\binom{k}{3}}=\frac{-L_{8n+6}-F_{8n+5}+5}{3}$$

and

$$\sum_{k=0}^{4n+3} L_{3k}\,(-1)^{\binom{k}{3}} = \frac{-9L_{12n+9} - 8L_{12n+8} + 20}{20},$$

respectively.

When $U_n = P_n$ and so $V_n = Q_n$, we get for $r = 1$,

$$\sum_{k=0}^{4n+3} (-1)^{\binom{k}{3}}\, Q_k = -P_{2n+2}Q_{2n+1}.$$

Now we give finite sums that alternate according to $(-1)^{\binom{k}{4}}$ for generalized Fibonacci and Lucas sequences.

Theorem 3. *For even* r,

$$\begin{aligned}\sum_{k=0}^{n} (-1)^{\binom{k}{4}}\, U_{kr} &= \frac{U_{nr} - U_{(n+1)r} + U_r}{2 - V_r} \\ &\quad - \frac{2U_{4r}V_r}{2 - V_{8r}}\left(V_r + V_{2r} - V_{(n+2)r} - V_{(n+3)r}\right),\end{aligned}$$

$$\begin{aligned}\sum_{k=0}^{n} (-1)^{\binom{k}{4}}\, V_{kr} &= \frac{V_{nr} - V_{(n+1)r} - V_r + 2}{2 - V_r} \\ &\quad - \frac{2U_{4r}V_r}{2 - V_{8r}}\left(U_r + U_{2r} - U_{(n+2)r} - U_{(n+3)r}\right)\end{aligned}$$

and for odd r,

$$\begin{aligned}\sum_{k=0}^{n} (-1)^{\binom{k}{4}}\, U_{kr} &= \frac{U_{nr} + U_{(n+1)r} - U_r}{V_r} \\ &\quad - \frac{2\Delta U_r U_{4r}}{2 - V_{8r}}\left(U_r + U_{2r} - U_{(n+2)r} - U_{(n+3)r}\right),\end{aligned}$$

$$\begin{aligned}\sum_{k=0}^{n} (-1)^{\binom{k}{4}}\, V_{kr} &= \frac{V_{nr} + V_{(n+1)r} + V_r - 2}{V_r} \\ &\quad - \frac{2\Delta U_r U_{4r}}{2 - V_{8r}}\left(V_r + V_{2r} - V_{(n+2)r} - V_{(n+3)r}\right).\end{aligned}$$

The next result present sums that alternate according to $(-1)^{\binom{k}{5}}$.

Theorem 4. *For even r*

$$\sum_{k=0}^{n}(-1)^{\binom{k}{5}}U_{kr}=\frac{U_{nr}-U_{(n+1)r}+U_r}{2-V_r}-\frac{2U_{4r}V_r}{2-V_{8r}}\left(V_{2r}-V_{(n+3)r}\right),$$

$$\sum_{k=0}^{n}(-1)^{\binom{k}{5}}V_{kr}=\frac{V_{nr}-V_{(n+1)r}-V_r+2}{2-V_r}-\frac{2\Delta U_{4r}V_r}{2-V_{8r}}\left(U_{2r}-U_{(n+3)r}\right)$$

and for odd r,

$$\sum_{k=0}^{n}(-1)^{\binom{k}{5}}U_{kr}=\frac{U_{nr}+U_{(n+1)r}-U_r}{V_r}-\frac{2\Delta U_{4r}U_r}{2-V_{8r}}\left(U_{2r}-U_{(n+3)r}\right),$$

$$\sum_{k=0}^{n}(-1)^{\binom{k}{5}}V_{kr}=\frac{V_{nr}+V_{(n+1)r}+V_r-2}{V_r}-\frac{2\Delta U_{4r}U_r}{2-V_{8r}}\left(V_{2r}-V_{(n+3)r}\right).$$

For example we get

$$\sum_{k=0}^{8n+7}(-1)^{\binom{k}{5}}F_k=-\frac{1}{3}F_{4n+4}L_{4n+2}$$

and

$$\sum_{k=0}^{8n+7}(-1)^{\binom{k}{5}}L_k=-\frac{5}{3}F_{4n+4}F_{4n+2}.$$

Finally we give a general result about the sums alternate according to $(-1)^{\binom{n}{k}}$ where the summation index k of the form 2^m with $m>1$.

Theorem 5. *For any integer r,*

$$\sum_{k=0}^{n}(-1)^{\binom{k}{2^m}}U_{kr}=\frac{U_{nr}-(-1)^r\left(U_{(n+1)r}-U_r\right)}{2\left[r\text{ is even}\right]-(-1)^rV_r}$$
$$-\frac{2}{2-V_{2^{m+1}r}}\sum_{j=1}^{2^m}\left((-1)^{jr}U_{jr}+U_{(2^m-1+j)r}+U_{(n+1-j)r}-U_{(n+2^m+j)r}\right)$$

and

$$\sum_{k=0}^{n}(-1)^{\binom{k}{2^m}}V_{kr}=\frac{V_{nr}-(-1)^r\left(V_{(n+1)r}+V_r-2\right)}{2\left[\text{r is even}\right]-(-1)^rV_r}$$
$$-\frac{2}{2-V_{2^{m+1}r}}\sum_{j=1}^{2^m}\left((-1)^{jr-1}V_{jr}+V_{(2^m-1+j)r}+V_{(n+1-j)r}-V_{(n+2^m+j)r}\right).$$

As an example,

$$\sum_{k=0}^{2^{m+1}(n+1)-1} (-1)^{\binom{k}{2^m}} F_k = -\frac{\Delta F_{2^m-1}}{L_{2^m-1}} F_{2^m(n+1)} F_{2^m(n+1)+1},$$

$$\sum_{k=0}^{2^{m+1}(n+1)-1} (-1)^{\binom{k}{2^m}} L_k = -\frac{\Delta F_{2^m-1}}{L_{2^m-1}} F_{2^m(n+1)} L_{2^m(n+1)+1}.$$

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