On the rainbow restrained domination number

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Abstract

A *2-rainbow dominating function* of a graph G is a function f from the vertex set $V(G)$ to the set of all subsets of the set $\{1,2\}$ such that for any vertex $v \in V(G)$ with $f(v) = \emptyset$ the condition $\bigcup_{u\in N(v)} f(u) = \{1,2\}$ is fulfilled, where $N(v)$ is the open neighborhood of v. A rainbow dominating function f is said to be a *rainbow restrained domination function* if the induced subgraph of G by the vertices with label $\emptyset$, has no isolated vertex. The weight of a rainbow restrained dominating function is the value $w(f) = \sum_{u \in V(G)} |f(u)|$. The minimum weight of a rainbow restrained dominating function of G is called the *rainbow restrained domination number* of G. In this paper we continue the study of the rainbow restrained domination number. First we classify all graphs G, of order n, whose rainbow restrained domination number is $n-1$. Then we establish an upper bound on the rainbow restrained domination number of trees.

Keywords: domination, rainbow dominating function, rainbow domination number, rainbow restrained dominating function, rainbow restrained domination number

1 Introduction

In this paper, G is a simple graph with vertex set $V = V(G)$ and edge set $E = E(G)$. The order $|V|$ of G is denoted by $n = n(G)$. For every vertex $v \in V$, the *open neighborhood* $N(v)$ is the set $\{u \in V(G) \mid uv \in E(G)\}$ and the *closed neighborhood* of v is the set $N[v] = N(v) \cup \{v\}$. The *degree* of a vertex $v \in V$ is $\deg(v) = |N(v)|$. The *minimum* and *maximum degree* of a graph G are denoted by $\delta = \delta(G)$ and $\Delta = \Delta(G)$, respectively. The *open neighborhood* of a set $S \subseteq V$ is the set $N(S) = \bigcup_{v \in S} N(v)$, and the *closed neighborhood* of S is the set $N[S] = N(S) \cup S$. A *tree* is an acyclic connected graph. For two integers $r, s \geq 1$, a double star $S(r,s)$ is a tree with exactly two vertices that are not leaves, with one adjacent to r leaves and the other to s leaves. A vertex adjacent to a leaf is a *support vertex*. A support vertex adjacent to at least two leaves is called a *strong support vertex*. If $A \subseteq V(G)$, then $G[A]$ is the subgraph induced by A. If $A, B \subseteq V(G)$, then $E(A,B)$ is the set of edges between A and B. We write K_n for the *complete graph* of order n, P_n for a *path* of order n and C_n for a *cycle* of length n. Consult [6, 9], for terminology and notation on graph theory not defined here.

For a positive integer k, a *k-rainbow dominating function* (kRDF) of a graph G is a function f from the vertex set $V(G)$ to the set of all subsets of the set $\{1,2,\ldots,k\}$ such that for any vertex $v \in V(G)$ with $f(v) = \emptyset$ the condition $\bigcup_{u \in N(v)} f(u) = \{1,2,\ldots,k\}$ is fulfilled. The *weight* of a kRDF f is the value $\omega(f) = \sum_{v \in V} |f(v)|$. The *$k$-rainbow domination number* of a graph G, denoted by $\gamma_{rk}(G)$, is the minimum weight of a kRDF of G. A *$\gamma_{rk}(G)$-function* is a k-rainbow dominating function of G with weight $\gamma_{rk}(G)$. Note that $\gamma_{r1}(G)$ is the classical domination number $\gamma(G)$. The k-

rainbow domination number was introduced by Brešar, Henning, and Rall [2] and has been studied by several authors (see for example [3, 4, 5, 7, 8, 10, 11]).

A 2-rainbow dominating function $f : V \longrightarrow \mathcal{P}(\{1,2\})$ can be represented by the ordered partition $(V_0, V_1, V_2, V_{1,2})$ (or $(V_0^f, V_1^f, V_2^f, V_{1,2}^f)$ to refer f) of V, where $V_0 = \{v \in V \mid f(v) = \emptyset\}$, $V_1 = \{v \in V \mid f(v) = \{1\}\}$, $V_2 = \{v \in V \mid f(v) = \{2\}\}$, $V_{1,2} = \{v \in V \mid f(v) = \{1,2\}\}$. In this representation, its weight is $\omega(f) = |V_1| + |V_2| + 2|V_{1,2}|$.

A 2-rainbow dominating function $f = (V_0, V_1, V_2, V_{1,2})$ is called a *rainbow restrained dominating function* (RRDF) if the induced subgraph $G[V_0]$ has no isolated vertices. The *rainbow restrained domination number* of G, denoted by $\gamma_{rr}(G)$, is the minimum weight of an RRDF on G. A $\gamma_{rr}(G)$*-function* is an RRDF of G with $\omega(f) = \gamma_{rr}(G)$. The rainbow restrained domination number was investigated by Amjadi et al. in [1]. If $G_1, G_2, \cdots, G_s$ are the components of G, then $\gamma_{rr}(G) = \sum_{i=1}^{s} \gamma_{rr}(G_i)$. Hence, it is sufficient to study $\gamma_{rr}(G)$ for connected graphs.

In this paper, we continue the study of the rainbow restrained domination numbers. We first classify all connected graphs G of order n with $\gamma_{rr}(G) = n - 1$. Then we establish an upper on the rainbow restrained domination number of trees.

We make use of the following two results in this paper. The proofs can be found in [1].

Theorem A Let G be a connected graph of order $n \geq 2$. Then $\gamma_{rr}(G) = n$ if and only if $G \simeq K_{1,n-1}, C_4, C_5$ or $G = P_n$ for $n = 2, 3, 4, 5, 6$.

Observation 1 If H is a subgraph of G, then $\gamma_{rr}(G) \leq \gamma_{rr}(H) + |V(G)| - |V(H)|$.

A *subdivision* of an edge uv is obtained by removing the edge uv, adding a new vertex w, and adding edges uw and wv. The *subdivision graph* $S(G)$ is the graph obtained from G by subdividing each edge of G. The graph $S(K_{1,t})$ for $t \geq 2$, is called a *healthy spider* S_t, while a *wounded spider* S_t is the graph formed by subdividing at most $t-1$ of the edges of a star $K_{1,t}$ for $t \geq 2$. Clearly stars are wounded spiders. A *spider* is a healthy or wounded spider.

Example 2 (a) $\gamma_{rr}(P_n) = n$ for $1 \leq n \leq 6$ and $\gamma_{rr}(P_n) = \lceil \frac{2n+1}{3} \rceil + 1$ for $n \geq 7$.

(b) $\gamma_{rr}(C_n) = 2\lceil \frac{n}{3} \rceil$ when $n \not\equiv 2 \pmod 3$ and $\gamma_{rr}(C_n) = 2\lceil \frac{n}{3} \rceil + 1$ otherwise.
(c) $\gamma_{rr}(S(r,s)) = r+s$ for $2 \le r \le s$, $\gamma_{rr}(S(r,s)) = r+s+1$ for $r = 1, s \ge 2$ and $\gamma_{rr}(S(r,s)) = r+s+2$ for $1 = r = s$.
(d) If T is a spider different from stars, P_4 and P_5, then $\gamma_{rr}(T) = |V(T)| - 1$.

Examples 2 (a) and (b) can be found in [1] and (c) and (d) are easy to prove.

2 Graphs with large rainbow restrained domination number

In [1] the authors characterize all graphs whose rainbow restrained domination number is equal their order (Theorem A). In this section we characterize all graphs G of order n with $\gamma_{rr}(G) = n - 1$. We start with the following lemmas.

Lemma 3 Let G be a connected graph of order n. If G has two adjacent vertices each of degree at least 3, then $\gamma_{rr}(G) \le n - 2$.

Proof. Assume x and y are adjacent vertices each of degree at least three in G. Let $x_1, x_2 \in N(x)\setminus\{y\}$ and $y_1, y_2 \in N(y)\setminus\{x\}$. If $|\{x_1,x_2\}\cap\{y_1,y_2\}| \ge 1$, then let $x_1 = y_1$ and define $f : V(G) \to \mathcal{P}(\{1,2\})$ by $f(x) = f(y) = \emptyset, f(x_1) = \{1\}$ and $f(u) = \{2\}$ otherwise. If $\{x_1,x_2\}\cap\{y_1,y_2\} = \emptyset$, then define $f(x) = f(y) = \emptyset, f(x_1) = f(y_1) = \{1\}$ and $f(u) = \{2\}$ otherwise. It is easy to see that in each case, f is an RRDF of G of weight $n-2$ that implies $\gamma_{rr}(G) \le n-2$. □

Lemma 4 Let G be a connected graph of order n with $\mathrm{diam}(G) \ge 6$. Then

$$\gamma_{rr}(G) \le n + 1 + \left\lceil \frac{-\mathrm{diam}(G)}{3} \right\rceil.$$

In particular, if $\mathrm{diam}(G) \ge 9$, then $\gamma_{rr}(G) \le n - 2$.

Proof. Let P be a diametral path in G. By Observation 1 and Example 2, we obtain

$$\begin{aligned}\gamma_{rr}(G) &\le (n - \mathrm{diam}(G) - 1) + \left\lceil \tfrac{2(\mathrm{diam}(G)+1)+1}{3} \right\rceil + 1 \\ &= n + 1 + \left\lceil \tfrac{-\mathrm{diam}(G)}{3} \right\rceil.\end{aligned}$$

$$
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&= n + 1 + \left\lceil \tfrac{-\operatorname{diam}(G)}{3} \right\rceil .
\end{aligned}
\qquad \square
$$

Lemma 5 Let G be a connected graph of order n. If G has a path $x_1x_2\ldots x_k$ ($k \ge 6$) such that $|N(x_1) \setminus \{x_2, x_3, \ldots, x_k\}| \ge 2$, then $\gamma_{rr}(G) \le n-2$.

Proof. Let $x, y \in N(x_1) \setminus \{x_2, x_3, \ldots, x_k\}$. Define $f : V(G) \to \mathcal{P}(\{1,2\})$ by $f(x) = \{1\}, f(x_3) = f(x_6) = \{1,2\}, f(x_1) = f(x_2) = f(x_4) = f(x_5) = \emptyset$ and $f(u) = \{2\}$ otherwise. Obviously, f is an RRDF of G of weight $n-2$ and hence $\gamma_{rr}(G) \le n-2$. $\square$

Lemmas 3 and 5 lead to the next result immediately.

Corollary 6 Let G be a connected graph of order n that is not a cycle. If G has a cycle of length at least 7, then $\gamma_{rr}(C_n) \le n-2$.

Now we introduce a family of trees. Let $\mathcal{F}$ be the family of trees consisting of

(a) P_7, P_8, P_9 and $S(1,s)$ for $s \ge 2$,

(b) spiders except stars, P_4 and P_5,

(c) trees that can be obtained from two disjoint stars $K_{1,r}$ and $K_{1,s}$ ($s \ge 2, r \ge 1$), by adding a new vertex v and joining it to the central vertices of stars,

(d) trees that can be obtained from two disjoint stars $K_{1,r}$ and $K_{1,s}$ ($s \ge 3, r \ge 1$), by adding an edge joining one leaf of $K_{1,r}$ to one leaf of $K_{1,s}$,

(e) trees that can be obtained from a star $K_{1,r}$ ($r \ge 2$) and a spider S different from star, by adding an edge joining one leaf of $K_{1,r}$ to the central vertex of S,

(f) trees that can be obtained from P_4 and a spider S different from a star, by adding an edge joining one leaf of P_4 to the central vertex of S,

(g) trees that can be obtained from P_7 (respectively P_9) by adding pendant edges at the central vertex of P_7 (respectively P_9) or joining the central vertex of P_7 (respectively P_9) to exactly one leaf of $q \ge 0$ disjoint complete graphs K_2,

(h) trees that can be obtained from P_8 by adding pendant edges at one of the central vertices of P_8, say x, or joining x to exactly one leaf of $q \geq 0$ disjoint complete graphs K_2.

Theorem 7 Let T be a tree of order n. Then $\gamma_{rr}(T) = n - 1$ if and only if $T \in \mathcal{F}$.

Proof. One side is clear. Let $\gamma_{rr}(T) = n - 1$. It follows from Lemma 4 that $\text{diam}(T) \leq 8$. Assume that $v_1 v_2 \ldots v_k$ is a diametral path in T. First let $\text{diam}(T) = 8$. By Lemma 5, $\deg(v_2) = \deg(v_3) = \deg(v_4) = \deg(v_6) = \deg(v_7) = \deg(v_8) = 2$. If T has a path $v_5 z_1 z_2 z_3$ where $z_1 \notin \{v_4, v_6\}$, then the function f defined by $f(z_3) = f(v_5) = f(v_2) = f(v_8) = \{1, 2\}, f(z_1) = f(z_2) = f(v_3) = f(v_4) = f(v_6) = f(v_7) = \emptyset$ and $f(x) = \{1\}$ otherwise, is an RRDF of T of weight $n - 2$, a contradiction. We deduce from this fact and Lemma 3 that each neighbor of v_5 except v_4, v_6, if any, is a leaf or a support vertex of degree 2. Thus $T = P_9$ or T satisfies (g) and hence $T \in \mathcal{F}$. Now let $\text{diam}(T) = 7$. By Lemma 5, $\deg(v_2) = \deg(v_3) = \deg(v_6) = \deg(v_7) = 2$. If $\deg(v_4) = \deg(v_5) = 2$, then $T = P_8 \in \mathcal{F}$. Assume, without loss of generality, that $\deg(v_5) \geq 3$. Then $\deg(v_4) = 2$. As above we can see that each neighbor of v_5 except v_4, v_6, is a leaf or a support vertex of degree 2. Thus T satisfies (h) and hence $T \in \mathcal{F}$. Hence let $\text{diam}(T) \leq 6$. By Theorem A, $\text{diam}(T) \geq 3$. Let $v_1 v_2 \ldots v_k$ be a diametral path in T and let $\deg(v_2)$ be as large as possible. We consider the following cases.

Case 1. $\text{diam}(T) = 3$.
We deduce from Example 2 that $T = S(1, s)$ for some positive integer $s \geq 2$ and so $T \in \mathcal{F}$.

Case 2. $\text{diam}(T) = 4$.
If $\deg(v_2) = 2$, then by the choice of the diametral path, all support vertices on diametral paths have degree two implying that T is a spider, except P_5, and so $T \in \mathcal{F}$. Assume $\deg(v_2) \geq 3$. By Lemma 3, we have $\deg(v_3) = 2$. Then clearly the components of $T - v_3$ are stars and hence T satisfies (c) and hence $T \in \mathcal{F}$.

Case 3. $\text{diam}(T) = 5$.
First let $\deg(v_2) = 2$. By the choice of the diametral path, all support vertices on diametral paths have degree two. Since $\gamma_{rr}(P_6) = n$, $\deg(v_3) \geq 3$ or $\deg(v_4) \geq 3$. Assume, without loss of generality, that $\deg(v_3) \geq 3$. Then $\deg(v_4) = 2$. Rooting T at v_6, we can see that T_{v_3} is a spider different from stars. Thus T satisfies (e) and so $T \in \mathcal{F}$. Now let $\deg(v_2) \geq 3$. By Lemma 3, we have $\deg(v_3) = 2$. If $\deg(v_5) \geq 3$, then $\deg(v_4) = 2$ and clearly T

satisfies (d) that yields $T \in \mathcal{F}$. Assume $\deg(v_5) = 2$. Similarly, we may assume that all support vertices adjacent to v_4 have degree 2. Rooting T at v_1, we see that T_{v_4} is a spider. Thus T satisfies (e) and so $T \in \mathcal{F}$.

Case 4. $\mathrm{diam}(T) = 6$.
By Lemma 5, we have $\deg(v_2) = \deg(v_6) = 2$. By the choice of the diametral path, we deduce that all support vertices in $N(v_3) \cup N(v_5)$ but v_4, have degree 2.

First let $\deg(v_3) \geq 3$ (the case $\deg(v_5) \geq 3$ is similar). It follows from Lemma 3 that $\deg(v_4) = 2$. Rooting T at v_7, implies that T_{v_3} is a spider. We claim that $\deg(v_5) = 2$ that in the case T satisfies (f) and so $T \in \mathcal{F}$. Assume to the contrary that $\deg(v_5) \geq 3$. Let $x \in N(v_3) - \{v_2, v_4\}$ and $y \in N(v_5) - \{v_6, v_4\}$. Then the function f defined by $f(x) = f(y) = \{1\}, f(v_1) = f(v_7) = \{1, 2\}, f(v_2) = f(v_3) = f(v_5) = f(v_6) = \emptyset$ and $f(u) = \{2\}$ otherwise, is an RRDF of T of weight $n - 2$, which is a contradiction.

Now let $\deg(v_3) = \deg(v_5) = 2$. If $\deg(v_4) = 2$, then $T = P_7 \in \mathcal{T}$. Let $\deg(v_4) \geq 3$. If there is a path $v_4 x_1 x_2 x_3$ in T where $x_1 \notin \{v_3, v_5\}$, then by the arguments above we may assume that $\deg(x_i) = 2$ for $i = 1, 2$. It is easy to see that the function f defined by $f(x) = \emptyset$ for $x \in \{v_2, v_3, v_5, v_6, x_2, x_1\}$ and $f(v_4) = f(v_1) = f(v_7) = f(x_3) = \{1, 2\}$, is an RRDf of G of weight less than $n - 1$, a contradiction. On the other hand, we deduce from Lemma 3 that each support vertex adjacent to v_4 has degree two and hence T satisfies (g), and the proof is complete. □

The next result is an immediate consequence of Example 2 and Observation 1.

Corollary 8 For $n \geq 3$, $\gamma_{rr}(C_n) = n - 1$ if and only if $n = 3, 7, 8$. Moreover, if G has a cycle different from C_3, C_4, C_5, C_7 and C_8, then $\gamma_{rr}(G) \leq n - 2$.

Fig. 1: The graphs G_1 and G_2

Let $\mathcal{H}_1$ be the family consisting of C_3 and the graphs obtained from $C_3 = uvw$ by adding pendant edges at u or joining u to one leaf of $q \geq 0$ disjoint complete graphs K_2. (Figure 2.)

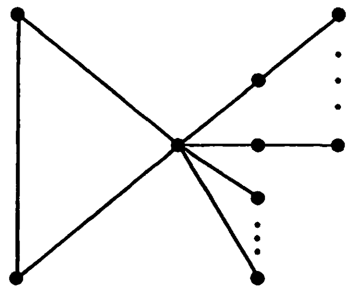


Fig. 2: Family $\mathcal{H}_1$

Lemma 9 Let G be a connected graph of order n. If G has a triangle, then $\gamma_{rr}(G) = n - 1$ if and only if $G \in \mathcal{H}_1$.

Proof. If $G \in \mathcal{H}_1$, then clearly $\gamma_{rr}(G) = n - 1$. Conversely, let $\gamma_{rr}(G) = n-1$. If $G = C_3$, then we are done. Let $G \neq C_3$. Suppose uvw is a triangle in G. Since G is connected, we may assume that $\deg(u) \geq 3$. It follows from Lemma 3 that $\deg(v) = \deg(w) = 2$. Now let $x \in N(u) - \{v, w\}$. By Lemma 3, $\deg(x) \leq 2$. If $\deg(x) = 1$, we are done. Assume that $\deg(x) = 2$ and $y \in N(x) - \{u\}$. We claim that $\deg(y) = 1$. Assume to the contrary that $\deg(y) \geq 2$ and let $z \in N(y) \setminus \{x\}$ (note that $z = u$ is possible). Define the function f by $f(v) = f(w) = f(x) = f(y) = \emptyset, f(u) = f(z) = \{1,2\}$ and $f(a) = \{1\}$ otherwise. It is easy to see that f is an RRDF of G of weight at most $n - 2$ which is a contradiction. Thus $\deg(y) = 1$. This implies that $G \in \mathcal{H}_1$. □

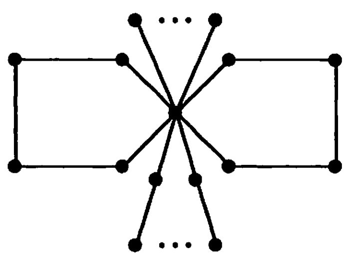


Fig 3: Family $\mathcal{H}_2$

Let $G_1, G_2, \ldots, G_k$ be mutually pairwise vertex-disjoint graphs, $k \geq 2$, and $u_i \in V(G_i)$ for $i = 1, 2 \ldots, k$. The *multiple coalescence* $G_1 \circ G_2 \circ \ldots \circ G_k$ of $G_1, G_2, \ldots, G_k$ on u_1 is the graph obtained from the union of these graphs by identifying the vertices $u_1, u_2, \ldots, u_k$. Let $\mathcal{H}_2$ be a family consisting of graphs obtained from multiple coalescence of two copies of C_5 and some copies of P_2 and P_3. (Figure 3.)

Lemma 10 Let G be a connected graph of order n with at least two distinct cycles. Then $\gamma_{rr}(G) = n - 1$ if and only if $G \in \{G_1, G_2\} \cup \mathcal{H}_2$.

Proof. Let $\gamma_{rr}(C_n) = n - 1$. By Corollaries 6 and 8 and Lemma 9, all cycles of G have length 4 or 5. Let $C_r = (x_1 x_2 \ldots x_r)$ and $C_s = (y_1 y_2 \ldots y_s)$ be two distinct cycles of G. We may assume $r \leq s$.

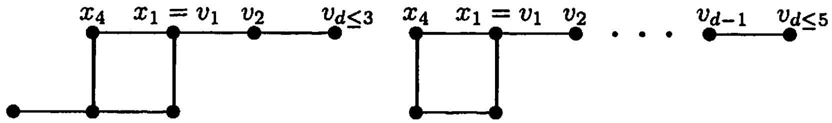
First assume that C_r and C_s are vertex-disjoint. By Lemma 3, there is no edge joining C_r and C_s. Let $x_1 z_1 \ldots z_k y_1$ be a path joining $V(C_r)$ and $V(C_s)$ such that $z_1, \ldots, z_k \notin V(C_r) \cup V(C_s)$. Then the path $x_1 z_1 \ldots z_k y_1 \ldots y_s$ satisfies the condition of Lemma 5, a contradiction.

Assume now C_r and C_s have exactly one vertex in common, say $x_1 = y_1$. If $r = 4$, then $f_1 = (\{x_1, x_2, x_4\}, \{y_2\}, V(G) \setminus \{x_1, x_2, x_3, x_4, y_2\}, \{x_3\})$ is an RRDF on G with $w(f_1) < n - 1$, a contradiction. Let $r = s = 5$. By Lemmas 3 and 5, all vertices of $V(C_r) \cup V(C_s) - \{x_1\}$ have degree 2 in G. If G has order 9, then $G = C_5 \circ C_5 \in \mathcal{H}_2$. We claim that $d(x_1, z) \leq 2$ for each $z \in V(G)$. Assume to the contrary that $d(x_1, z) \geq 3$ for some $z \in V(G)$. Then clearly $z \in V(G) - (V(C_r) \cup V(C_s))$. Let $x_1 z_1 z_2 \ldots z_k$ be a (x_1, z)-path of length $d(x_1, z)$ where $z = z_k$. Obviously, $z_i \notin V(C_r) \cup V(C_s)$ for each i. Then the function $f_2 = (\{x_2, x_3, y_2, y_3, z_1, z_2\}, V(G) - \{x_1, x_2, x_3, x_4, y_2, y_3, y_4, z_1, z_2, z_3\}, \emptyset, \{x_1, x_4, y_4, z_3\})$, is an RRDF on G with $w(f_2) < n - 1$, a contradiction. This proves the claim. Also by Lemma 3, each vertex adjacent to x_1 has degree at most 2. Thus G is a multiple coalescence of two copies of C_5 and copies of P_2 and P_3 on x_1. Hence $G \in \mathcal{H}_2$.

Henceforth, we assume that any pair of cycles in G have at least two vertices in common. It is easy to see that if G has two cycles with exactly two vertices in common, then G will have two cycles with at least three vertices in common. We may assume, without loss of generality, that C_r and C_s have at least three vertices in common. First assume $|V(C_r) \cap V(C_s)| = 4$. Suppose $x_i = y_i$ for $i = 1, 2, 3, 4$. Then $r = s = 5$, otherwise G will have a triangle which is a contradiction. By Lemma 3, we deduce that $\deg(x_2) = \deg(x_3) = \deg(x_5) = \deg(y_5) = 2$. If x_4 has a neighbor outside $V(C_r) \cup V(C_s)$, say z, then the function $p : V(G) \to \mathcal{P}(\{1, 2\})$

defined by $p(z) = \{1\}, p(x_1) = \{1,2\}$, $f(x_4) = f(x_5) = f(y_5) = \emptyset$, and $p(x) = \{2\}$ otherwise, is an RRDF of G of weight $n-2$, a contradiction. Thus $\deg(x_4) = 3$. Similarly, $\deg(x_1) = 3$. Therefore G is the union of two copies of C_5 having 4 vertices in common and hence $G = G_2$.

Suppose now $|V(C_r) \cap V(C_s)| = 3$ and let no pair of cycles in G have four vertices in common. Suppose $x_i = y_i$ for $i = 1,2,3$. If $r \geq 5$, then the function $h_1 = (\{x_4, x_5, y_4, y_5\}, \{x_2\}, V(G) - (V(C_r) \cup V(C_s)), \{x_1, x_3\})$ is an RRDF on G of weight $n-2$, a contradiction. Therefore $r = 4$. If $s = 5$, then two distinct cycles $(x_1x_2x_3y_4y_5)$ and $(x_1x_4x_3y_4y_5)$ have four vertices in common, a contradiction. Henceforth, $s = 4$. By Lemma 3, the vertices x_2, x_4, y_4 have degree 2 in G. If x_1 has a neighbor x not in $V(C_r) \cup V(C_s)$, then the function $h_2 = (\{x_1, x_2, x_4\}, \{x\}, V(G) - \{x_1, x_2, x_3, x_4, x\}, \{x_3\})$ is an RRDF on G of weight $n-2$, a contradiction. Therefore $\deg(x_1) = 3$. Similarly, $\deg(x_3) = 3$. Thus $G = G_1$ and the proof is complete. □



$\mathcal{G}_1$ $\mathcal{G}_2$

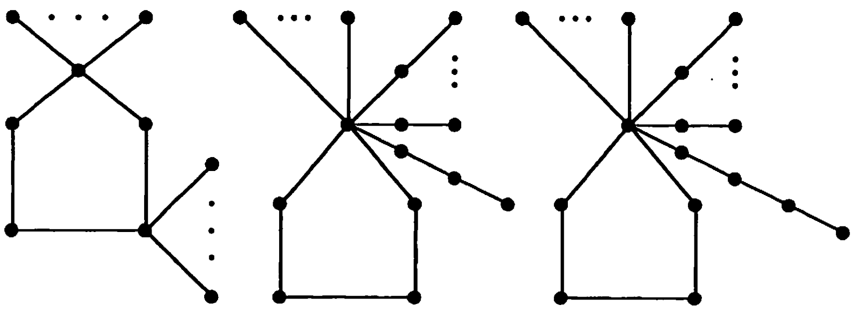


Fig 4: Family $\mathcal{H}_3$

Lemma 11 Let G be a connected graph of order n with exactly one cycle. Then $\gamma_{rr}(G) = n-1$ if and only if $G \in \mathcal{G}_1 \cup \mathcal{G}_2 \cup \mathcal{H}_3$.

Proof. Let $C_r = (x_1x_2 \ldots x_r)$ be the unique cycle of G. Clearly, $r \in \{4,5\}$. Since $\gamma_{rr}(G) = n-1$, $G \ncong C_r$. Let $\deg(x_1) \geq 3$. We consider two cases.

Case 1. Assume that $r = 4$.
If x_1 has two neighbors in $V(G) - V(C_r)$, say y, z, then the function $h : V(G) \to \mathcal{P}(\{1,2\})$ defined by $h(x_1) = h(x_2) = h(x_4) = \emptyset, h(x_3) = \{1,2\}, h(y) = \{2\}$ and $h(u) = \{1\}$ otherwise, is an RRDF of G of weight $n-2$, a contradiction. Thus $\deg(x_1) = 3$. Similarly, we can see that $2 \leq \deg(x_i) \leq 3$ for $i = 2,3,4$. Suppose that $P = x_1z_2 \ldots z_k$ is a path in G such that $z_j \notin V(C_r)$ for each $2 \leq j \leq k$. By Lemma 5, $k \leq 5$. Since G is unicyclic, we deduce that $\deg(z_k) = 1$. By Lemma 3, $\deg(z_2) \leq 2$. If $\deg(z_3) \geq 3$ and $y \in N(z_3) - \{z_2, z_4\}$, then the function h_1 defined by $h_1(z_2) = h_1(z_3) = h_1(x_2) = h_1(x_3) = \emptyset, h_1(x_1) = h_1(x_4) = \{1,2\}, h_1(y) = \{2\}$ and $h_1(u) = \{1\}$ otherwise, is an RRDF of G of weight less than $n-1$, a contradiction. Thus $\deg(z_3) = 2$ if $k \geq 4$. Applying a similar argument, we can see that $\deg(z_4) = 2$ when $k = 5$. If there is another vertex of degree more than 2, it can only be x_3. If $\deg(x_3) = 2$, then clearly $G \in \mathcal{G}_2$. Let $\deg(x_3) \geq 3$. Let $Q = x_3t_2 \ldots t_m$ be a path in G such that $t_j \notin V(C_r)$ for each j. Assume, without loss of generality, that $k \geq m$. As above we can see that $\deg(x_3) = 3$ and $\deg(t_j) \leq 2$ for $2 \leq j \leq t$. If $k \geq 4$, then the function f defined by $f(x) = \emptyset$ when $x \in \{x_2, x_3, x_4, z_2, z_3\}$, $f(t_2) = f(x_1) = f(z_4) = \{1,2\}$ and $f(x) = \{2\}$ otherwise, is an RRDF of G of weight less than $n-1$, a contradiction. Hence $k \leq 3$. Next we will show that $k = 3$ implies $m = 2$. Otherwise the function f_1 defined by $f_1(x) = \emptyset$ when $x \in \{x_1, x_3, z_2, t_2\}, f_1(t_3) = f_1(z_3) = \{1,2\}, f_1(x_2) = \{1\}$ and $f_1(x) = \{2\}$ otherwise, is an RRDF of G of weight less than $n-1$, a contradiction. Thus $G \in \mathcal{G}_1$.

Case 2. Assume that $r = 5$.
Let $P = x_1z_2 \ldots z_k$ be a longest path in G, where $z_i \notin V(C_r)$ for each i. By Lemma 5, $k \leq 5$. By Lemma 3, $\deg(z_2) = 2$. Now we show that $\deg(z_3) = 2$ if $k = 4$ and $\deg(z_3) = \deg(z_4) = 2$ when $k = 5$. Indeed, if $u \in N(z_3) - \{z_2, z_4\}$, then the function g_1 defined by $g_1(x_1) = g_1(x_3) = \{1,2\}, g_1(z_2) = g_1(z_3) = g_1(x_4) = g_1(x_5) = \emptyset, g_1(u) = \{1\}$ and $g_1(x) = \{2\}$ otherwise, is an RRDF of G of weight less than $n-1$, a contradiction. Also, if $u \in N(z_4) - \{z_3, z_5\}$, then the function g_2 defined by $g_2(u) = \{1\}, g_2(z_3) = g_2(z_4) = g_2(x_1) = g_2(x_5) = \emptyset, g_2(z_2) = g_2(x_4) = \{1,2\}$ and $g_2(x) = \{2\}$ otherwise, is an RRDF of G of weight less than $n-1$, a contradiction again. If $P_1 = x_1u_2 \ldots u_m$ is a path in G where $u_2 \notin \{z_2, x_2, x_5\}$, then it is easy to verify that $\min\{k, m\} \leq 3$. In G has no vertex of degree least 3 but x_1, then clearly $G \in \mathcal{H}_3$. Now let G has another vertex of degree at least 3, say w. Then w can only be one of the x_3 or x_4. Assume, without loss of generality, that $w = x_3$. Let $Q = x_3t_2 \ldots t_s$ be a path in G such that

$t_j \notin V(C_r)$ for each j. Suppose that $k \geq s$. If $k \geq 3$, then the function g_3 defined by $g_3(z_3) = g_3(x_5) = \{1,2\}, g_3(t_2) = \{1\}, g_3(x_1) = g_3(z_2) = g_3(x_3) = g_3(x_4) = \emptyset$ and $g_3(x) = \{2\}$ otherwise, is an RRDF of G of weight less than $n-1$, a contradiction. Thus $\max\{k,m\} = 2$. It follows that $G \in \mathcal{H}_3$. This completes the proof. □

Our main theorem is an immediate consequence of Theorem 7, Lemmas 9, 10 and 11.

Theorem 12 Let G be a connected graph of order n. Then $\gamma_{rr}(G) = n-1$ if and only if $G \in \mathcal{F} \cup \mathcal{G}_1 \cup \mathcal{G}_2 \cup \mathcal{H}_1 \cup \mathcal{H}_2 \cup \mathcal{H}_3 \cup \{G_1, G_2\}$.

3 Bounds on trees

In this section, we establish an upper and a lower bound on the rainbow restrained domination number in trees. If T is a tree, then let $s(T)$ and $\ell(T)$ be the number of support vertices and leaves, respectively.

Theorem 13 Let T be a tree of order $n \geq 3$. Then

$$\gamma_{rr}(T) \leq \left\lceil \frac{2n + 4s(T) + \ell(T) - 5}{3} \right\rceil.$$

This bound is sharp for stars, and for paths P_n such that $n \equiv 0 \pmod 3$ or $n \equiv 2 \pmod 3$ with $n \geq 8$.

Proof. We proceed by induction on n. By Example 2, the statement holds for all trees of order $n = 3, 4$. Let $n \geq 5$ and assume that for every tree T of order at least 3 and less than n the result is valid. Let T be a tree of order $n \geq 5$. If T is the star $K_{1,n-1}$, then by Theorem A, we have $\gamma_{rr}(T) = n = \lceil \frac{2n+4s(T)+\ell(T)-5}{3} \rceil$. If T is the double star $S(r,s)$ then $s(T) = 2, \ell(T) = r+s$ and therefore $\gamma_{rr}(T) \leq n < \lceil \frac{3n+1}{3} \rceil = \lceil \frac{2n+4s(T)+\ell(T)-5}{3} \rceil$. Henceforth, we may assume that $\mathrm{diam}(T) \geq 4$. In the following let $f = (V_0, V_1, V_2, V_{1,2})$ be a $\gamma_{rr}(T')$-function. We proceed further with a series of claims that we may assume satisfied by the tree.

Claim 1. T has no strong support vertex.
Proof. Let T have a strong support vertex u and let v, w are two leaves adjacent to u. Set $T' = T - v$. Then $|V(T')| = n-1$, $s(T') = s(T)$ and $\ell(T') = \ell(T) - 1$. On the other hand, $f = (V_0, V_1 \cup \{v\}, V_2, V_{1,2})$ is

an RRDF of T implying that $\gamma_{rr}(T) \leq \gamma_{rr}(T') + 1$. It follows from the induction hypothesis that

$$\begin{aligned} \gamma_{rr}(T) &\leq \gamma_{rr}(T') + 1 \\ &\leq \lceil \tfrac{2(n-1)+4s(T')+\ell(T')-5}{3} \rceil + 1 \\ &= \lceil \tfrac{2n+4s(T)+\ell(T)-8}{3} \rceil + 1 \\ &= \lceil \tfrac{2n+4s(T)+\ell(T)-5}{3} \rceil, \end{aligned}$$

as desired. (■)

Let $v_1v_2\ldots v_D$ be a diametral path in T and root T at v_D. From Claim 1, we deduce that $\deg(v_2) = \deg(v_{D-1}) = 2$ and v_3 is only adjacent to leaves or support vertices of degree 2. If $\mathrm{diam}(T) = 4$, then T is a spider and we have $s(T) \geq 2$ and $s(T) + \ell(T) \geq n - 1$. It follows that $\gamma_{rr}(T) \leq n \leq \lceil \frac{2n+4s(T)+\ell(T)-5}{3} \rceil$. Henceforth, we assume that $\mathrm{diam}(T) \geq 5$.

Claim 2. $\deg(v_3) = 2$.

Proof. Suppose that $\deg(v_3) \geq 3$ and let $T' = T - \{v_1, v_2\}$. Obviously, $f = (V_0, V_1 \cup \{v_1, v_2\}, V_2, V_{1,2})$ is an RRDF of T and hence $\gamma_{rr}(T) \leq \gamma_{rr}(T') + 2$. Since $|V(T')| = n - 2$, $s(T') = s(T) - 1$ and $\ell(T') = \ell(T) - 1$, it follows from the induction hypothesis that

$$\begin{aligned} \gamma_{rr}(T) &\leq \gamma_{rr}(T') + 2 \\ &\leq \lceil \tfrac{2(n-2)+4s(T')+\ell(T')-5}{3} \rceil + 2 \\ &= \lceil \tfrac{2n+4s(T)+\ell(T)-14}{3} \rceil + 2 \\ &< \lceil \tfrac{2n+4s(T)+\ell(T)-5}{3} \rceil. \qquad (■) \end{aligned}$$

Claim 3. $\deg(v_4) = 2$.

Proof. Suppose that $\deg(v_4) \geq 3$ and let $T' = T - \{v_3, v_2, v_1\}$. Clearly, f can be extended to an RRDF of T by assigning $\{1\}$ to v_1, v_2, v_3 and so $\gamma_{rr}(T) \leq \gamma_{rr}(T') + 3$. Since $|V(T')| = n - 3$, $s(T') = s(T) - 1$ and $\ell(T') = \ell(T) - 1$, we deduce from the induction hypothesis that

$$\begin{aligned} \gamma_{rr}(T) &\leq \gamma_{rr}(T') + 3 \\ &\leq \lceil \tfrac{2(n-3)+4s(T')+\ell(T')-5}{3} \rceil + 3 \\ &= \lceil \tfrac{2n+4s(T)+\ell(T)-16}{3} \rceil + 3 \\ &\leq \lceil \tfrac{2n+4s(T)+\ell(T)-5}{3} \rceil. \qquad (■) \end{aligned}$$

Claim 4. $\deg(v_5) = 2$.

Proof. Suppose that $\deg(v_5) \geq 3$ and let $T' = T - \{v_4, v_3, v_2, v_1\}$. Then $|V(T')| = n - 4$, $s(T') = s(T) - 1$ and $\ell(T') = \ell(T) - 1$. On the other hand,

f can be extended to an RRDF of T by assigning $\{1\}$ to v_1, v_2, v_3, v_4. Thus $\gamma_{rr}(T) \leq \gamma_{rr}(T') + 4$. As above, we have $\gamma_{rr}(T) \leq \lceil \frac{2n+4s(T)+\ell(T)-5}{3} \rceil$. (■)

If v_5 is a support vertex, then $T = P_6$ and

$$\gamma_{rr}(T) = 6 = \lceil \frac{2n + 4s(T) + \ell(T) - 5}{3} \rceil.$$

Hence suppose that v_5 is not a support vertex.

Claim 5. $\deg(v_6) = 2$.

Proof. Suppose that $\deg(v_6) \geq 3$ and let $T' = T - \{v_5, v_4, v_3, v_2, v_1\}$. Then $|V(T')| = n-5$, $s(T') = s(T) - 1$ and $\ell(T') = \ell(T) - 1$. On the other hand, f can be extended to an RRDF of T by assigning $\{1\}$ to v_1, v_2, v_3, v_4, v_5 and so $\gamma_{rr}(T) \leq \gamma_{rr}(T') + 5$. It follows from the induction hypothesis that $\gamma_{rr}(T) \leq \lceil \frac{2n+4s(T)+\ell(T)-5}{3} \rceil$. (■)

If v_6 is a support vertex, then $T = P_7$ and so

$$\gamma_{rr}(T) = 6 < \lceil \frac{2n + 4s(T) + \ell(T) - 5}{3} \rceil.$$

Hence we may assume that v_6 is not a support vertex.

Claim 6. $\deg(v_7) = 2$.

Proof. Suppose that $\deg(v_7) \geq 3$ and let $T' = T - \{v_6, v_5, v_4, v_3, v_2, v_1\}$. Then $|V(T')| = n - 6$, $s(T') = s(T) - 1$ and $\ell(T') = \ell(T) - 1$. If $v_7 \in V_0$, then f can be extended to an RRDF of T by assigning $\{1\}$ to v_1, $\{1,2\}$ to v_2, v_5 and $\emptyset$ to v_3, v_4, v_6, and if $v_7 \notin V_0$, then f can be extended to an RRDF of T by assigning $\{1,2\}$ to v_1, v_4, v_7 and $\emptyset$ to v_2, v_3, v_5, v_6, implying that $\gamma_{rr}(T) \leq \gamma_{rr}(T')+5$. By the induction hypothesis, we obtain $\gamma_{rr}(T) \leq \gamma_{rr}(T') + 5 \leq \lceil \frac{2n+4s(T)+\ell(T)-5}{3} \rceil$. (■)

We now return to the proof of the theorem. If T is a path of order $n \geq 8$, then it follows from Example 2 (a) $\gamma_{rr}(T) = \lceil \frac{2n+4}{3} \rceil \leq \lceil \frac{2n+4s(T)+\ell(T)-5}{3} \rceil$. Now suppose that T is not a path, and let v_{k+1} be a vertex of degree at least 3 such that k is minimum. Then $k \geq 7$, and by symmetry we know that $v_{D-1}, v_{D-2}, \ldots, v_{D-6}$ are vertices of degree two. Let $T' = T - \{v_1, v_2, \ldots, v_k\}$. Then $|V(T')| = n-k$, $s(T') = s(T)-1$, $\ell(T') = \ell(T)-1$. If $k \equiv 1, 2 \pmod 3$, then it follows by Example 2 and the induction hypothesis, that

$$\begin{aligned}
\gamma_{rr}(T) &\leq \lceil \tfrac{2k+4}{3} \rceil + \left\lceil \tfrac{2(n-k)+4s(T)+\ell(T)-10}{3} \right\rceil \\
&\leq \left\lceil \tfrac{2n+4s(T)+\ell(T)-5}{3} \right\rceil.
\end{aligned}$$

Let now $k = 3t$ for an integer $t \geq 3$. If $v_{k+1} \in V_0$, then f can be extended to an RRDF of T by assigning $\{1\}$ to v_1, $\{1,2\}$ to $v_2, v_5, \ldots, v_{3t-1}$ and

$\emptyset$ to the remaining vertices of $\{v_1, v_2, \ldots, v_{3t}\}$, and if $v_{k+1} \notin V_0$, then f can be extended to an RRDF of T by assigning $\{1,2\}$ to $v_1, v_4, \ldots, v_{3t+1}$ and $\emptyset$ to the remaining vertices of $\{v_1, v_2, \ldots, v_{3t}\}$. This implies $\gamma_{rr}(T) \leq \gamma_{rr}(T') + 2t + 1$. By the induction hypothesis, we obtain

$$\gamma_{rr}(T) \leq \gamma_{rr}(T') + 2t + 1 \leq \lceil \frac{2n + 4s(T) + \ell(T) - 5}{3} \rceil.$$

This completes the proof. □

Observation 14 Let T be a tree of order $n \geq 2$. Then

$$\gamma_{rr}(T) \geq \ell(T).$$

If $\mathrm{diam}(T) \geq 3$, then $\gamma_{rr}(T) = \ell(T)$ if and only if each vertex of T is a leaf or a strong support vertex.

Proof. If g is an RRDF on T, then $g(v) \neq \emptyset$ for each leaf v of T. Therefore $\gamma_{rr}(T) \geq \ell(T)$, and the lower bound is proved.

Now let $\mathrm{diam}(T) \geq 3$. Assume first that each vertex of T is a leaf or a strong support vertex. Let $S = \{u_1, u_2, \ldots, u_k\}$ be the set of strong support vertices. Since $\mathrm{diam}(T) \geq 3$, the subtree $T[S]$ is connected and contains at least two vertices. Let v_i be a leaf adjacent to u_i for $i \in \{1, 2, \ldots, k\}$, and define $f : V(T) \to \mathcal{P}(\{1,2\})$ by $f(u_i) = \emptyset$, $f(v_i) = \{1\}$ for $i \in \{1, 2, \ldots, k\}$ and $f(x) = \{2\}$ otherwise. Then f is an RRDF on T of weight $\ell(T)$ and thus $\gamma_{rr}(T) = \ell(T)$.

Conversely, assume that $\gamma_{rr}(T) = \ell(T)$. Let h be an RRDF on T. Suppose that T contains a vertex w which is neither a leaf nor a support vertex. If $h(w) \neq \emptyset$, then we obtain the contradiction $\gamma_{rr}(T) \geq \ell(T) + 1$. If $h(w) = \emptyset$, then w has a neighbor y such that $h(y) \neq \emptyset$. Since y is not a leaf, we obtain the contradiction $\gamma_{rr}(T) \geq \ell(T) + 1$. Therefore each vertex of T is a leaf or a support vertex.

Suppose that u is a support vertex adjacent to exactly one leaf v. If $h(u) \neq \emptyset$, $\gamma_{rr}(T) \geq \ell(T) + 1$, a contradiction. If $h(u) = \emptyset$, then $h(x) = \{1,2\}$ for $x \in N(u)$ and so $\gamma_{rr}(T) \geq \ell(T) + 1$, a contradiction again. Hence each support vertex is strong, and the proof is complete. □

The star $K_{1,n-1}$ shows that the condition $\mathrm{diam}(T) \geq 3$ in Observation 14 for the characterization of trees with $\gamma_{rr}(T) = \ell(T)$ is necessary.

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