DOMINATION NUMBERS OF m-CACTUS CHAINS

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Abstract: In this paper we present explicit formulas for domination numbers of equidistant m-cactus chains and find the corresponding minimum dominating sets. For an arbitrary m-cactus chain, we establish the lower and the upper bound for its domination number. We find some extremal chains with respect to this graph invariant.

Key words: *m-cactus chain, minimum dominating set, domination number, perfect dominating set*

2000 subject classifications: Primary 05C69

1 Introduction and terminology

Cactus graph is a connected graph in which any two simple cycles have at most one vertex in common. The study of such graphs started in 1950's under the name Husimi trees, after a paper by Kodi Husimi [9]. Later, in 1953, Harary and Uhlenbeck wrote a paper [8], in which they used the term cactus graph for a graph in which every cycle is a triangle. Since then, the study of cactus graphs has attracted a significant attention because some NP-hard facility location problems can be solved in polynomial time for cactus graphs [1, 12].
An *m-cactus graph* is a cactus in which all cycles have m vertices.
Finite *m-cactus chain* is an m-cactus graph consisting of cycles C_m^1, $C_m^2, \ldots, C_m^h$, $h \geq 2$, with the following properties:
(i) For $i = 1, \ldots, h-1$, C_m^i and C_m^{i+1} have a common vertex,
(ii) each vertex belongs to at most two cycles.
Examples of finite m-cactus chains are given in Figure 1.
With G_m^h we denote an m-cactus chain of length h. We write $G_m^h = C_m^1 C_m^2 \cdots C_m^h$, where C_m^1 and C_m^h are terminal cycles. Subgraph $C_m^k C_m^{k+1} \cdots C_m^{k+t}$, $k \geq 1$, $t \geq 0$, $k+t \leq h$, is called a *subchain* of G_m^h.
Let $c_i = min\{d(y,w) : y \in V(C_m^i), w \in V(C_m^{i+2})\}$, $i = 1, 2, \ldots, h-2$. We say that c_i is the distance between cycles C_m^i and C_m^{i+2}.
An m-cactus chain in which $c_1 = c_2 = \ldots = c_{h-2} = c$, $1 \leq c \leq \lfloor \frac{m}{2} \rfloor$ is called an *equidistant m-cactus chain*, and is denoted with EG_m^h. See Figure 1b.

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The notion of m-cactus chain appeared in [4], where the authors considered hexagonal cactus chains.
A subset D of the vertex-set of G is called a *dominating set* if every vertex v not in D is adjacent to at least one vertex of D. The *domination number* $\gamma(G)$ is the cardinality of any smallest dominating set.
A dominating set D of a graph G is *perfect* if each vertex of G is dominated by exactly one vertex in D. A perfect dominating set of G is necessarily a minimum dominating set of G as well.
The domination number is one of the most studied simple graph invariants. Several books ([6, 7]) are written on this invariant alone, and many classes of graphs were investigated with respect to it [5, 10, 11].
This article deals with determination of minimum dominating sets of EG_m^h, and proving its minimum cardinality, i.e. determination of domination numbers of EG_m^h. For an arbitrary G_m^h, bounds for γ are established and extremal chains were found.

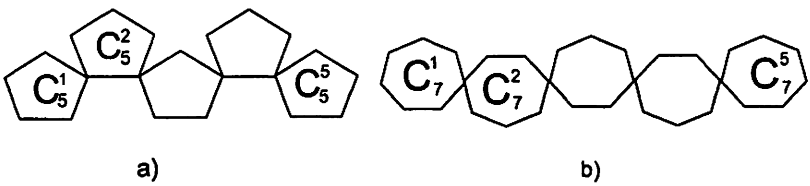

FIGURE 1. a) G_5^5 and b) EG_7^5

2 Domination numbers of EG_m^h

We start by labeling the vertices of EG_m^h in the way shown in Fig. 2.

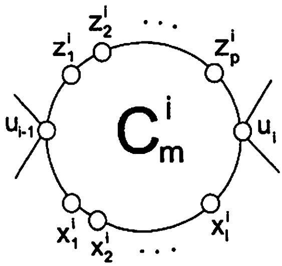

FIGURE 2. Labeling of vertices of EG_m^h.

Evidently, $p + l + 2 = m$. We take $p \leq l$ and $c = d(u_{i-1}, u_i)$, $i = 1, \ldots, h$. The following well-known result will be used repeatedly in our proofs.

Proposition 1 Let C_m be a cycle and P_m a path with m vertices. Then

$$\gamma(C_m) = \gamma(P_m) = \left\lceil \frac{m}{3} \right\rceil.$$

Now we can state our main results.

Theorem 1 Let EG^h_m be the equidistant cactus chain of length $h \geq 1$ and $m \geq 3$. Then

$$\gamma(EG^h_m) = \begin{cases} h\left\lceil \frac{m}{3} \right\rceil - (h-1), & \begin{cases} m = 1 \pmod 3, \\ m, c = 0 \pmod 3, \\ m = 2 \pmod 3 \text{ with} \\ c = 0, 2 \pmod 3, \end{cases} \\ h\frac{m}{3} - \left\lfloor \frac{h}{2} \right\rfloor, & m = 0 \pmod 3 \text{ and } c = 1, 2 \pmod 3, \\ h\left(\left\lceil \frac{m}{3} \right\rceil - 1\right) + \left\lceil \frac{h+1}{3} \right\rceil, & m = 2 \pmod 3 \text{ and } c = 1 \pmod 3. \end{cases}$$

Proof: The case $h = 1$ is settled in Proposition 1. We continue with the proof for $h \geq 2$.

Case 1 $m = 1 \pmod 3$.

Let $c = 0 \pmod 3$. Then $l = 0 \pmod 3$ and $p = 2 \pmod 3$. We consider the set $D_i = \left\{z^i_{3j-2} : j = 1, 2, \ldots, \left\lceil \frac{p}{3} \right\rceil\right\} \cup \left\{x^i_{3j-1} : j = 1, 2, \ldots, \frac{l}{3}\right\}$.
Set $P_0 = \left(\cup^h_{i=1} D_i\right) \cup \{u_h\}$ is a dominating set of EG^h_m.

If $c = 1 \pmod 3$, then $l = 2 \pmod 3$ and $p = 0 \pmod 3$.
Set $P_1 = \left(\cup^h_{i=1} D_i\right) \cup \{u_h\}$, where
$D_i = \left\{z^i_{3j-1} : j = 1, 2, \ldots, \frac{p}{3}\right\} \cup \left\{x^i_{3j-2} : j = 1, 2, \ldots, \left\lceil \frac{l}{3} \right\rceil\right\}$, is a dominating set of EG^h_m.

For $c = 2 \pmod 3$ we have $l, p = 1 \pmod 3$. Set $P_2 = \left(\cup^h_{i=1} D_i\right) \cup \{u_0\}$, where $D_i = \{u_i\} \cup \left\{z^i_{3j} : j = 1, 2, \ldots, \left\lfloor \frac{p}{3} \right\rfloor\right\} \cup \left\{x^i_{3j} : j = 1, 2, \ldots, \left\lfloor \frac{l}{3} \right\rfloor\right\}$, is a dominating set of EG^h_m.

By calculating the cardinality of sets P_r, $r = 0, 1, 2$, we obtain

$$\gamma(EG^h_m) \leq |P_r| = h\left\lceil \frac{m}{3} \right\rceil - h + 1.$$

Examples of EG_m^h with $m = 1 \pmod 3$ and corresponding dominating sets P_r, $r = 0, 1, 2$, are presented in Figure 3.

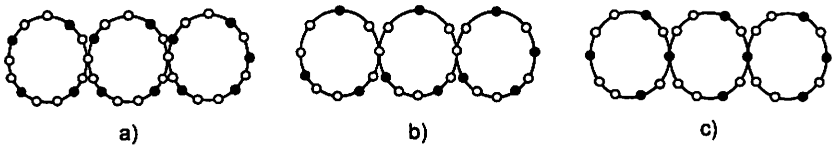

FIGURE 3. Minimum dominating set of a) C_{13}^3 with $c = 6$, b) C_{10}^3 with $c = 4$ and c) C_{10}^3 with $c = 5$.

In the sequel we prove that P_r, $r = 0, 1, 2$ has the smallest cardinality among all dominating sets of EG_m^h.

Subcase 1.1 $c \neq 2 \pmod 3$.
For $c = 0 \pmod 3$ and $h \geq 1$, the set $P_0 \setminus \{u_h\}$ is a perfect dominating set of $EG_m^h \setminus \{u_h\}$, and therefore is the minimum one. To dominate u_h we need at least one more dominating vertex. The same conclusion is obtained for case $c = 1 \pmod 3$.
We conclude $\gamma(EG_m^h) = h\left\lceil \frac{m}{3} \right\rceil - h + 1$.

Subcase 1.2 $c = 2 \pmod 3$.

Lemma 1 There exists a minimum dominating set D such that $\{u_i : i = 0, 1, \ldots, h\} \subset D$.

Proof. Let D be a minimum dominating set such that $u_s \notin D$ for some fixed $s \in \{0, 1, \ldots, h\}$.
We first consider the case $s \neq 0, h$. If $u_s \notin D$, then u_s is dominated by at least one adjacent vertex. If we assume that C_m^s contains a vertex that dominates u_s, then from Proposition 1, C_m^s is dominated by at least $\lceil \frac{m}{3} \rceil$ vertices. Since $c = 2 \pmod 3$, either $u_{s-1} \in D$ or $u_{s-1} \notin D$, but it is dominated by vertex from C_m^s. Let $T = D \cap C_m^s$ and $u_s \notin T$. We define $D' = D \setminus T \cup T'$, where $T' = \{u_{s-1}, u_s\} \cup \{z_{3j}^s : j = 1, \ldots, \lfloor \frac{p}{3} \rfloor\}$ $\cup \{x_{3j}^s : j = 1, \ldots, \lfloor \frac{l}{3} \rfloor\}$. Since $|T| = |T'|$, we have $|D'| = |D|$, so D' is also a minimum dominating set of EG_m^h.
The case when u_s is dominated by some adjacent vertex from C_m^{s+1} is symmetric to the previous one. For $s = 0$, only C_m^1 contains at least one adjacent vertex that dominates u_0. Case $s = h$ is symmetric to the case $s = 0$. □

Now, let D be a minimum dominating set that satisfies Lemma 1. Then $\forall i = 1, \ldots, h$, vertices z_1^i, x_1^i, z_p^i and x_l^i are dominated by the set

$\{u_i : i = 1, \ldots, h\}$. To dominate the remaining $l + p - 4$ vertices in C_m^i, we need at least $\lceil \frac{l-2}{3} \rceil + \lceil \frac{p-2}{3} \rceil = \lceil \frac{m}{3} \rceil - 2$ vertices.
It follows that $|D| = h\left(\lceil \frac{m}{3} \rceil - 2\right) + h + 1 = h\lceil \frac{m}{3} \rceil - h + 1$.

Case 2 $m = 2 \pmod 3$.

Let $c = 0 \pmod 3$. Then $p = 2 \pmod 3$ and $l = 1 \pmod 3$.
Set $Q_0 = \left(\cup_{i=1}^h D_i\right) \cup \{u_0\}$, where $D_i = \{u_i\} \cup \left\{z_{3j}^i : j = 1, 2, \ldots, \lfloor \frac{p}{3} \rfloor\right\}$ $\cup \left\{x_{3j}^i : j = 1, 2, \ldots, \lfloor \frac{l}{3} \rfloor\right\}$, is a dominating set of EG_m^h.
It follows that $\gamma(EG_m^h) \le |Q_0| = h\left\lceil \frac{m}{3} \right\rceil - h + 1$.

Now, let $c = 1 \pmod 3$. Then $l, p = 0 \pmod 3$. We consider the set
$D_i = \{u_{1+3i}\} \cup \left\{z_{3j-1}^{1+3i}, z_{3j-1}^{3+3i}, z_{3j}^{2+3i} : j = 1, \ldots, \frac{p}{3}, s = 1, 3\right\}$
$\cup \left\{x_{3j-2}^{1+3i}, x_{3j}^{2+3i}, x_{3j-1}^{3+3i} : j = 1, \ldots, \frac{l}{3}\right\}$.
For $h = 0 \pmod 3$ set $(Q_1)_0 = \left(\cup_{i=0}^{\frac{h}{3}-1} D_i\right) \cup \{u_h\}$ is a dominating set of EG_m^h.
For $h = 1 \pmod 3$ dominating set is
$(Q_1)_1 = \left(\cup_{i=0}^{\lfloor \frac{h}{3} \rfloor - 1} D_i\right) \cup \{u_h\} \cup \left\{z_{3j-1}^h : j = 1, \ldots, \frac{p}{3}\right\} \cup \left\{x_{3j-2}^h : j = 1, \ldots, \frac{l}{3}\right\}$.
For $h = 2 \pmod 3$ set
$(Q_1)_2 = \left(\cup_{i=0}^{\lfloor \frac{h}{3} \rfloor - 1} D_i\right) \cup \{u_{h-1}\} \cup \left\{z_{3j-1}^{h-1}, z_{3j}^h : j = 1, \ldots, \frac{p}{3}\right\}$
$\cup \left\{x_{3j-2}^{h-1}, x_{3j}^h : j = 1, \ldots, \frac{l}{3}\right\}$ is a dominating set of EG_m^h.
Sets $(Q_1)_r$, $r = 0, 1, 2$ are a dominating sets of EG_m^h and

$$\gamma(EG_m^h) \le |(Q_1)_r| = h\left(\left\lceil \frac{m}{3} \right\rceil - 1\right) + \left\lceil \frac{h+1}{3} \right\rceil, \quad r = 0, 1, 2.$$

Finally, let $c = 2 \pmod 3$. We have $p = 1 \pmod 3$ and $l = 2 \pmod 3$.
Set $Q_2 = \left(\cup_{i=1}^h D_i\right) \cup \{u_0\}$, where $D_i = \{u_i\} \cup \left\{z_{3j}^i : j = 1, 2, \ldots, \lfloor \frac{p}{3} \rfloor\right\}$ $\cup \left\{x_{3j}^i : j = 1, 2, \ldots, \lfloor \frac{l}{3} \rfloor\right\}$, is a dominating set of EG_m^h.
We conclude $\gamma(EG_m^h) \le |Q_2| = h\left\lceil \frac{m}{3} \right\rceil - h + 1$.

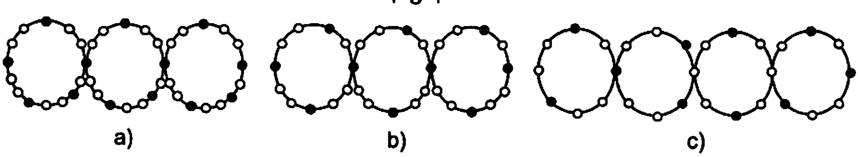

FIGURE 4. Minimum dominating set of a) C_{14}^3 with $c = 6$, b) C_{11}^3 with $c = 5$ and c) C_8^6 with $c = 4$.

Let us prove that Q_0, Q_2 and $(Q_1)_r$, $r = 0, 1, 2$ are the smallest dominating sets of EG_m^h.

Subcase 2.1 $c \neq 1 \,(\mathrm{mod}\, 3)$

Lemma 2 If D is a minimum dominating set of EG_m^h, then D contains all cut-vertices.

Proof. Let D be a minimum dominating set such that $u_s \notin D$ for some fixed $s \in \{1, \ldots, h-1\}$. Then D contains at least one vertex adjacent to u_s. Let T be the set such that $D \cap C_m^s C_m^{s+1} = T$ and T contains at least one vertex adjacent to u_s.
We have $|T| = 2\left\lceil \frac{m}{3} \right\rceil$ and $C_m^s C_m^{s+1}$ is dominated by vertices inside of $C_m^s C_m^{s+1}$. Let $T' = \{u_{s-1}, u_s, u_{s+1}\} \cup \{z_{3j}^s, z_{3j}^{s+1} : j = 1, \ldots, \left\lfloor \frac{p}{2} \right\rfloor\}$
$\cup \{x_{3j}^s, x_{s+1} : j = 1, \ldots, \left\lfloor \frac{l}{2} \right\rfloor\}$. Set $D' = D \setminus T \cup T'$ also dominates EG_m^h. Since $|T'| < |T|$, we have $|D'| < |D|$, and this is a contradiction to the assumption that D is a minimum dominating set. Therefore, every minimum dominating set necessarily contains all cut-vertices. □

From Lemma 2 we conclude that we need at least $\left\lceil \frac{p-2}{3} \right\rceil + \left\lceil \frac{l-2}{3} \right\rceil = \left\lceil \frac{m}{3} \right\rceil - 2$ dominating vertices for C_m^j, $j = 2, \ldots, h-1$. To dominate $C_m^1 \setminus \{u_1, x_l^1, z_p^1\}$ we need at least $\left\lceil \frac{m}{3} \right\rceil - 1$ vertices. The same number of vertices is necessary to dominate $C_m^h \setminus \{u_{h-1}, x_1^h, z_1^h\}$.
We conclude $\gamma(G_m^h) = h\left\lceil \dfrac{m}{3} \right\rceil - h + 1$.

Subcase 2.2 $c = 1 \,(\mathrm{mod}\, 3)$.
Let $t = \left\lfloor \frac{h}{3} \right\rfloor + 1$. We partition G_m^h into blocks B_j, $j = 1, \ldots, t$. For $j \neq t$ we define $B_j = C_m^{3j-2} C_m^{3j-1} C_m^{3j} \setminus \{u_{3j}\}$. The structure of B_t depends on h.

Lemma 3 For any dominating set D we have $|D \cap B_j| \geq 3\left\lceil \frac{m}{3} \right\rceil - 2$, $j = 1, \ldots, t-1$.

Proof. Let $j = 1$. Let x_l^3 and z_p^3 be dominated by vertices from adjacent block. To dominate $C_m^1 C_m^2$ we need at least $2\left\lceil \frac{m}{3} \right\rceil - 1$ vertices. In this case $u_2 \notin D$. (If we assume that $u_2 \in D$, then we would need at least $2\left\lceil \frac{m}{3} \right\rceil$ vertices to dominate $C_m^1 C_m^2$.) To dominate $C_m^3 \setminus \{u_2, u_3, x_l^3, z_p^3\}$ we need at least $\left\lceil \frac{l-1}{3} \right\rceil + \left\lceil \frac{p-1}{3} \right\rceil = \left\lceil \frac{m}{3} \right\rceil - 1$ vertices. Let $j \neq 1$. Let us assume that vertices u_{3j-3}, x_l^{3j} and z_p^{3j} are dominated by vertices from adjacent blocks. To dominate $C_m^{3j-2} \setminus \{u_{3j-3}, u_{3j-2}\}$, we need $\left\lceil \frac{m}{3} \right\rceil - 1$ perfect dominating vertices. This follows from the fact that $p, l = 0 \,(\mathrm{mod}\, 3)$. Then, to dominate $C_m^{3j-1} \setminus \{z_p^{3j-1}, u_{3j-1}\}$, we need $\left\lceil \frac{m}{3} \right\rceil - 1$ perfect dominating vertices. $C_m^{3j} \setminus \{z_p^{3j}, x_l^{3j}\} \cup \{z_p^{3j-1}, u_{3j-1}\}$ is dominated by at least $\left\lceil \frac{m}{3} \right\rceil$ vertices.
We conclude $|D \cap B_j| \geq 3\left\lceil \frac{m}{3} \right\rceil - 2$, $j = 1, \ldots, t-1$. □

Let $h = 0(mod)3$. Then $B_t = \{u_h\}$.

Lemma 4 If $|D \cap B_t| = 0$, then there exists a block B_s, $s \in \{1, \dots, t-1\}$, such that $|D \cap B_s| \geq 3\left\lceil \frac{m}{3} \right\rceil - 1$, for any dominating set D.

Proof. Let $|D \cap B_t| = 0$. Then D contains at least one of the vertices x_l^h and z_p^h. We consider three cases:
1° $z_p^h \in D$, $x_l^h \notin D$,
2° $z_p^h \notin D$ and $x_l^h \in D$, and
3° $z_p^h, x_l^h \in D$.
All cases imply that $|D \cap C_m^h| \geq \left\lceil \frac{m}{3} \right\rceil$. First two cases imply that either $u_{h-1} \in D$ or $u_{h-1} \notin D$. If $u_{h-1} \notin D$, then u_{h-1} is dominated by some vertex from C_m^h. The last case implies $u_{h-1} \in D$.
Let $u_{h-1} \in D$. If u_{h-3} is dominated by vertices from adjacent block, we have $|D \cap B_{t-1}| = 3\left\lceil \frac{m}{3} \right\rceil - 2$. If u_{h-3} is dominated by some vertex inside of B_{t-1}, then $|D \cap B_{t-1}| \geq 3\left\lceil \frac{m}{3} \right\rceil - 1$ and we are done with the proof.
If $u_{h-1} \notin D$, then, even if u_{h-3} is dominated by some vertex from adjacent block, we obtain $|D \cap B_{t-1}| \geq 3\left\lceil \frac{m}{3} \right\rceil - 1$. However, this lower bound is not the minimum one.
Let us return to the case when $u_{h-1} \in D$ and u_{h-3} is dominated by some vertex from adjacent block. Then, u_{h-3} is dominated by at least one of the vertices x_l^{h-3} and z_p^{h-3} from B_{t-2}. We continue with the proof inductively and either obtain a block B_s such that $|D \cap B_s| \geq 3\left\lceil \frac{m}{3} \right\rceil - 1$ for some $s \in \{2, \dots, t-1\}$ or $|D \cap B_j| = 3\left\lceil \frac{m}{3} \right\rceil - 2$, $j = 2, \dots, t-1$. But then u_3 is dominated by at least one of the vertices x_l^3 and z_p^3. Since all vertices from B_1 are dominated only by vertices inside of B_1, we conclude that $|D \cap B_1| \geq 3\left\lceil \frac{m}{3} \right\rceil - 1$. □

From Lemmas 3 and 4 we conclude that for $h = 0 \pmod 3$ we have
$|D| \geq (t-2)\left(3\left\lceil \frac{m}{3} \right\rceil - 2\right) + 3\left\lceil \frac{m}{3} \right\rceil - 1 = h\left(\left\lceil \frac{m}{3} \right\rceil - 1\right) + \left\lceil \frac{h+1}{3} \right\rceil$.

For $h = 1 \pmod 3$ we have $B_t = C_m^h$. Since only u_{h-1} can be dominated by vertices from adjacent block, we conclude that $|D \cap B_t| \geq \left\lceil \frac{m}{3} \right\rceil$ for any dominating set D.
If $h = 2 \pmod 3$, then $B_t = C_m^{h-1} C_m^h$. If u_{h-2} is dominated by some vertex from B_{t-1}, then $C_m^{h-1} \setminus \{u_{h-2}, u_{h-1}\}$ is perfectly dominated by $\left\lceil \frac{m}{3} \right\rceil - 1$ vertices. To dominate C_m^h we need at least $\left\lceil \frac{m}{3} \right\rceil$ vertices. We obtain $|D \cap B_t| \geq 2\left\lceil \frac{m}{3} \right\rceil - 1$.

By using Lemma 3 and the above conclusions, for $h = 1, 2 \pmod 3$ we have $|D| \geq h\left(\left\lceil \frac{m}{3} \right\rceil - 1\right) + \frac{h+1}{3}$ for any dominating set D.

Case 3 $m = 0\,(\text{mod}\,3)$.

For $c = 0\,(\text{mod}\,3)$ we have $p, l = 2\,(\text{mod}\,3)$. If we consider the set $R_0 = \left(\cup_{i=1}^{h} D_i\right) \cup \{u_0\}$, where $D_i = \{u_i\} \cup \left\{z_{3j}^{i} : j = 1, 2, \ldots, \left\lfloor \frac{p}{3} \right\rfloor\right\}$ $\cup \left\{x_{3j}^{i} : j = 1, 2, \ldots, \left\lfloor \frac{l}{3} \right\rfloor\right\}$, then R_0 is dominating set of EG_m^h and $\gamma(EG_m^h) \leq |R_0| = h\frac{m}{3} - h + 1$.

Let $c = 1\,(\text{mod}\,3)$. We have $p = 0\,(\text{mod}\,3)$ and $l = 1\,(\text{mod}\,3)$.
We consider the sets $D_{2i-1} = \{u_{2i-1}\} \cup \left\{z_{3j-2}^{2i-1} : j = 1, 2, \ldots, \frac{p}{3}\right\}$ $\cup \left\{x_{3j-1}^{2i-1} : j = 1, 2, \ldots, \left\lfloor \frac{l}{3} \right\rfloor\right\}$ and
$D_{2i} = \left\{z_{3j}^{2i} : j = 1, 2, \ldots, \frac{p}{3}\right\} \cup \left\{x_{3j}^{2i} : j = 1, 2, \ldots, \left\lfloor \frac{l}{3} \right\rfloor\right\}$.
Then $R_1 = \left(\cup_{i=1}^{\lceil \frac{h}{2} \rceil} D_{2i-1}\right) \cup \left(\cup_{i=1}^{\lfloor \frac{h}{2} \rfloor} D_{2i}\right)$ is a dominating set of EG_m^h and $\gamma(EG_m^h) \leq |R_1| = h\frac{m}{3} - \left\lfloor \frac{h}{2} \right\rfloor$.

If $c = 2\,(\text{mod}\,3)$, then $p = 1\,(\text{mod}\,3)$ and $l = 0\,(\text{mod}\,3)$.
Set $R_2 = \left(\cup_{i=1}^{\lceil \frac{h}{2} \rceil} D_{2i-1}\right) \cup \left(\cup_{i=1}^{\lfloor \frac{h}{2} \rfloor} D_{2i}\right)$, with
$D_{2i-1} = \{u_{2i-1}\} \cup \left\{z_{3j-1}^{2i-1} : j = 1, 2, \ldots, \left\lfloor \frac{p}{3} \right\rfloor\right\} \cup \left\{x_{3j-2}^{2i-1} : j = 1, 2, \ldots, \frac{l}{3}\right\}$
and $D_{2i} = \left\{z_{3j}^{2i} : j = 1, 2, \ldots, \left\lfloor \frac{p}{3} \right\rfloor\right\} \cup \left\{x_{3j}^{2i} : j = 1, 2, \ldots, \frac{l}{3}\right\}$,
is a dominating set of EG_m^h and $\gamma(EG_m^h) \leq |R_2| = h\frac{m}{3} - \left\lfloor \frac{h}{2} \right\rfloor$.

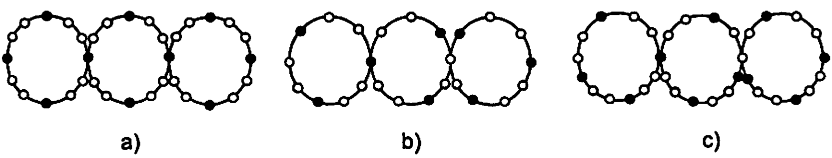

FIGURE 5. Minimum dominating sets of a) C_{12}^3 with $c = 6$, b) C_9^3 with $c = 4$ and c) C_{12}^3 with $c = 5$.

Let us prove that sets R_r, $r = 0, 1, 2$, have the minimum cardinality among all dominating sets of EG_m^h.

Subcase 3.1 $c = 0\,(\text{mod}\,3)$.

Lemma 5 Set R_0 is the unique minimum dominating set of EG_m^h and $(R_0)_{h-1} \subset (R_0)_h$ $\forall h \geq 2$.

Proof. Set R_0 is a perfect dominating set of EG_m^h and therefore is the minimum one. Uniqueness of R_0 follows from the fact that $\{u_i : i = 1, \ldots, h-1\} \subset D$ for any minimum dominating set D of EG^h_m. To prove this, we use the same approach as in the proof of Lemma 2. Since vertices z^j_1, x^j_1, z^j_p and x^j_l are already dominated by u_{j-1}, u_j, there is the unique dominating set of vertices that dominates remaining vertices in C^j_m, $j = 2, \ldots, h-1$. From $u_1, u_{h-1} \in D$, it follows that for undominated vertices in C^1_m and C^h_m we need altogether $2\frac{m}{3} - 2$ perfect dominating vertices. Therefore, $D = R_0$ and $(R_0)_{h-1} \subset (R_0)_h$, $\forall h \geq 2$. □

From Lemma 5 it follows that $\gamma(EG^h_m) = |R_0| = h\dfrac{m}{3} - h + 1$.

Subcase 3.2 $c = 1 \pmod 3$.

We partition EG^h_m into blocks B_j, $j = 1, 2, \ldots, t$, where $t = \lfloor \frac{h}{2} \rfloor + 1$. We define $B_j = C^{2j-1}_m C^{2j}_m \setminus \{u_{2j}\}$, $j = 1, 2, \ldots, t-1$. If $h = 0 \pmod 2$, then $B_t = \{u_h\}$. For $h = 1 \pmod 2$ we have $B_t = C^h_m$.
In a similar way as in the previous cases we conclude that for any dominating set D we have $|D \cap B_j| \geq 2\frac{m}{3} - 1$, $j = 1, \ldots, t-1$. If $h = 0 \pmod 2$, then $|D \cap (B_{t-1} \cup B_t)| \geq 2\frac{m}{3} - 1$. For $h = 1 \pmod 2$ $|D \cap B_t| \geq \frac{m}{3}$.
We conclude $\gamma(EG^h_m) \geq h\dfrac{m}{3} - \left\lfloor \dfrac{h}{2} \right\rfloor$.

Subcase 3.3 $c = 2 \pmod 3$.

The proof that R_2 is a dominating set of minimum cardinality is similar to the proof for the subcase 3.1. We simply interchange symbols l and p, and x and z. □

Corollary 1 For $m = 1 \pmod 3$ and $m = 2 \pmod 3$ with $c \neq 1 \pmod 3$ there exists a minimum dominating set D_h of EG^h_m with the property $\{u_i : i = 0, 1, \ldots, h\} \subset D_h$. Then $D_{h-1} \subset D_h$ $\forall h \geq 2$.

Proof. Let $m = 1 \pmod 3$. For $c = 2 \pmod 3$, we proved that P_2 is a minimum dominating set of EG^h_m. Since it contains all vertices from the set $\{u_i : i = 0, 1, \ldots, h\}$, its existence is proved.
Let $c = 0 \pmod 3$. We consider the set
$S_i = \{u_i\} \cup \{z^i_{3j} : j = 1, 2, \ldots, \lfloor \frac{p}{3} \rfloor\} \cup \{x^i_{3j} : j = 1, 2, \ldots, \frac{l}{3}\}$.
Set $P'_0 = (\cup^h_{i=1} S_i) \cup \{u_0\}$ is a dominating set of EG^h_m.
Let $c = 1 \pmod 3$. A dominating set of EG^h_m is $P'_1 = (\cup^h_{i=1} S_i) \cup \{u_0\}$, where $S_i = \{u_i\} \cup \{z^i_{3j} : j = 1, 2, \ldots, \frac{p}{3}\} \cup \{x^i_{3j} : j = 1, 2, \ldots, \lfloor \frac{l}{3} \rfloor\}$.
We have $|P'_0| = |P'_1| = h\lceil \frac{m}{3} \rceil - h + 1$. The second assertion of the corollary follows from the construction of sets P'_0, P'_1 and P_2.
Assertion for the case $m = 2 \pmod 3$ with $c \neq 1 \pmod 3$ follows from the

definition of dominating sets Q_0 and Q_2. These are the minimum dominating sets that satisfy Lemma 2 and $(Q_0)_{j-1} \subset (Q_0)_j$, $(Q_2)_{j-1} \subset (Q_2)_j$, $j = 2, \ldots, h$. □

3 Domination numbers of G_m^h

In this section we present some results about dominating sets and domination numbers of an arbitrary m-cactus chain G_m^h and find some extremal chains regarding to domination numbers. For $h \geq 2$ we will consider G_m^h as a chain obtained from G_m^{h-1} by adding one new cycle to C_m^{h-1}. With D_h we denote the minimum dominating set of G_m^h, $h \geq 1$.

Proposition 2 $\gamma(G_m^{h-1}) + \lceil \frac{m}{3} \rceil \geq \gamma(G_m^h) \geq \gamma(G_m^{h-1}) + \lceil \frac{m}{3} \rceil - 1$.

Proof. The first inequality is obvious. To prove the second inequality, notice that vertices from the minimum dominating set of G_m^{h-1} can dominate at most 3 vertices in C_m^h of G_m^h. For the remaining vertices in C_m^h we need at least $\lceil \frac{m}{3} \rceil - 1$ dominating vertices. □

Theorem 2 Let G_m^h be an arbitrary m-cactus chain of length $h \geq 2$. Then either $\gamma(G_m^h) = \gamma(G_m^{h-1}) + \lceil \frac{m}{3} \rceil - 1$ or $\gamma(G_m^h) = \gamma(G_m^{h-1}) + \lceil \frac{m}{3} \rceil$.
If $\gamma(G_m^h) = \gamma(G_m^{h-1}) + \lceil \frac{m}{3} \rceil$, then $\gamma(G_m^{h+1}) = \gamma(G_m^h) + \lceil \frac{m}{3} \rceil - 1$.

Proof. Let $m = 1 \pmod 3$. If we choose T_h to be a dominating set of G_m^h such that $\{u_i : i = 1, \ldots, h-1\} \subset T_h$, then we obtain
$|T_h \cap (C_m^j \setminus \{u_{j-1}, u_j\})| \geq \lceil \frac{m}{3} \rceil - 2 \; \forall j = 2, \ldots, h-1$,
$|T_h \cap (C_m^1 \setminus \{u_1\})| \geq \lceil \frac{m}{3} \rceil - 1$ and $|T_h \cap (C_m^h \setminus \{u_{h-1}\})| \geq \lceil \frac{m}{3} \rceil - 1$. Since $|T_h \cap C_m^j| \leq \lceil \frac{m}{3} \rceil$, $\forall j = 1, \ldots, h$, we conclude $|T_h| = h \lceil \frac{m}{3} \rceil - h + 1$. From Proposition 2 it follows that $T_h = D_h$. Since D_h contains all cut-vertices, we have $\gamma(G_m^h) = \gamma(G_m^{h-1}) + \lceil \frac{m}{3} \rceil - 1$.
The same approach is used for the case $m = 2 \pmod 3$ with $c_i \neq 1 \pmod 3$ $\forall i = 1, \ldots, h-2$.
Let us prove the theorem for case $m = 0 \pmod 3$. Adding one new cycle to G_m^{h-1} results in $m-1$ new vertices. Let $\{u_{h-1}\} = C_m^{h-1} \cap C_m^h$.
We consider the following cases:
1° $u_{h-1} \in D_{h-1}$. Then u_{h-1} dominates two more vertices in C_m^h. For the remaining $m-3$ vertices we need at most $\frac{m}{3} - 1$ vertices. Therefore, $|D_h| \leq |D_{h-1}| + \frac{m}{3} - 1$. Since $|D_h \cap C_m^j| = \frac{m}{3}$, $\forall j = 1, \ldots, h$, we conclude that $|D_{h-1}| \leq |D_h| - \frac{m}{3} - 1$. We obtain $|D_h| = |D_{h-1}| + \frac{m}{3} - 1$, that is, $\gamma(G_m^h) = \gamma(G_m^{h-1}) + \lceil \frac{m}{3} \rceil - 1$.
2° $u_{h-1} \notin D_{h-1}$. If there exists another minimum dominating set D'_{h-1} such that $u_{h-1} \in D'_{h-1}$, then we take into consideration D'_{h-1} instead of D_{h-1} and continue as in the previous case. Otherwise, u_{h-1} is domi-

nated by at least one vertex from C_m^{h-1}. Then $|D_h| \leq |D_{h-1}| + \frac{m}{3}$, since we have $m-1$ undominated vertices in C_m^h. From $|D_h \cap C_m^h| = \frac{m}{3}$ it follows that $|D_{h-1}| \leq |D_h| - \frac{m}{3}$. We conclude $|D_h| = |D_{h-1}| + \frac{m}{3}$ and $\gamma(G_m^h) = \gamma(G_m^{h-1}) + \lceil \frac{m}{3} \rceil$.
If $\gamma(G_m^h) = \gamma(G_m^{h-1}) + \lceil \frac{m}{3} \rceil$, then at least $m-2$ vertices from C_m^h are not dominated by D_{h-1}. But then obviously $u_{h-1} \notin D_{h-1}$. From case 2° we obtain $|D_h| = |D_{h-1}| + \frac{m}{3}$ and dominating vertices in C_m^h can be chosen in a way so that $u_h \in D_h$, $\{u_h\} = C_m^h \cap C_m^{h+1}$. Now we have case 1° from which we conclude $\gamma(G_m^{h+1}) = \gamma(G_m^h) + \lceil \frac{m}{3} \rceil - 1$.
In a similar way we prove the theorem for $m = 2 \,(\mathrm{mod}\, 3)$ with
$c_i = 1 \,(\mathrm{mod}\, 3)$, $\forall i = 1, \ldots, h-2$. □

Corollary 2 $\gamma(G_m^h) = h \lceil \frac{m}{3} \rceil - h + 1$ for $m = 1 \,(\mathrm{mod}\, 3)$ and for
$m = 2 \,(\mathrm{mod}\, 3)$ with $c_i \neq 1 \,(\mathrm{mod}\, 3)$, $i = 1, \ldots, h-2$. □

Corollary 3 Let $m \neq 2 \,(\mathrm{mod}\, 3)$. There exists a minimum dominating set D_h of G_m^h such that $D_{j-1} \subset D_j$ $\forall j = 2, \ldots, h$. □

Theorem 3 Let $m \geq 4$. Then $h \lceil \frac{m}{3} \rceil - (h-1) \leq \gamma(G_m^h) \leq h \lceil \frac{m}{3} \rceil - \lfloor \frac{h}{2} \rfloor$.

Proof. The first inequality follows from Proposition 2. The second inequality follows from Theorem 2 by constructing a chain G_m^h such that
$\gamma(G_m^j) = \gamma(G_m^{j-1}) + \lceil \frac{m}{3} \rceil - 1$, for $j = 0 \,(\mathrm{mod}\, 2)$, and
$\gamma(G_m^j) = \gamma(G_m^{j-1}) + \lceil \frac{m}{3} \rceil$, for $j = 1 \,(\mathrm{mod}\, 2)$, $j = 3, \ldots, h$. □

An example of an m-cactus chain with the smallest domination number is EG_m^h with $m = 1 \,(\mathrm{mod}\, 3)$.
Examples of extremal m-cactus chains with the greatest γ are illustrated in Figure 6.

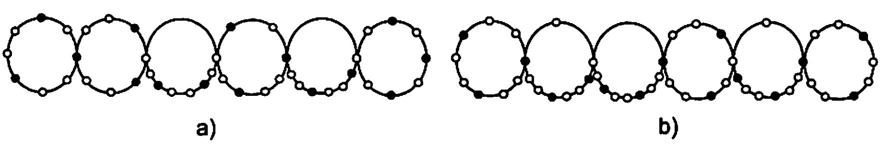

FIGURE 6. Extremal m-cactus chains with the corresponding minimum dominating sets: a) C_8^6 and b) C_9^6.

Acknowledgment Partial support of the Ministry of Science, Education and Sport of the Republic of Croatia (Grants No. 177-0000000-0884 and 037-0000000-2779) is gratefully acknowledged.

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