On S-(p,q)-Dyck paths

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Abstract

Bizley [*J. Inst. Actuar.* **80** (1954), 55-62] studied a generalization of Dyck paths from $(0,0)$ to (pn, qn) ($\gcd(p,q)=1$), which never go below the line $py = qx$ and are made of steps in $\{(0,1),(1,0)\}$, called step set, and calculated the number of such paths. In this paper, we mainly generalize Bizley's results to an arbitrary step set S. We call these paths *S-(p,q)-Dyck paths*, and give explicit enumeration formulas of such paths. In addition, we provide a proof of these formulas by the method raised in Gessel [*J. Combin. Theory Ser. A* **28** (1980), no. 3, 321-337]. As applications, we calculate some examples which generalize the classical Schröder and Motzkin numbers.

Keywords: lattice paths; generalized Dyck paths; Motzkin numbers; generating functions.

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1 Introduction

Define an *S-(p,q)-Dyck path*, denoted by S-(p,q)-path for short, to be a lattice path from $(0,0)$ to (pn, qn) which never goes below $py = qx$ with step set S, where $p, q, n \in \mathbb{N}$, $\gcd(p,q) = 1$, and S is a multiset in $\mathbb{N}^2 \setminus \{(0,0)\}$ (regard repeated elements as steps with different colors).

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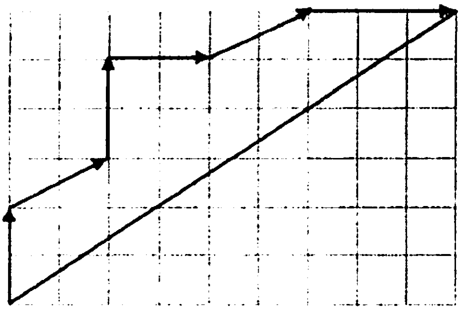

Figure 1: A {(0,2), (2,1), (2,0),(3,0)}-(3,2)-path.

With the above notation, $\{(0,1),(1,0)\}$-$(1,1)$-paths represent the classical Dyck paths, and many previous studies are concerned with S-(p,q)-paths. Examples include S-$(1,1)$-paths for certain special step sets [17, 11], $\{(0,1),(1,0)\}$-$(1,k)$-paths where k is an arbitrary positive integer [10], $\{(k,k),(0,2),(2,0)\}$-$(1,1)$-paths by some rotations [13], $\{(0,1),(1,0)\}$-(p,q)-paths [18], and general S-(p,q)-paths of which solutions are given with generating functions satisfying corresponding equations [12, 3].

The earliest study related to S-(p,q)-paths seems to appear in [9] described as a ballot problem: suppose an election results in pn votes for A and qn votes for B, where p,q,n are positive integers and $\gcd(p,q)=1$. In how many ways can votes be cast so that A's vote is always at least p/q times B's? The author gave without proof the result

$$\sum_{n_1+2n_2+3n_3+\cdots=n} \prod_{i=1}^{\infty} \frac{F_i^{n_i}}{n_i!}, \tag{1.1}$$

where $F_i = \frac{1}{pi+qi}\binom{pi+qi}{pi}$. Clearly this is the number of $\{(0,1),(1,0)\}$-(p,q)-paths from $(0,0)$ to (pn,qn). Denote (1.1) by ϕ_n. As usual, for a multivariant formal power series $f(x_1,\cdots,x_n)$, denote the coefficient of $x_1^{i_1}\cdots x_n^{i_n}$ by $[x_1^{i_1}\cdots x_n^{i_n}](f(x_1,\cdots,x_n))$. Bizley [2] showed that $\phi_n = [x^n](e^{\sum_{i=1}^{\infty} F_i x^i}-1)$ with the generating function method, and thus

$$\sum_{i=1}^{\infty} \phi_i x^i = e^{\sum_{i=1}^{\infty} F_i x^i} - 1.$$

Two additional formulas are given in [2]:

(i) The number of $\{(0,1),(1,0)\}$-(p,q)-paths from $(0,0)$ to (pn,qn) intersecting with the diagonal only at the ends is

$$\sum_{n_1+2n_2+3n_3+\cdots=n}\prod_{i=1}^{\infty}(-1)^{\sum_{i=1}^{\infty}n_i+1}\frac{F_i^{n_i}}{n_i!}. \tag{1.2}$$

(ii) The number of $\{(0,1),(1,0)\}$-(p,q)-paths from $(0,0)$ to (pn,qn) intersecting with the diagonal exactly t times ($t\le n$) except $(0,0)$ is $[x^n](1-e^{-\sum_{i=1}^{\infty}F_ix^i})^t$.

Recently [5] gave another proof of (1.1) by algebraic calculation with the recurrence relations.

In this paper, we are inspired to generalize Bizley's conclusions to an arbitrary step set S, that is, give explicit formulas of S-(p,q)-paths.

In what follows, we assume without special explanation that the step set $S=\{\mathbf{u}_i=(a_i,b_i)\mid i\in\gamma\}$ is fixed in $\mathbb{N}^2\setminus\{(0,0)\}$, p and q are fixed co-prime positive integers, all the variables are non-negative integers, and all the paths are planar lattice paths.

Notation 1.1. *In the paths from $(0,0)$ to (pn,qn), let $\mathcal{C}^S_{pn,qn}$ denote the set of all S-(p,q)-Dyck paths, let $\mathcal{B}^S_{pn,qn}$ denote the set of all S-(p,q)-Dyck paths intersecting with the diagonal only at the ends, and $\mathcal{C}^S_{pn,qn,t}$ denote the set of all S-(p,q)-Dyck paths intersecting with the diagonal exactly t times except $(0,0)$. Let $C^S_{pn,qn}$, $B^S_{pn,qn}$ and $C^S_{pn,qn,t}$ denote $|\mathcal{C}^S_{pn,qn}|$, $|\mathcal{B}^S_{pn,qn}|$ and $|\mathcal{C}^S_{pn,qn,t}|$, respectively.*

Definition 1.2. *Define the* class *of a path π to be a sequence $\alpha=(\alpha_i)_{i\in\gamma}$ (without loss of generality we regard the elements in S as ordered, thus α is a sequence), denoted by $\alpha(\pi)$, where α_i is the number of $\mathbf{u}_i$-steps in π.*

The class describes the distribution of steps in π. All the $\{(0,1),(1,0)\}$-(p,q)-paths from $(0,0)$ to (pn,qn) have a unique class (qn,pn), while there may be various classes for general S.

Notation 1.3. *For $\alpha\in\mathbb{N}^\gamma$, let $C^S_\alpha, C^S_{\alpha,t}$ and B^S_α denote the number of paths in $\bigcup_{n=0}^{\infty}\mathcal{C}^S_{pn,qn}$, $\bigcup_{n=0}^{\infty}\mathcal{C}^S_{pn,qn,t}$ and $\bigcup_{n=0}^{\infty}\mathcal{B}^S_{pn,qn}$ with class α, respectively.*

Notation 1.4. *Let*

$$\mathcal{A}^n_{p,q}=\{(\alpha_i)_{i\in\gamma}\in\mathbb{N}^\gamma\mid\sum_{i\in\gamma}\alpha_i(a_i,b_i)=(pn,qn)\}$$

and $\mathcal{A}_{p,q}=\bigcup_{n=0}^{\infty}\mathcal{A}^n_{p,q}$, denoted by $\mathcal{A}^n$ and $\mathcal{A}$ for short.

Notation 1.5. *For a multiset M in $\mathbb{N}$ satisfying $\sum_{m\in M} m < \infty$, let*

$$a_M = \frac{1}{\sum_{m\in M} m} \frac{(\sum_{m\in M} m)!}{\prod_{m\in M} m!}$$

and $a_{pn,qn} = \sum_{\alpha\in\mathcal{A}^n} a_\alpha$. (Define $a_M = 0$ when $\sum_{m\in M} m = 0$.)

For convenience, we omit S in the following when it causes no confusion. E.g., S-(p,q)-path and $C^S_{pn,qn}$ are denoted by (p,q)-path and $C_{pn,qn}$.

Our main result, Theorem 2.1, gives the explicit formulas of $C_{pn,qn}$, $B_{pn,qn}$, C_α, B_α, $C_{pn,qn,t}$ and $C_{\alpha,t}$ by calculating their generating functions, that is, the numbers of (p,q)-paths from $(0,0)$ to (pn,qn) with restricted conditions of none, intersecting with the diagonal only at the ends, with class α, intersecting with the diagonal only at the ends with class α, intersecting with the diagonal exactly t times except $(0,0)$, and intersecting with the diagonal exactly t times except $(0,0)$ with class α. Thus our theorem generalize Bizley's results as the $S = \{(0,1),(1,0)\}$ case. Although, as Bizley said, these formulas are not particularly convenient for computation, there still seems to be no better explicit expressions up to now except for special (pn,qn)s.

This paper is organized as follows. In Section 2, by generating functions method, we prove Theorem 2.1 and obtain some conclusions including recursive relations of these enumeration results. When $n = 1$, Theorem 2.1 becomes a much simpler form Theorem 2.4, which deals with paths from $(0,0)$ to (p,q) and deduces a useful corollary Lemma 2.6. In Section 3, we give another proof of Theorem 2.1 by decomposing paths and generating functions with Gessel's method [7], and calculate some further enumeration problems. In section 4, we show some applications of our conclusions by several examples, including the generalized Catalan numbers, Schröder numbers and Motzkin numbers derived from Lemma 2.6.

2 Main results

Let u^α denote $\prod_{i\in\gamma} u_i^{\alpha_i}$. We define the generating functions of the enumerative sequences:

$$A(x) = \sum_{i=1}^{\infty} a_{pi,qi} x^i, \ B(x) = \sum_{i=1}^{\infty} B_{pi,qi} x^i,$$

$$C(x) = \sum_{i=1}^{\infty} C_{pi,qi} x^i, \ C_t(x) = \sum_{i=1}^{\infty} C_{pi,qi,t} x^i,$$

and the multivariable generating functions by further considering the classes of the paths:

$$A(x,u) = \sum_{i\geq 1, \alpha\in \mathcal{A}^i} a_\alpha x^i u^\alpha, \ B(x,u) = \sum_{i\geq 1, \alpha\in \mathcal{A}^i} B_\alpha x^i u^\alpha,$$

$$C(x,u) = \sum_{i\geq 1, \alpha\in \mathcal{A}^i} C_\alpha x^i u^\alpha, C_t(x,u) = \sum_{i\geq 1, \alpha\in \mathcal{A}^i} C_{\alpha,t} x^i u^\alpha.$$

Theorem 2.1. *For $n \geq t > 0$, $\alpha \in \mathcal{A}^n_{p,q}$, there holds:*

(i) $C(x) = e^{A(x)} - 1$. *Thus*

$$C_{pn,qn} = \sum_{n_1+2n_2+3n_3+\cdots=n} \prod_{i=1}^{\infty} \frac{a_{pi,qi}^{n_i}}{n_i!}.$$

(ii) $B(x) = 1 - e^{-A(x)}$. *Thus*

$$B_{pn,qn} = \sum_{n_1+2n_2+3n_3+\cdots=n} (-1)^{1+\sum_{i=1}^{\infty} n_i} \prod_{i=1}^{\infty} \frac{a_{pi,qi}^{n_i}}{n_i!}.$$

(iii) $C(x,u) = e^{A(x,u)} - 1$. *Thus*

$$C_\alpha = \sum_{\sum_{\lambda\in\mathcal{A}} \lambda n_\lambda = \alpha} \prod_{\lambda\in\mathcal{A}} \frac{a_\lambda^{n_\lambda}}{n_\lambda!}.$$

(iv) $B(x,u) = 1 - e^{-A(x,u)}$. *Thus*

$$B_\alpha = \sum_{\sum_{\lambda\in\mathcal{A}} \lambda n_\lambda = \alpha} (-1)^{1+\sum_{\lambda\in\mathcal{A}} n_\lambda} \prod_{\lambda\in\mathcal{A}} \frac{a_\lambda^{n_\lambda}}{n_\lambda!}.$$

(v) $C_t(x) = (1 - e^{-A(x)})^t$. *Thus*

$$C_{pn,qn,t} = \sum_{0\leq k\leq t, n_1+2n_2+3n_3+\cdots=n} (-1)^{k+\sum_{i=1}^{\infty} n_i} \binom{t}{k} k^{\sum_{i=1}^{\infty} n_i} \prod_{i=1}^{\infty} \frac{a_{pi,qi}^{n_i}}{n_i!}.$$

(vi) $C_t(x,u) = (1 - e^{-A(x,u)})^t$. *Thus*

$$C_{\alpha,t} = \sum_{0\leq k\leq t, \sum_{\lambda\in\mathcal{A}} \lambda n_\lambda = \alpha} (-1)^{k+\sum_{\lambda\in\mathcal{A}} n_\lambda} \binom{t}{k} k^{\sum_{\lambda\in\mathcal{A}} n_\lambda} \prod_{\lambda\in\mathcal{A}} \frac{a_\lambda^{n_\lambda}}{n_\lambda!}.$$

To prove Theorem 2.1, first we define a relation "$\sim$", as [5] did, on the lattice paths with step set S as follows: given two paths $\pi = \mathbf{u}_{\mathbf{i}_1} \cdots \mathbf{u}_{\mathbf{i}_a}$ and ϕ, say $\pi \sim \phi$ if and only if there exists some $1 \leqslant k \leqslant a$ such that $\pi_k \triangleq \mathbf{u}_{\mathbf{i}_{k+1}} \cdots \mathbf{u}_{\mathbf{i}_a} \mathbf{u}_{\mathbf{i}_1} \cdots \mathbf{u}_{\mathbf{i}_k}$ is just ϕ. Obviously, "$\sim$" is an equivalence relation. Let $[\pi]$ denote the equivalence class of π and $per(\pi) \triangleq \min\{r | r > 0, \pi = \pi_r\}$ denote the period of π. Then $per(\pi) \mid \sum_{i \in \gamma} \alpha_i$, where $(\alpha_i)_{i \in \gamma}$ is the class of π.

We say that a path π from $(0,0)$ to (pn, qn) have a *lowest point*, as [2] defined, at a lattice point $X = (x_1, x_2)$ on π (we mean that X is an endpoint of some step in π) if $px_2 - qx_1 = \min\{py_2 - qy_1\}$ where the *min* extending over all lattice points $Y = (y_1, y_2)$ on π. E.g., the path in Figure 2 have lowest points at X_1, X_2 and X_3. Obviously, for a path π from $(0,0)$ to (pn, qn), all those paths in $[\pi]$ have the same number of lowest points which is no more than n (the starting and ending points are seen as the same point), denoted by $lp(\pi)$.

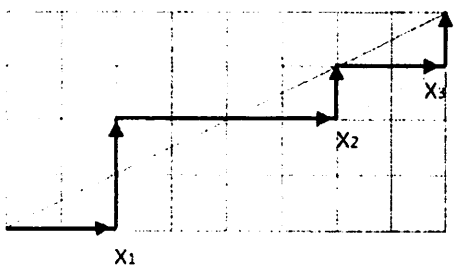

Figure 2: A (2,1)-path with 3 lowest points.

The following lemma is the key to proving Theorem 2.1:

Lemma 2.2. *For $n \geq t > 0$, $\alpha \in \mathcal{A}^n_{p,q}$, we have:*

(i) $a_\alpha = \sum_{t=1}^{n} \frac{1}{t} C_{\alpha,t}$.

(ii) $a_{pn,qn} = \sum_{t=1}^{n} \frac{1}{t} C_{pn,qn,t}$.

Proof. Note that $C_{pn,qn,t} = \sum_{\alpha \in \mathcal{A}^n} C_{\alpha,t}$ and $a_{pn,qn} = \sum_{\alpha \in \mathcal{A}^n} a_\alpha$, we only need to prove (i). It is equivalent to

$$\frac{(\sum_{i\in\gamma}\alpha_i)!}{\prod_{i\in\gamma}\alpha_i!} = \sum_{t=1}^{n}\frac{\sum_{i\in\gamma}\alpha_i}{t}C_{\alpha,t}. \tag{2.1}$$

Let $a = \sum_{i\in\gamma}\alpha_i$, $\mathcal{P}_\alpha$ and $\mathcal{C}_\alpha$ denote the sets of paths with class α and (p,q)-paths with class α, respectively. Then the left side of (2.1) is $|\mathcal{P}_\alpha|$. Note that a (p,q)-path π intersecting with the diagonal exactly t times except $(0,0)$ has t lowest points, and there are t (p,q)-paths in $\pi_1,\cdots,\pi_a$. We have

$$\begin{aligned}
\sum_{t=1}^{n}\frac{a}{t}C_{\alpha,t} &= \sum_{t=1}^{n}\frac{1}{t}\sum_{\pi\in\mathcal{P}_\alpha}1_{\{\pi\in\mathcal{C}_\alpha,lp(\pi)=t\}}\sum_{k=1}^{a}\sum_{\phi\in\mathcal{P}_\alpha}1_{\{\phi=\pi_k\}}\\
&= \sum_{t=1}^{n}\frac{1}{t}\sum_{k=1}^{a}\sum_{\pi,\phi\in\mathcal{P}_\alpha}1_{\{\pi\in\mathcal{C}_\alpha,lp(\pi)=t,\phi=\pi_k\}}\\
&= \sum_{\phi\in\mathcal{P}_\alpha}\sum_{t=1}^{n}1_{\{lp(\phi)=t\}}\sum_{k=1}^{a}\sum_{\pi\in\mathcal{P}_\alpha}\frac{1}{t}1_{\{\pi\in\mathcal{C}_\alpha,\pi=\phi_{a-k}\}}\\
&= \sum_{\phi\in\mathcal{P}_\alpha}\sum_{k=1}^{a}\frac{1}{lp(\phi)}1_{\{\phi_{a-k}\in\mathcal{C}_\alpha\}}\\
&= |\mathcal{P}_\alpha|.
\end{aligned}$$

Accordingly, (2.1) is true. ∎

Now we could prove Theorem 2.1 by generating functions method:

Proof of Theorem 2.1. We only need to prove the relationships between these generating functions, which the formulas of $C_{pn,qn}$, $B_{pn,qn}$, C_α, B_α, $C_{pn,qn,t}$ and $C_{\alpha,t}$ could be easily derived from.

By the definitions, any (p,q)-path in $\mathcal{C}_{pn,qn,t}$ could be divided into t paths in $\bigcup_{i=1}^{\infty}\mathcal{B}_{pi,qi}$ uniquely. Thus

$$C_{pn,qn,t} = \sum_{n_1+n_2+\cdots+n_t=n}\prod_{i=1}^{t}B_{pn_i,qn_i} = [x^n](B(x))^t.$$

Applying Lemma 2.2 (ii), we have

$$a_{pn,qn} = \sum_{t=1}^{n}\frac{1}{t}C_{pn,qn,t} = \sum_{t=1}^{n}\frac{1}{t}[x^n](B(x))^t = -[x^n]\ln(1-B(x)). \tag{2.2}$$

The last equality holds since when $t > n$,

$$[x^n](B(x))^t = [x^n](x^t((B(x)/x)^t)) = 0.$$

Therefore $A(x) = -\ln(1 - B(x))$. This prove Theorem 2.1 (ii).

Since $C_{pn,qn,t} = [x^n](B(x))^t$ and Theorem 2.1 (ii), we have

$$C_t(x) = (B(x))^t = (1 - e^{-A(x)})^t.$$

This is Theorem 2.1 (v).

Similarly to (2.2), we have

$$C_{pn,qn} = \sum_{t=1}^{n} C_{pn,qn,t} = [x^n] \sum_{t=1}^{n} (B(x))^t = [x^n](B(x)(1 - B(x))^{-1}).$$

Then $C(x) = B(x)(1 - B(x))^{-1} = e^{A(x)} - 1$, which is Theorem 2.1 (i).

The proofs of Theorem 2.1 (iii), (iv) and (vi) are similar. Any path in $\mathcal{C}_{pn,qn,t}$ with class α could be divided into t paths in $\bigcup_{j=1}^{\infty} \mathcal{B}_{pj,qj}$ uniquely such that the sum of the classes of these paths is α, thus

$$C_{\alpha,t} = \sum_{\alpha^1+\alpha^2+\cdots+\alpha^t=\alpha} \prod_{i=1}^{t} B_{\alpha^i} = [x^n u^\alpha](B(x,u))^t.$$

Applying Lemma 2.2 (i), we have

$$a_\alpha = \sum_{t=1}^{n} \frac{1}{t} C_{\alpha,t} = \sum_{t=1}^{n} \frac{1}{t} [x^n u^\alpha](B(x,u))^t = -[x^n u^\alpha] \ln(1 - B(x,u)). \quad (2.3)$$

Then $A(x,u) = -\ln(1 - B(x,u))$. This prove Theorem 2.1 (iv).

Since $C_{\alpha,t} = [x^n u^\alpha](B(x,u))^t$ and Theorem 2.1 (iv), we have

$$C_t(x,u) = (B(x,u))^t = (1 - e^{-A(x,u)})^t.$$

This is Theorem 2.1 (vi).

Similarly to (2.3), we have

$$C_\alpha = \sum_{t=1}^{n} C_{\alpha,t} = [x^n u^\alpha] \sum_{t=1}^{n} (B(x,u))^t = [x^n u^\alpha] B(x,u)(1 - B(x,u))^{-1}.$$

Then $C(x,u) = B(x,u)(1 - B(x,u))^{-1} = e^{A(x,u)} - 1$, that is Theorem 2.1 (iii).

This completes the proof. ∎

Remark 2.3. *From Theorem 2.1 (i)-(iv), one could get the inverse formulas. E.g., since Theorem 2.1 (iv) implies $A(x,u) = -\ln(1 - B(x,u))$, we have*

$$a_\alpha = \sum_{\sum_{\lambda\in\mathcal{A}} \lambda\beta_\lambda = \alpha} a_\beta \prod_{\lambda\in\mathcal{A}} B_\lambda^{\beta_\lambda},$$

where $\beta = (\beta_\lambda)_{\lambda\in\mathcal{A}}$.

When $n = 1$, note that there are no integral points on the segment $\{(x, \frac{qx}{p}) \mid 0 < x < p, x \in \mathbb{R}\}$, and the equations $\sum_{i=1}^{\infty} in_i = 1$ and $\sum_{\lambda\in\mathcal{A}} \lambda n_\lambda = \alpha$ have a unique solution, we get the following corollary from Theorem 2.1 immediately:

Theorem 2.4. *For $\alpha \in \mathcal{A}^1_{p,q}$, we have:*

(i) *The number of (p,q)-paths from $(0,0)$ to (p,q) is*

$$C_{p,q} = B_{p,q} = a_{p,q} = \sum_{\alpha\in\mathcal{A}^1} \frac{1}{\sum_{i\in\gamma} \alpha_i} \frac{(\sum_{i\in\gamma} \alpha_i)!}{\prod_{i\in\gamma} \alpha_i!}.$$

(ii) *The number of (p,q)-paths from $(0,0)$ to (p,q) with class α is*

$$C_\alpha = B_\alpha = a_\alpha = \frac{1}{\sum_{i\in\gamma} \alpha_i} \frac{(\sum_{i\in\gamma} \alpha_i)!}{\prod_{i\in\gamma} \alpha_i!}.$$

Remark 2.5. *Theorem 2.4 implies that the formulas of (p,q)-paths from $(0,0)$ to (p,q) have pretty forms. It could also derive from Lemma 2.2 letting $n = 1$, where the combinatorial meaning of $(\sum_{i\in\gamma} \alpha_i) C_\alpha = \frac{(\sum_{i\in\gamma} \alpha_i)!}{\prod_{i\in\gamma} \alpha_i!}$ becomes that for each path π from $(0,0)$ to (p,q) with class α, there exists unique (p,q)-path in $[\pi]$.*

The following lemma deduced from Theorem 2.4 shows that $(m,1)$-paths from $(0,0)$ to (mn,n) has a close relation with $(mn+1,n)$-paths:

Lemma 2.6. *Suppose $(1,0) \in S$, $m,n > 0$. For any $\mathbf{u_i} = (a_i, b_i) \in S \setminus \{(1,0)\}$, we have $mb_i \geq a_i$. Then $C_{mn,n} = a_{mn+1,n}$ and $C_{\alpha'} = a_\alpha$, where $\alpha \in \mathcal{A}^1_{mn+1,n}$,*

$$\alpha_i' = \begin{cases} \alpha_i - 1, & \text{if } \mathbf{u_i} = (1,0), \\ \alpha_i, & \text{otherwise.} \end{cases}$$

Proof. $\forall\ 0 < i < n$, the segment $\{(x,i) \mid mi < x \leq mi + \frac{i}{n}, x \in \mathbb{R}\}$ contains no integral points. Hence, there are no integral points in the segment $\{(x, \frac{nx}{mn+1}) \mid 0 < x < mn+1, x \in \mathbb{R}\}$ and the interior of the area surround by $y = \frac{x}{m}, y = \frac{nx}{mn+1}$ and $y = n$. From the geometric meaning of the conditions, any $(mn+1, n)$-path from $(0,0)$ to $(mn+1, n)$ is composed of a $(m,1)$-path from $(0,0)$ to (mn, n) and a $(1,0)$-step. Since $\gcd(mn+1, n) = 1$, we get the conclusion immediately from Theorem 2.4. ∎

Remark 2.7. *The notation of $(m,1)$-paths is an important generalization of Dyck paths. In recent years much work has been done in this field, such as [10], [19]. In section 4, we will apply Lemma 2.6 to such problems, and obtain some results which seem hard to get by other methods.*

From Theorem 2.1 (i) and Lemma 2.6, we have:

Theorem 2.8. *Suppose $(1,0) \in S$, $m, n > 0$. For any $\mathbf{u}_i = (a_i, b_i) \in S \setminus \{(1,0)\}$, we have $mb_i \geq a_i$. Then*

$$\sum_{n_1+2n_2+3n_3+\cdots=n} \prod_{i=1}^{\infty} \frac{a_{mi,i}^{n_i}}{n_i!} = \sum_{\alpha \in \mathcal{A}^1_{mn+1,n}} a_\alpha.$$

Theorem 2.9. *For $n \geq t > 0$, $\alpha \in \mathcal{A}^n_{p,q}$, there holds:*

(i) $C_{pn,qn} = \sum_{i=1}^{n} \frac{i}{n} a_{pi,qi} C_{p(n-i),q(n-i)}$.

(ii) $B_{pn,qn} = a_{pn,qn} - \sum_{i=1}^{n-1} \frac{i}{n} a_{pi,qi} B_{p(n-i),q(n-i)}$.

(iii) $C_{pn,qn} = \sum_{i=1}^{n} B_{pi,qi} C_{p(n-i),q(n-i)}$.

(iv) $C_\alpha = \sum_{\lambda+\nu=\alpha} \frac{\sum_{i\in\gamma} \lambda_i b_i}{qn} a_\lambda C_\nu$, *where λ, $\nu \in \mathcal{A}$.*

(v) $B_\alpha = a_\alpha - \sum_{\lambda+\nu=\alpha} \frac{\sum_{i\in\gamma} \lambda_i b_i}{qn} a_\lambda B_\nu$, *where λ, $\nu \in \mathcal{A}$.*

(vi) $C_{pn,qn,t} = \sum_{i=1}^{n-t+1} B_{pi,qi} C_{p(n-i),q(n-i),t-1}$ $(t > 1)$.

(vii) $C_{\alpha,t} = \sum_{\lambda+\nu=\alpha} B_\lambda C_{\nu,t-1}$, *where λ, $\nu \in \mathcal{A}$ $(t > 1)$.*

Proof. It follows from Theorem 2.1 (i) that

$$\ln(1 + \sum_{n=1}^{\infty} C_{pn,qn} x^n) = \sum_{i=1}^{\infty} a_{pi,qi} x^i,$$

take the derivatives, we get

$$\sum_{n=1}^{\infty} n C_{pn,qn} x^n = (1 + \sum_{n=1}^{\infty} C_{pn,qn} x^n) \sum_{i=1}^{\infty} i a_{pi,qi} x^i.$$

This proves (i) by comparing the coefficient of x^n on both sides of the equality. Similarly, (ii) could be derived from Theorem 2.1 (ii).

The conclusions of (iii), (vi) and (vii) may be easily obtained from the combinatorial meanings.

Theorem 2.1 (iii) deduces that

$$\ln(1 + \sum_{n\geq 1,\alpha\in\mathcal{A}^n} C_\alpha x^n u^\alpha) = \sum_{i\geq 1,\alpha\in\mathcal{A}^i} a_\alpha x^i u^\alpha.$$

Take the partial derivatives for variable x, we get

$$\sum_{n\geq 1,\alpha\in\mathcal{A}^n} nC_\alpha x^n u^\alpha = (\sum_{i\geq 1,\alpha\in\mathcal{A}^i} ia_\alpha x^i u^\alpha)(1 + \sum_{n\geq 1,\alpha\in\mathcal{A}^n} C_\alpha x^n u^\alpha). \quad (2.4)$$

Note that if $\alpha \in \mathcal{A}^n$, $n = \sum_{i\in\gamma} \alpha_i b_i/q$. We prove (iv) by comparing the coefficients of $x^n u^\alpha$ on both sides of (2.4). Similarly, (v) could be derived from Theorem 2.1 (iv). ∎

3 Another proof of Theorem 2.1

In this section, we provide another proof and a generalization of Theorem 2.1 applying Gessel's method [7]. We also calculate some similar enumeration problems. First, we review the relevant definitions and results in [7].

Definition 3.1. *Define the height of a point (a, b) to be $a - b$. For a path π from (m_1, n_1) to (m_2, n_2), define the height of π to be the height of $(m_2 - m_1, n_2 - n_1)$, which is defined to be the endpoint of π and denoted by $(e_1(\pi), e_2(\pi))$. Define the hight of the empty path, which has no steps and no points, to be zero. and height zero.*

Definition 3.2. *Define a minus-path to be either the empty path or a path the height of whose endpoint is negative and less than that of any other point. Define a zero-path to be a path of height zero all of whose points have non-negative height. Define a plus-path to be a path all of whose points have positive height.*

Clearly a non-empty zero-path never goes above the diagonal $y = x$, which corresponds to a Dyck path by inversing the steps in it.

Definition 3.3. *For a set of paths P, define*

$$\Gamma(P) = \Gamma(P)(t,x) = \sum_{\pi \in P} t^{e_1(\pi)-e_2(\pi)} x^{e_2(\pi)}.$$

Let S^* be the set of all paths with step set S. It is easily seen that $\Gamma(\sigma_1\sigma_2) = \Gamma(\sigma_1)\Gamma(\sigma_2)$, $\Gamma(S^*) = \sum_{n=0}^{\infty}(\Gamma(S))^n = (1-\Gamma(S))^{-1}$.

[7, Theorem 4.1] shows that any element of $\mathbb{C}[[t, x/t]]$ with constant term 1 has a unique decomposition in $\mathbb{C}[[t, x/t]]$ $f = f_- f_0 f_+$, where

$$f_- = 1 + \sum_{i,j>0} a_{ij} x^i t^{-j}, \quad f_0 = 1 + \sum_{i>0} a_i x^i,$$

and

$$f_+ = 1 + \sum_{i \geq 0, j>0} a_{ij} x^i t^j.$$

Thus

$$f_- = e^{\sum_{i>0, j<0} b_{ij} x^i t^j}, \quad f_0 = e^{\sum_{i=1}^{\infty} b_{i0} x^i}, \quad f_+ = e^{\sum_{i\geq 0, j>0} b_{ij} x^i t^j},$$

where $\ln f = \sum_{i,j} b_{ij} x^i t^j$. [7, Lemma 4.3] showed that any path π has a unique factorization $\pi_- \pi_0 \pi_+$, where π_- is a minus-path, π_0 is a zero-path, and π_+ is a plus-path. These conclusions follow [7, Theorem 4.4] which proved that

$$\Gamma(S_-) = (\Gamma(S^*))_-, \ \Gamma(S_0) = (\Gamma(S^*))_0, \ \Gamma(S_+) = (\Gamma(S^*))_+,$$

where S_-, S_0, and S_+ are the sets of minus-paths, zero-paths, and plus-paths with step set S, respectively.

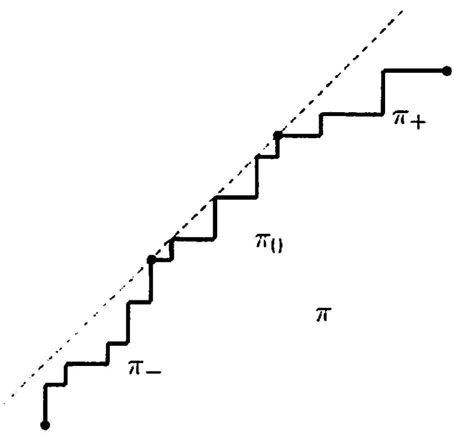

Figure 3: A path π with factorization $\pi = \pi_- \pi_0 \pi_+$.

We generalize these results to prove Theorem 2.1. For convenience, we still use the same notation. Define

$$\Gamma(P) = \Gamma(P)(t,x) = \sum_{\pi \in P} t^{qe_1(\pi)-pe_2(\pi)} x^{e_2(\pi)} u^{\alpha(\pi)}.$$

Thus $\Gamma(\sigma_1\sigma_2) = \Gamma(\sigma_1)\Gamma(\sigma_2)$, and $\Gamma(S^*) = (1-\Gamma(S))^{-1}$. Now the height of a point (a,b) is defined to be $qa - pb$. We could similarly get that any element of $\mathbb{C}[[t,u,x/t]]$ (the ring of formal power series in the variables t, $\{u_i\}_{i\in\gamma}$ and x/t on $\mathbb{C}$) with constant term 1 has a unique decomposition in $\mathbb{C}[[t,u,x/t]]$ $f = f_- f_0 f_+$, where

$$f_- = 1 + \sum_{i,j>0,\alpha} a_{ij\alpha} x^i t^{-j} u^\alpha, \quad f_0 = 1 + \sum_{i>0,\alpha} a_{i\alpha} x^i u^\alpha,$$

and

$$f_+ = 1 + \sum_{i\geq 0, j>0, \alpha} a_{ij\alpha} x^i t^j u^\alpha.$$

In fact,

$$f_- = e^{\sum_{i>0,j<0,\alpha} b_{ij\alpha} x^i t^j u^\alpha}, \quad f_0 = e^{\sum_{i\geq 1,\alpha} b_{i0\alpha} x^i u^\alpha}, \quad f_+ = e^{\sum_{i\geq 0, j>0,\alpha} b_{ij\alpha} x^i t^j u^\alpha},$$

where $\ln f = \sum_{i,j,\alpha} b_{ij\alpha} x^i t^j u^\alpha$. Thus $\ln f_0 = [t^0] \ln f$. Therefore we get the generalization of [7, Theorem 4.4] as the $u_i = 1 (i \in \gamma)$ case:

Theorem 3.4. *We have* $\Gamma(S_-) = (\Gamma(S^*))_-$, $\Gamma(S_0) = (\Gamma(S^*))_0$, *and* $\Gamma(S_+) = (\Gamma(S^*))_+$.

Now a non-empty zero-path π corresponds to a (p,q)-path by inversing the steps in π, thus

$$\begin{aligned}
\Gamma(S_0) &= \sum_{\pi \in S_0} t^{qe_1(\pi)-pe_2(\pi)} x^{e_2(\pi)} u^\alpha \\
&= 1 + \sum_{n=1}^{\infty} \sum_{\pi \in S_0, (e_1(\pi), e_2(\pi)) = (pn, qn)} x^{qn} u^{\alpha(\pi)} \\
&= 1 + C(x^q, u).
\end{aligned}$$

Now we can give the proof of Theorem 2.1 using Theorem 3.4. Since (v) and (vi) could be easily gotten from (ii) and (iv), respectively, we only need to prove (i)-(iv) :

Proof. From above, we get $\Gamma(S^*) = (1-\sum_{i\in\gamma} t^{qa_i-pb_i}x^{b_i}u_i)^{-1}$ and $\Gamma(S_0) = 1 + C(x^q, u)$. Applying Theorem 3.4, we have

$$\begin{aligned}
\ln\Gamma(S_0) &= \ln(\Gamma(S^*))_0 \\
&= [t^0]\ln\Gamma(S^*) \\
&= [t^0]\sum_{n=1}^{\infty}\frac{1}{n}\Big(\sum_{i\in\gamma} t^{qa_i-pb_i}x^{b_i}u_i\Big)^n \\
&= [t^0]\sum_{n=1}^{\infty}\sum_{\sum_{i\in\gamma}\alpha_i=n}\frac{1}{n}\frac{n!}{\prod_{i\in\gamma}\alpha_i!}t^{q\sum_{i\in\gamma}a_i\alpha_i-p\sum_{i\in\gamma}b_i\alpha_i}x^{\sum_{i\in\gamma}b_i\alpha_i}u^\alpha \\
&= \sum_{q\sum_{i\in\gamma}a_i\alpha_i=p\sum_{i\in\gamma}b_i\alpha_i>0}\frac{1}{\sum_{i\in\gamma}\alpha_i}\frac{(\sum_{i\in\gamma}\alpha_i)!}{\prod_{i\in\gamma}\alpha_i!}x^{\sum_{i\in\gamma}b_i\alpha_i}u^\alpha \\
&= \sum_{n=1}^{\infty}x^{qn}\sum_{\alpha\in\mathcal{A}''}a_\alpha u^\alpha \\
&= A(x^q, u).
\end{aligned}$$

Therefore $1 + C(x^q, u) = e^{A(x^q,u)}$, which prove Theorem 2.1 (iii).

Letting $u_i = 1 (i \in \gamma)$ in Theorem 2.1 (iii), we have

$$1 + C(x) = 1 + C(x,u)|_{u_i=1,i\in\gamma} = e^{A(x,u)}|_{u_i=1,i\in\gamma} = e^{A(x)}.$$

This prove Theorem 2.1 (i).

By decomposing (p, q)-paths as the proof of Theorem 2.1 does in section 2, one could prove (ii) and (iv) from (i) and (iii), respectively. This completes the proof. ∎

We can similarly get some further conclusions.

Let f be a function $S^* \to A$, where A is a commutative ring with identity 1. f satisfies:

(i) $f(\sigma_1\sigma_2) = f(\sigma_1)f(\sigma_2)$ for any path $\sigma_1, \sigma_2 \in S^*$.

(ii) $f(\pi) = 1$ if and only if π is the empty path.

For a set of paths P, define

$$\Gamma(P) = \Gamma(P)(t, A) = \sum_{\pi\in P} t^{qe_1(\pi)-pe_2(\pi)}f(\pi).$$

$\Gamma(S_-)$, $\Gamma(S_0)$ and $\Gamma(S_+)$ are defined similarly as above. One can similarly get

$$\Gamma(S_0) = \sum_{\pi \in S_0} f(\pi) = e^{\sum_{\alpha \in \mathcal{A}} a_\alpha \prod_{i \in \gamma} f^{\alpha_i}(\mathbf{u}_i)},$$

$$\Gamma(S_-) = e^{\sum_{c \geq 1, \alpha \in \mathcal{D}^{-c}} a_\alpha t^{-c} \prod_{i \in \gamma} f^{\alpha_i}(\mathbf{u}_i)} = e^{\sum_{\alpha \in \mathcal{D}^-} a_\alpha \prod_{i \in \gamma} (f(\mathbf{u}_i) \mathbf{t}^{\mathbf{q}\mathbf{a}_i - \mathbf{p}\mathbf{b}_i})^{\alpha_i}},$$

$$\Gamma(S_+) = e^{\sum_{c \geq 1, \alpha \in \mathcal{D}^c} a_\alpha t^c \prod_{i \in \gamma} f^{\alpha_i}(\mathbf{u}_i)} = e^{\sum_{\alpha \in \mathcal{D}^+} a_\alpha \prod_{i \in \gamma} (f(\mathbf{u}_i) \mathbf{t}^{\mathbf{q}\mathbf{a}_i - \mathbf{p}\mathbf{b}_i})^{\alpha_i}},$$

where

$$\mathcal{D}^c = \{\alpha \in \mathbb{N}^\gamma \mid q \sum_{i \in \gamma} a_i \alpha_i - p \sum_{i \in \gamma} b_i \alpha_i = c\} \ (c \in \mathbb{Z}),$$

and

$$\mathcal{D}^- = \bigcup_{c=1}^{\infty} \mathcal{D}^{-c}, \ \mathcal{D}^+ = \bigcup_{c=1}^{\infty} \mathcal{D}^c.$$

Let A be a ring of polynomials with complex coefficients in some variables and assign a variable as $f(\mathbf{u}_i)$ ($i \in \gamma$). E.g., let $f(\mathbf{u}_i)$ be v_i, $v_i t^{pb_i - qa_i}$, and $v_i t^{pb_i - qa_i}$ in the three equalities above, respectively. We get the following theorem:

Theorem 3.5. *We have:*

(i) $1 + \sum_{\alpha \in \mathcal{A}} C_\alpha v^\alpha = e^{\sum_{\alpha \in \mathcal{A}} a_\alpha v^\alpha}$.

(ii) $1 + \sum_{\alpha \in \mathcal{D}^-} C_\alpha v^\alpha = e^{\sum_{\alpha \in \mathcal{D}^-} a_\alpha v^\alpha}$, *where for* $\alpha \in \mathcal{D}^{-c}$, C_α *is defined to be the number of minus-paths starting from* $(0, 0)$ *with class* α. *Thus*

$$C_\alpha = \sum_{\sum_{\lambda \in \mathcal{D}^-} \lambda n_\lambda = \alpha} \prod_{\lambda \in \mathcal{D}^-} \frac{a_\lambda^{n_\lambda}}{n_\lambda !}.$$

(iii) $1 + \sum_{\alpha \in \mathcal{D}^+} C_\alpha v^\alpha = e^{\sum_{\alpha \in \mathcal{D}^+} a_\alpha v^\alpha}$, *where for* $\alpha \in \mathcal{D}^c$, C_α *is defined to be the number of plus-paths starting from* $(0, 0)$ *with class* α. *Thus*

$$C_\alpha = \sum_{\sum_{\lambda \in \mathcal{D}^+} \lambda n_\lambda = \alpha} \prod_{\lambda \in \mathcal{D}^+} \frac{a_\lambda^{n_\lambda}}{n_\lambda !}.$$

Theorem 2.1 (i) and (iii) are the $v_i = x^{b_i}$ and $v_i = x^{b_i} u_i$ ($i \in \gamma$) cases of Theorem 3.5 (i).

Notation 3.6. *Let*

$$E^+ = \{(s, t) \in \mathbb{N}^2 \mid qs - pt > 0\}.$$

For $(m, n) \in E^+$, *let* $\mathcal{C}^+_{m,n}$ *denote the set of all plus-paths from* $(0, 0)$ *to* (m, n), $C^+_{m,n} = |\mathcal{C}^+_{m,n}|$, *and* $a^+_{m,n} = \sum_{\pi \in \mathcal{C}^+_{m,n}} a_{\alpha(\pi)}$.

Notation 3.7. *Let*

$$E^- = \{(s,t) \in \mathbb{N}^2 \mid qs - pt < 0\}.$$

For $(m,n) \in E^-$*, let* $C^-_{m,n}$ *denote the set of all minus-paths from* $(0,0)$ *to* (m,n)*,* $C^-_{m,n} = |C^-_{m,n}|$*, and* $a^-_{m,n} = \sum_{\pi \in C^-_{m,n}} a_{\alpha(\pi)}$.

Letting $v_i = t^{a_i} x^{b_i}$ in Theorem 3.5 (ii) and (iii), we get the following theorem:

Theorem 3.8. *We have:*

(i) $1 + \sum_{(m,n)\in E^+} C^+_{m,n} t^m x^n = e^{\sum_{(m,n)\in E^+} a^+_{m,n} t^m x^n}$. *Thus*

$$C^+_{m,n} = \sum_{\sum_{\mu\in E^+} \mu d_\mu = (m,n)} \prod_{\mu \in E^+} \frac{(a^+_\mu)^{d_\mu}}{d_\mu!}.$$

(ii) $1 + \sum_{(m,n)\in E^-} C^-_{m,n} t^m x^n = e^{\sum_{(m,n)\in E^-} a^-_{m,n} t^m x^n}$. *Thus*

$$C^-_{m,n} = \sum_{\sum_{\mu\in E^-} \mu d_\mu = (m,n)} \prod_{\mu \in E^-} \frac{(a^-_\mu)^{d_\mu}}{d_\mu!}.$$

Remark 3.9. *One could generalize the theorems above to higher dimension cases. E.g., given* $d \geq 3$*,* $S = \{\mathbf{u}_i\}_{i\in\gamma}$ *is a multiset in* $\mathbb{N}^d \setminus \{(0,\cdots,0)\}$ *and* $\mathbf{p} = (p_1, \cdots, p_d) \in \mathbb{Z}^d$ *such that*

$$\max\{p_1, \cdots, p_d\} \min\{p_1, \cdots, p_d\} < 0.$$

Define C_α *to be the number of lattice paths in* $\mathbb{Z}^d$ *starting from* $(0,0)$ *with class* α *that never go below the hyperplane* $\mathbf{p} \cdot \mathbf{x} = \sum_{i\in\gamma} p_i x_i = 0$*. Define*

$$\mathcal{A}^{(d)} = \{(\alpha_i)_{i\in\gamma} \in \mathbb{N}^\gamma \mid \mathbf{p} \cdot (\sum_{i\in\gamma} \alpha_i \mathbf{u}_i) = 0\}.$$

Thus

$$1 + \sum_{\alpha\in\mathcal{A}^{(d)}} C_\alpha v^\alpha = e^{\sum_{\alpha\in\mathcal{A}^{(d)}} a_\alpha v^\alpha},$$

which is a d-dimensional generalization of Theorem 3.5 (i).

4 Applications

In this section, conclusions are obtained for several special sets S. Some of these results generalize the classical Schröder and Motzkin numbers.

Example 4.1. *Let* $S = \{(0,1),(1,0)\}$. *Then* $\mathcal{A}^n = \{(qn, pn)\}$, *and* $a_{pn,qn} = \frac{1}{pn+qn}\binom{pn+qn}{pn}$. *According to Theorem 2.1, the number of* (p,q)*-paths from* $(0,0)$ *to* (pn, qn) *is*

$$C_{pn,qn} = \sum_{n_1+2n_2+3n_3+\cdots=n} \prod_{i=1}^{\infty} \frac{a_{pi,qi}^{n_i}}{n_i!}.$$

This is the main result in [5]. Let $n = 1$, *we have*

$$C_{p,q} = a_{p,q} = \frac{1}{p+q}\binom{p+q}{p}.$$

Lemma 2.6 gives that

$$C_{mn,n} = a_{mn+1,n} = \frac{1}{mn+1+n}\binom{mn+1+n}{mn+1} = \frac{1}{mn+1}\binom{mn+n}{n},$$

which is the generalized Catalan number found in [6] and shown by many papers as the number of generalized Dyck paths, e.g., [10]. It is also a direct corollary of the generalized Ballot Problem considered in [1] which has various solutions as well (see, e.g., [4], [15, p.8],[14],[16, p.10], [20, p.10] and [8]).

Example 4.2. *Let* $S = \{(0,1),(1,0),(1,1)\}$. *Then*

$$\mathcal{A}^n = \{(qn-i, pn-i, i) \mid i \le \min\{pn, qn\}, i \in \mathbb{N}\}.$$

Applying Theorem 2.4, the number of (p,q)*-paths from* $(0,0)$ *to* (p,q) *with class* $(q-i, p-i, i)$ *(i.e., with* i *diagonal steps) is*

$$C_{(q-i,p-i,i)} = a_{(q-i,p-i,i)} = \frac{1}{p+q-i}\binom{p+q-i}{p-i,q-i,i},$$

and the number of (p,q)*-paths from* $(0,0)$ *to* (p,q) *is*

$$C_{p,q} = a_{p,q} = \sum_{i=0}^{q} \frac{1}{p+q-i}\binom{p+q-i}{p-i,q-i,i}.$$

In particular, let $(p,q) = (mn+1,n)$*, applying Lemma 2.6 we have that the number of* $(m,1)$*-paths from* $(0,0)$ *to* (mn,n) *with* i *diagonal steps is*

$$C_{(n-i,mn-i,i)} = a_{(n-i,mn+1-i,i)} = \frac{1}{mn-i+1}\binom{mn+n-i}{mn-i,n-i,i},$$

and the number of $(m,1)$*-paths from* $(0,0)$ *to* (mn,n) *is*

$$C_{mn,n} = \sum_{i=0}^{n} \frac{1}{mn-i+1}\binom{mn+n-i}{mn-i,n-i,i}.$$

The number $C_{mn,n}$ *is indeed the m-schröder number* S_n^m *introduced and calculated in [19].*

Example 4.3. *Let* $S = \{(0,1),(1,0),(k,k)\}$*. Similarly to Example 4.2, we have*

$$\mathcal{A}^n = \{(qn-ki, pn-ki, i) \mid i \le \frac{\min\{pn,qn\}}{k}, i \in \mathbb{N}\},$$

$$C_{(q-ki,p-ki,i)} = \frac{1}{p+q-(2k-1)i}\binom{p+q-(2k-1)i}{q-ki,p-ki,i},$$

$$C_{p,q} = \sum_{i=0}^{\lfloor \frac{q}{k} \rfloor} \frac{1}{p+q-(2k-1)i}\binom{p+q-(2k-1)i}{q-ki,p-ki,i}.$$

Especially, let $(p,q) = (mn+1,n)$*, we have that the number of* $(m,1)$*-paths from* $(0,0)$ *to* (mn,n) *with* i *diagonal steps is*

$$C_{(n-ki,mn-ki,i)} = \frac{1}{mn-ki+1}\binom{mn+n-(2k-1)i}{mn-ki,n-ki,i},$$

and the number of $(m,1)$*-paths from* $(0,0)$ *to* (mn,n) *is*

$$C_{mn,n} = \sum_{i=0}^{\lfloor \frac{n}{k} \rfloor} \frac{1}{mn-ki+1}\binom{mn+n-(2k-1)i}{mn-ki,n-ki,i}.$$

Thus we get a more widely generalization for Schröder path as the $k=1$ *and* $m=1$ *case.*

Example 4.4. *Let* $S = \{(0,2),(2,0),(2k+1,2k+1)\}$*. Similarly to Example 4.2 and 4.3, we have*

$$\mathcal{A}^n = \{(\frac{qn-(2k+1)i}{2}, \frac{pn-(2k+1)i}{2}, i) \mid i \le \frac{\min\{pn,qn\}}{2k+1},$$
$$i \equiv pn \equiv qn \pmod 2, i \in \mathbb{N}\},$$

$$C_{(\frac{q-(2k+1)i}{2},\frac{p-(2k+1)i}{2},i)}=\frac{1}{\frac{p+q}{2}-2ki}\binom{\frac{p+q}{2}-2ki}{\frac{q-(2k+1)i}{2},\frac{p-(2k+1)i}{2},i},$$

where $0\le i\le \frac{\min\{p,q\}}{2k+1}, i\equiv p\equiv q \pmod 2$. *(For convenience, we omit these conditions of the variables in the following discussion.)*

To calculate the number of $(m,1)$*-paths from* $(0,0)$ *to* (mn,n) *with* i *diagonal steps* $C_{(\frac{n-(2k+1)i}{2},\frac{mn-(2k+1)i}{2},i)}$, *we discuss the parities of* m *and* n:

- *Case 1:* m *is even while* n *is odd. Then* $mn+n$ *is odd. Hence*

$$\mathcal{A}^n_{m,1}=\emptyset,\ C_{(\frac{n-(2k+1)i}{2},\frac{mn-(2k+1)i}{2},i)}=0.$$

- *Case 2:* m *and* n *are both odd. Then* $\gcd(mn+2,n)=1$. *Similarly to the proof of Lemma 2.6, any integral point in the interior of the area surround by three lines* $y=\frac{x}{m}, y=\frac{nx}{mn+2}, y=n$ *and the segment* $\{(x,\frac{nx}{mn+2})\mid 0<x<mn+2, x\in\mathbb{R}\}$ *must be the form of* $(mj+1,j)$ $(0<j<n)$. *Since* $mj+1+j=(m+1)j+1$ *is odd, it can't belong to any (m,1)-path. Therefore*

$$\begin{aligned}C_{(\frac{n-(2k+1)i}{2},\frac{mn-(2k+1)i}{2},i)}&=C_{(\frac{n-(2k+1)i}{2},\frac{mn-(2k+1)i}{2}+1,i)}\\&=a_{(\frac{n-(2k+1)i}{2},\frac{mn-(2k+1)i}{2}+1,i)}\\&=\frac{1}{\frac{mn-(2k+1)i}{2}+1}\binom{\frac{mn+n}{2}-2ki}{\frac{n-(2k+1)i}{2},\frac{mn-(2k+1)i}{2},i}.\end{aligned}$$

- *Case 3:* m *is odd while* n *is even. Then* $\gcd(mn+2,n)=2$. *Similarly to Case 2, we have*

$$C_{(\frac{n-(2k+1)i}{2},\frac{mn-(2k+1)i}{2},i)}=C_{(\frac{n-(2k+1)i}{2},\frac{mn-(2k+1)i}{2}+1,i)}.$$

Applying Theorem 2.1 (iii) with

$$\alpha=(\frac{n-(2k+1)i}{2},\frac{mn-(2k+1)i}{2}+1,i),$$

we have

$$C_{(\frac{n-(2k+1)i}{2},\frac{mn-(2k+1)i}{2}+1,i)}=\sum_{\sum_{\lambda\in\mathcal{A}}\lambda n_\lambda=\alpha}\prod_{\lambda\in\mathcal{A}}\frac{a_\lambda^{n_\lambda}}{n_\lambda!}.\qquad(4.1)$$

Note that $(mn+2)/2+n/2=1+n(m+1)/2$ *is odd, the equation*

$$\alpha_1(0,2)+\alpha_2(2,0)+\alpha_3(2k+1,2k+1)=(\frac{mn+2}{2},\frac{n}{2})$$

has no integral solutions, thus $\sum_{\lambda\in\mathcal{A}}\lambda n_\lambda = \alpha$ *has a unique solution* $n_\alpha = 1, n_\lambda = 0, \lambda \neq \alpha$. *Therefore* (4.1) *follows that*

$$C_{(\frac{n-(2k+1)i}{2},\frac{mn-(2k+1)i}{2},i)} = a_\alpha = \frac{1}{\frac{mn-(2k+1)i}{2}+1}\binom{\frac{mn+n}{2}-2ki}{\frac{n-(2k+1)i}{2},\frac{mn-(2k+1)i}{2},i}.$$

- *Case 4: m and n are both even. The method for case 3 does not work in this case. Applying Theorem 2.1 (iii) with*

$$\alpha = (\frac{n-(2k+1)i}{2},\frac{mn-(2k+1)i}{2},i),$$

we have

$$C_\alpha = \sum_{\sum_{\lambda\in\mathcal{A}}\lambda n_\lambda=\alpha}\prod_{\lambda\in\mathcal{A}}\frac{a_\lambda^{n_\lambda}}{n_\lambda!}.$$

Finding a simpler formula for this case would be an interesting problem.

As a corollary, we get a generalization of the classical Motzkin numbers:

Theorem 4.5. *Given* $S = \{(0,2),(2,0),(2k+1,2k+1)\}$, *an even integer* $i \leq \lfloor\frac{n}{2k+1}\rfloor$, *and an odd integer m. Then the number of* $(m,1)$*-paths from* $(0,0)$ *to* (mn,n) *with i diagonal steps is*

$$\frac{1}{\frac{mn-(2k+1)i}{2}+1}\binom{\frac{mn+n}{2}-2ki}{\frac{n-(2k+1)i}{2},\frac{mn-(2k+1)i}{2},i},$$

and the number of $(m,1)$*-paths from* $(0,0)$ *to* (mn,n) *is*

$$\sum_{0\leq i\leq\lfloor\frac{n}{2k+1}\rfloor, i\equiv n \pmod 2}\frac{1}{\frac{mn-(2k+1)i}{2}+1}\binom{\frac{mn+n}{2}-2ki}{\frac{n-(2k+1)i}{2},\frac{mn-(2k+1)i}{2},i}.$$

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