Entire Chromatic Number of 1-Tree *

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Abstract

The entire chromatic number $\chi_c(G)$ of a plane graph $G(V, E, F)$ is the minimum number of colors such that any two distinct adjacent or incident elements receive different colors in $V(G) \cup E(G) \cup F(G)$. A plane graph G is called a 1-tree if there is a vertex $u \in V(G)$ such that $G - u$ is a forest. In the paper, it is proved that if G is a 2-connected 1-tree with $\Delta(G) \geq 6$, then the entire chromatic number of G is $\Delta(G) + 1$, where $\Delta(G)$ is the maximum degree of G.

Key words: plane graph; 1-tree; entire chromatic number

1 Introduction

In this paper, In this paper,all graphs are all simple plane graphs. Let $G(V, E, F)$ be a plane graph, where $V(G)$, $E(G)$, $F(G)$, $|G|$, $\Delta(G)$ and $\delta(G)$ are the vertex set,the edge set, the face set, the size, the maximum degree and the minimum degree of G, respectively. We use $N_G(u)$ to denote the neighbor set of a vertex u in G and $d(v) = |N_G(u)|$ to denote the degree of v. A *k-vertex* in a graph G is a vertex of degree k. Let $V_k(G)$ denote the set of all k-vertices in G, where $k = 0, 1, \cdots, \Delta(G)$. Two vertices in $V(G)$ are adjacent if they are joined by an edge, two edges in $E(G)$ are adjacent if they have a common vertex and two faces in $F(G)$ are adjacent if their boundaries have at least one common edge. We say that a vertex(an edge) is incident with a face if it forms a part of the boundary of the face. Also the vertices u and v are incident with the edge uv. A k-face is a face incident with k vertices. The other definitions and notations can be found in [1].

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Definition 1.1. *A plane graph G is called a* 1*-tree if there is a vertex $u \in V(G)$ such that $G - u$ is a forest, where u is called a crown of G. Clearly a tree must be a* 1*-tree. If a* 1*-tree G is not a tree, we call G a proper* 1*-tree.*

Definition 1.2. *A plane graph $G(V, E, F)$ is k-total (entire) colorable if the elements of $V(G) \cup E(G)(V(G) \cup E(G) \cup F(G))$ can be colored with k colors such that any two distinct adjacent or incident elements receive different colors. The total (entire) chromatic number $\chi_T(G)(\chi_c(G))$ is defined as the minimum number k for which G is k-total (entire) colorable.*

In 1973, Kronk and Mitchem [6] posed the following conjecture(ECC):

Conjecture 1. For every plane graph G, $\Delta(G) + 1 \leq \chi_c(G) \leq \Delta(G) + 4$.

The conjecture was proved for $\Delta \leq 3$[6], $\Delta = 4$ or 5 [11], $\Delta = 6$[8] and $\Delta \geq 7$[3]. So the conjecture has been proved completely. In fact when the maximum degree is very large this upper bound can be reduced. Borodin [2] proved $\chi_c(G) \leq \Delta(G) + 2$ for any plane graph G having $\Delta(G) \geq 12$. Zhang *et al.* [18] proved that a 2-connected outerplane graph G with $\Delta(G) \geq 7$ has the entire chromatic number $\Delta(G) + 1$. Wang [13] improved the result to $\Delta(G) \geq 6$. Borodin and Woodall [4] proved that an outerplane graph G with $\Delta(G) \geq 6$ satisfies $\chi_c(G) = \Delta(G) + 1$ or $\Delta(G) + 2$ according as G does or does not possess a special matching. Other related papers can be found in [7, 14, 9, 12, 13, 15, 16, 17, 18]. This paper will prove that if G is a 2-connected 1-tree with $\Delta(G) \geq 6$, then the entire chromatic number of G is $\Delta(G) + 1$.

2 A property of 2-connected 1-trees

First, we introduce some useful lemmas.

Lemma 2.1. [10] *If F is a forest, then $|V_1(F)| \geq \Delta(F)$.*

Lemma 2.2. [10] *If G is a* 1*-tree, then $\delta(G) \leq 2$.*

Let T be a tree with $\Delta(T) \geq 2$. A vertex v of T with $d_T(v) \geq 2$ is called a *sub-pendent vertex* if v is adjacent to at most one vertex of degree at least 2. Let $S(T)$ denote the set of all sub-pendent vertices of T. We have

$$S(T) = \bigcup_{0 \leq i \leq 1} V_i(T - V_1(T)).$$

Let Q_p denote a plane graph of order p with vertices u, v, x_1, x_2, $\cdots$, x_{p-2} and the edges ux_1, ux_2, $\cdots$, ux_{p-2}, vx_1, vx_2, $\cdots$, vx_{p-2}, and let $\overline{Q}_p = Q_p + uv$. Then $\Delta(Q_p) = p - 2$ and $\Delta(\overline{Q}_p) = p - 1$.

Lemma 2.3. [10] *If G is a tree with $\Delta(T) \geq 2$, then $|S(T)| \geq 1$. Moreover, $|S(T)| = 1$ iff T is a star, that is, $T \cong K_{1,p(G)-1}$.*

Lemma 2.4. *If G is a 2-connected 1-tree with $\Delta(G) \geq 3$, then at least one of the following results holds.*

(1) $G \in \{Q_p, \overline{Q}_p, p = 3, 4, ...\}$.

(2) *There are two adjacent 2-vertices x and u;*

(3) *There is a 4-face $uxvy$ such that $xy \notin E(G)$, $d_G(u) = d_G(v) = 2$, $d_G(y) = \Delta(G)$ and $3 \leq d_G(x) \leq \lceil \frac{\Delta(G)+1}{2} \rceil$;*

(4) *There is a 3-face uxy such that $d_G(u) = 2$, $d_G(x) = 3$ and $d_G(y) = \Delta(G)$;*

(5) *There are two adjacent 3-faces uxy and vxy such that $d_G(u) = d_G(v) = 2$, $d_G(y) = \Delta(G)$ and $4 \leq d_G(x) \leq \lceil \frac{\Delta(G)+1}{2} \rceil$;*

(6) *There is a a 4-face $uxvy$ adjacent to 3-face uxy such that $d_G(u) = d_G(v) = 2$, $d_G(y) = \Delta(G)$ and $4 \leq d_G(x) \leq \lceil \frac{\Delta(G)+1}{2} \rceil$.*

Proof. Let y be a crown of G and $T = G - y$. Then T is a tree. Since G is a 2-connected, $\delta(G) \leq 2$, $\Delta(T) \geq 2$ and $V_1(T) \subseteq N_G(y)$. By lemma 2.1, we have $d_G(y) = \Delta(G)$.

Suppose that T has only one vertex of degree at least 2, that is, $|S(T)| = 1$, then $T \cong K_{1,p(G)-1}$ by Lemma 2.3. Let $S(T) = \{w\}$. If $yw \notin E(G)$, then $G \cong Q_p$; otherwise $G \cong \overline{Q}_p$. This implies that (1) holds for G. So in the following, we assume that $|S(T)| \geq 2$.

Let x be a vertex in $S(T)$ such that $d_T(x) = \min\{d_T(z) | z \in S(T)\}$. It follow from $|S(T)| \geq 2$ and the minimality of x that $d_G(x) \leq \lceil \frac{d_G(t)+1}{2} \rceil \leq \lceil \frac{\Delta(G)+1}{2} \rceil$. We consider the following cases.

Case 1. $xy \notin E(G)$.

If $d_T(x) = 2$, then there exists a vertex $u \in V_1(T) \bigcap N_G(x) \bigcap N_G(y)$ such that $d_G(u) = 2$, which implies that (2) holds. Otherwise, there exists two vertices $u, v \in V_1(T) \bigcap N(x)$ such that $d_G(u) = d_G(v) = 2$ and u, x, v, y form a 4-face, which implies that (3) holds.

Case 2. $xy \in E(G)$.

Then $d_G(x) \geq 3$, and there exists a vertex $u \in V_1(T) \bigcap N_G(x) \bigcap N_G(y)$ such that $d_G(u) = 2$ and u, x, y form a 3-face. If $d_G(x) = 3$, then (4) holds. Otherwise, $d_G(x) \geq 4$. If xy is incident with two 3-faces, then (5) holds. Otherwise, there exists a vertex $v \in V_1(T) \bigcap N_G(x) \bigcap N_G(y)$ such that $d_G(v) = 2$ and u, x, v, y form a 4-face, which implies (6). □

3 The main results

Given an entire coloring σ of a plane graph G, we use x_σ denote the set of colors which are colored on the vertex x and its incident edges.

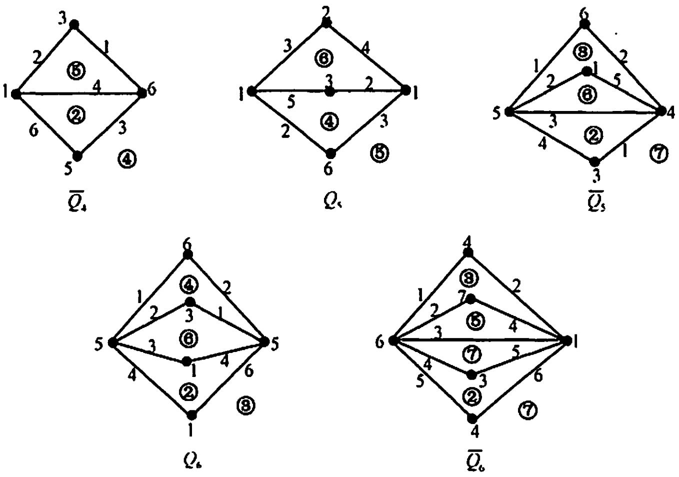
Lemma 3.1. [18] *For fan F_p with $p \geq 6$, we have $\chi_c(F_p) = p$.*

Lemma 3.2. [5] *Let C_n be a cycle of length n. We have*

$$\chi_T(C_n) = \begin{cases} 3, & \text{if } n \equiv 0(mod3); \\ 4, & \text{if } n \geq 3 \text{ and } n \not\equiv 0(mod3). \end{cases}$$

Lemma 3.3. *If $G \in \{Q_p, \overline{Q}_p : p \geq 4\}$, then*

$$\chi_c(G) = \begin{cases} 6, & \text{if } G \in \{Q_4, \overline{Q}_4, Q_5, Q_6\}, \\ 7, & \text{if } G \in \{\overline{Q}_5, \overline{Q}_6\}, \\ \Delta(G) + 1, & \text{if } p \geq 7. \end{cases}$$



Proof. The proof of the case $p \leq 6$ can be seen in Figure 1, we omit the detail here. So we assume that $p \geq 7$. Let $q = p-2$. we will give a $(\Delta(G)+1)$-entire coloring σ of G: $V(G) \cup E(G) \cup F(G) \mapsto C = \{1, 2, \cdots, \Delta(G) + 1\}$ as follows.

Suppose that $G \cong Q_p$ for some $p(p \geq 7)$. Then $\Delta(G) = q$. Let $\sigma(u) = \sigma(v) = q + 1$, and for any $i(1 \leq i \leq q)$, $\sigma(ux_i) = i$, $\sigma(vx_i) = i + 1$, $\sigma(x_i) = i + 2$, $\sigma(f_{ux_i vx_{i+1}}) = i + 4$, $\sigma(f_{out}) = 4$,where all the subscripts in the paper are taken modulo q.

Suppose that $G \cong \overline{Q}_p$ for some $p(p \geq 7)$. Then $\Delta(G) = q+1$. Without loss of generality, assume that u, v, x_1 form a triangle. Let $\sigma(uv) = q+2$, $\sigma(u) = q+1$, $\sigma(v) = 1$, $\sigma(ux_i) = i(1 \leq i \leq q)$, $\sigma(vx_i) = i+1(1 \leq i \leq q)$, $\sigma(x_i) = q+2(1 \leq i \leq q)$, $\sigma(f_{uvx_1}) = 3$, $\sigma(f_{ux_ivx_{i+1}}) = i+3$, $i = 1, ..., q-3$, $\sigma(f_{ux_{q-2}vx_{q-1}}) = 2$, $\sigma(f_{ux_{q-1}vx_q}) = 3$ and $\sigma(f_{out}) = 4$.

Hence, we get an entire coloring of G, and we complete the proof of the lemma. □

Lemma 3.4. *Let G be a 2-connected 1-tree and $\Delta(G) = 2$. Then $\chi_c(G) \leq 6$.*

Proof. Since G is 2-connected, G is a cycle and it is very easy to check that the result holds. □

Theorem 3.5. *Let t be an integer at least 7 and G be a 2-connected 1-tree with $\Delta(G) < t$. $\chi_c(G) \leq t$.*

Proof. The proof is carried out by induction on $|V(G)|+|E(G)|$. If $|V(G)|+|E(G)| \leq 8$, then $G = K_3$ or C_4, and the result is obvious. Let G be a 2-connected 1-tree with $|V(G)| + |E(G)| \geq 9$. In the following, we shall construct a new 2-connected 1-tree G' from G with $|V(G')| + |E(G')| < |V(G)| + |E(G)|$ and $\Delta(G') < t$. By the induction hypothesis, we extend a t-entire coloring ϕ of G' to an entire coloring φ of G with t colors, and we complete the proof of the theorem.

Let $S = \{1, 2, ..., t\}$. Note that $\lceil \frac{\Delta(G)+1}{2} \rceil \leq \lceil \frac{t+1}{2} \rceil \leq t-3$. By Lemma 3.4, we assume that $\Delta(G) \geq 3$. By Lemma 3.3 and Lemma 2.4, we just consider the following cases.

Case 1. There are two adjacent 2-vertices u and v.

Let $N(u) = \{u_1, v\}$, $N(v) = \{v_1, u\}$ and f_1, f_2 be the two faces incident with uv. Now we construct G' by setting $G' = G - u + u_1v$. Then G' is a 2-connected 1-tree. Giving a t-entire coloring ϕ of G', we color the rest elements of G by letting $\varphi(uu_1) = \phi(u_1v)$, $\varphi(u) \in S\backslash\{\phi(u_1), \phi(u_1v), \phi(v), \phi(f_1), \phi(f_2)\}$, $\varphi(uv) \in S\backslash\{\phi(vv_1), \varphi(u), \phi(u_1v), \phi(v), \phi(f_1), \phi(f_2)\}$.

Case 2. There is a 4-face $f = uxvy$ such that $xy \notin E(G)$, $d_G(u) = d_G(v) = 2$, $d_G(y) = \Delta(G)$ and $3 \leq d_G(x) \leq \lceil \frac{\Delta(G)+1}{2} \rceil$.

Let f_1, f_2 be the two faces adjacent to f such that f_1 is incident with u. We construct G' by setting $G' = G - u + xy$. Then G' has a t-entire coloring ϕ. Since $\lceil \frac{\Delta(G)+1}{2} \rceil \leq t-3$, $x_\phi \leq t-2$ (note $\phi(xy) \in x_\phi$). We color the rest elements of G according to the following steps: First, let $\varphi(uy) = \phi(xy)$. Then, if $\phi(xy) \neq \phi(f_2)$, then $\varphi(xv) = \phi(xy)$ and $\varphi(xu) = \phi(xv)$; otherwise, if $\phi(f_1) \in x_\phi$, then let $\varphi(xv) = \phi(xv)$ and $\varphi(xu) \in S\backslash(x_\phi \cup \{\phi(f)\}$; otherwise, if $\phi(vy) \neq \phi(f_1)$, then $\varphi(xv) = \phi(f_1)$ and

$\varphi(xu) = \phi(xv)$; otherwise, let $\varphi(xv) \in S\backslash(x_\phi \cup \{\phi(f_1)\}$ and $\varphi(xu) = \phi(xv)$. Finally, we recolor v and color u.

Case 3. There is a 3-face $f = uvw$ such that $d_G(u) = 2$, $d_G(v) = 3$ and $d_G(w) = \Delta(G)$.

We use f_1, f_2 to denote the two faces adjacent to f such that f_1 is incident with u. Let $G' = G-u$, $\{v'\} = N_G(v)\backslash\{u, w\}$ and $S' = S\backslash\{\phi(f_1), w_\phi\}$. We color the rest elements of G according to the following steps: If $S' \neq \emptyset$, then $\varphi(wu) \in S'$ and $\varphi(vw) = \phi(vw)$; otherwise, $\varphi(wu) = \phi(wv)$ and $\varphi(vw) = \phi(f_1)$. $\varphi(f) \in S\backslash\{\phi(v), \phi(w), \phi(f_1), \phi(f_2), \varphi(vw), \varphi(uw)\}$, $\varphi(uv) \in S\backslash\{\varphi(f), \phi(f_1), \phi(v), \varphi(uw), \varphi(vw), \phi(vv')\}$, $\varphi(u) \in S\backslash\{\varphi(f), \phi(f_1), \phi(v), \phi(w), \varphi(uw), \varphi(uv)\}$.

Case 4. There are two adjacent 3-faces $f_1 = uxy$ and $f_2 = vxy$ such that $d_G(u) = d_G(v) = 2$, $d_G(y) = \Delta(G)$ and $4 \leq d_G(x) \leq \lceil\frac{\Delta(G)+1}{2}\rceil$.

Let f_3 be the face incident with u and f_1 and f_4 the face incident with v and f_2. Let $G' = G - \{u, v\}$. Thus f_1 and f_2 are deleted, too. Then G' has an t-entire coloring ϕ. Let $\{a_1, a_2\} \subseteq S\backslash y_\phi$. Since $d_G(x) \leq t-3$, $d_{G'}(x) \leq t-5$. Let $\{b_1, b_2\} \subseteq S\backslash(x_\phi \cup \{\phi(f_3), \phi(f_4)\})$. Without loss of generality, assume that $a_1 \notin \{\phi(f_3), b_1\}$ and $a_2 \notin \{\phi(f_4), b_2\}$. We color the rest elements of G according to the following steps: Firstly, let $\varphi(uy) = a_1$, $\varphi(vy) = a_2$, $\varphi(ux) = b_1$ and $\varphi(vx) = b_2$. Secondly, if $a_2 \neq \phi(f_3)$, then $\varphi(f_2) = \phi(f_3)$ and $\varphi(f_1) \in S\backslash\{\phi(x), \phi(y), \phi(xy), \phi(f_3), a_1, b_1\}$; otherwise, if $a_1 \neq \phi(f_4)$, then $\varphi(f_1) = \phi(f_4)$ and $\varphi(f_2) \in S\backslash\{\phi(x), \phi(y), \phi(xy), \phi(f_4), a_2, b_2\}$; otherwise, if $\phi(y) \notin \{b_1, b_2\}$, then $\varphi(f_1) = b_2$ and $\varphi(f_2) = b_1$; otherwise, without loss of generality, assume that $b_1 = \phi(y)$, and then let $\varphi(f_1) = b_2$ and $\varphi(f_2) \in S\backslash\{\phi(x), \phi(y), \phi(xy), \phi(f_4), a_2, b_2\}$. Finally, we color u and v.

Case 5. There is a a 4-face $f_1 = uxvy$ adjacent to 3-face $f_2 = uxy$ such that $d_G(u) = d_G(v) = 2$, $d_G(y) = \Delta(G)$ and $4 \leq d_G(x) \leq \lceil\frac{\Delta(G)+1}{2}\rceil$.

Let f_3 be the face incident with v and f_1 and f_4 the face incident with xy and f_2. By the induced hypothesis, $G' = G - \{u, v\}$ has an t-entire coloring ϕ. Let $\{a_1, a_2\} \subseteq S\backslash y_\phi$ and $\{b_1, b_2\} \subseteq S\backslash(x_\phi \cup \{\phi(f_3), \phi(f_4)\})$. Without loss of generality, assume that $a_1 \notin \{\phi(f_3), b_1\}$ and $a_2 \notin \{\phi(f_4), b_2\}$. We color the rest elements of G as follows. Let $\varphi(vy) = a_1$, $\varphi(uy) = a_2$, $\varphi(vx) = b_1$, $\varphi(ux) = b_2$, $\varphi(f_1) = \phi(xy)$ and $\varphi(f_2) \in S\backslash\{\phi(x), \phi(y), \phi(xy), \phi(f_4), a_2, b_2\}$, $\varphi(u) \in S\backslash\{\phi(x), \phi(y), \varphi(f_1), \varphi(f_2), a_2, b_2\}$ and $\varphi(v) \in S\backslash\{\phi(x), \phi(y), \phi(f_3), \varphi(f_1), a_1, b_1\}$.

Hence, we have $\chi_c(G) \leq t$. □

Corollary 3.6. *Let G be a 2-connected 1-tree with $\Delta(G) \geq 6$, then* $\chi_c(G) = \Delta(G) + 1$.

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