ZAGREB ECCENTRICITY INDICES OF CYCLES RELATED GRAPHS

Tufan TURACI
Department of Mathematics, Faculty of Science, Karabük University
78050, Karabük/TURKEY
E-mail: tufanturaci@karabuk.edu.tr

Abstract
Graph theory, with its diverse applications in theoretical computer science and in natural (Chemistry, Biology) in particular is becoming an important component of the mathematics. Recently, the concepts of new zagreb eccentricity indices were introduced. These indices were defined for any graph G, as follows: $M_1^*(G) = \sum_{e_{uv} \in E(G)}[\varepsilon_G(u) + \varepsilon_G(v)]$, $M_1^{**}(G) = \sum_{v \in V(G)}[\varepsilon_G(v)]^2$ and $M_2^*(G) = \sum_{e_{uv} \in E(G)}[\varepsilon_G(u)\varepsilon_G(v)]$, where $\varepsilon_G(u)$ is eccentricity value of vertex u in the graph G. In this paper, new zagreb eccentricity indices $M_1^*(G)$, $M_1^{**}(G)$ and $M_2^*(G)$ of cycles related graphs namely gear, friendship and corona graphs are determined. Then, a programming code finding values of new zagreb indices of any graph is offered.

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1. Introduction

A topological representation of a molecule can be carried out molecular graph. The concept of topological indices is very important in chemical graph theory. This theory applies graph theory in mathematical modeling of chemical structure. A topological index is the graph invariant number calculated from a graph representing a molecule.

The vulnerability value of a communication network shows the resistance of the network after the disruption of some centers or connection lines until a communication breakdown. So, vulnerability values are the important for network robustness. Also, these new zagreb indices can be vulnerability parameters for networks that modeled by the graphs.

In this paper, we consider simple finite undirected graphs without loops and multiple edges. Let $G = (V(G), E(G))$ be a simple undirected graph of order n and size m. We begin by recalling some standard definitions using throughout this paper. For any vertex $v \in V(G)$, *the open neighborhood of* v is $N_G(v) = \{u \in V(G) | uv \in E(G)\}$ and *closed neighborhood of* v is $N_G[v] = N_G(v) \cup \{v\}$. The *degree of vertex* v in G denoted by $d_G(v)$, that is the size of its open neighborhood [2]. The *distance* $d(u,v)$ between two vertices u and v in G is the length of a shortest path between them. The *diameter* of G, denoted by $diam(G)$ is the largest distance between two vertices in $V(G)$. *Eccentricity of vertex u* in G denoted by $\varepsilon_G(u)$, that is the largest between *vertex u* and any other *vertex v* of G, $\varepsilon_G(u) = max_{v\in V(G)} d(u,v)$ [4]. The *complement* $\bar{G}$ of a graph G has $V(G)$ as its vertex sets, but two vertex are adjacent in $\bar{G}$ if only if they are not adjacent in G [2]. There are several topological indices which are graph stable number calculated from a molecule that is represented by a graph.

The *Wiener index* is the first distance based topological index [5]. The aim of *Wiener index* is to the sum of half of the distances between every pair of vertices of G and is defined as: $W(G) = \frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n} d(v_i, v_j)$. After that the *Zagreb indices* were introduced by Gutman and Trinajstić [6]. They are defined as: $M_1(G) = \sum_{v\in V(G)}(d_G(v))^2$ and $M_2(G) = \sum_{e_{uv}\in E(G)} d_G(u)d_G(v)$. Then, *new versions of Zagreb eccentricity indices have* been introduced by Ghorbani and Hosseinzadeh [13]. They are defined as: $M_1^*(G) = \sum_{e_{uv}\in E(G)}[\varepsilon_G(u) + \varepsilon_G(v)]$, $M_1^{**}(G) = \sum_{v\in V(G)}[\varepsilon_G(v)]^2$ and $M_2^*(G) = \sum_{e_{uv}\in E(G)}[\varepsilon_G(u)\varepsilon_G(v)]$. They have also studied of the well-known graphs. These graphs are complete graph, path graph, cycle graph, star graph and wheel graph [13]. You can see different studies about distance based topological indices in [3,7,8,10,11,12,14,15].

Let u and v be any two vertices of G. If these two vertices are adjacent in G, then the edge between these two vertices is denoted by e_{uv} in G. We shall use e_{uv} for the some edges.

Our aim in this paper is to consider the computing the new Zagreb eccentricity indices of cycles related graphs. In section 2, definitions are given and $M_1^*(G)$, $M_1^{**}(G)$ and $M_2^*(G)$ of gear graphs, friendship graphs and corona graphs are computed, respectively. Then, an algorithm is offered for computing the new Zagreb indices of any given graph in Section 3.

2. Main Results

In this section some new Zagreb indices of cycles related graphs namely gear, friendship and corona graphs are computed.

Definition 2.1 [2,9] The wheel W_n with n spokes is a graph that contains an *n-cycle* and one additional central vertex c that is adjacent to all vertices of the cycle.

Definition 2.2 [1] Gear graph is a wheel graph with a vertex added between each pair adjacent graph vertices of the outer cycle. Gear graph G_n has *(2n+1)* vertices and *3n* edges.

Theorem 2.1 *Let G_n be a gear graph. Then,* $M_1^*(G_n) = 19n$, $M_2^*(G_n) = 30n$ *and* $M_1^{**}(G_n) = 25n + 4$.

Proof. Let the vertex set G_n be $V(G_n) = V_1 \cup V_2 \cup V_3$, where: $V_1 = \{v_i \in V(G_n) | \, d_{G_n}(v_i) = n, i = 1\}$, $V_2 = \{v_i \in V(G_n) | \, d_{G_n}(v_i) = 3, \; 2 \leq i \leq n+1\}$ and $V_3 = \{v_i \in V(G_n) | \, d_{G_n}(v_i) = 2, \; n+2 \leq i \leq 2n+1\}$.

Let u, v and w be any vertices of V_1, V_2 and V_3, respectively. It is clear that $d(u,v) = 1$and $d(u,w) = 2$. So, we obtain $\varepsilon_{G_n}(u) = 2$. Furthermore, value of $d(v,w)$ can be equal to 1, 2 and 3 for $n > 4$. So, $\varepsilon_{G_n}(v) = 3$ for every $v \in V_2$. Let z be any vertex which is further from the vertex w of V_3. Then, we obtain $d(w,z) = 4$. Because this path that from vertices $w$and $z$is $zN_G(z)uN_G(w)w$. So, we have $\varepsilon_{G_n}(w) = 4$ for every $w \in V_3$. Furthermore, let the edges set of G_n be $E(G_n) = E_1 \cup E_2$, where: $E_1 = \{e_{uv} \in E(G_n) | u \in V_1, v \in V_2\}$ and $E_2 = \{e_{uv} \in E(G_n) | u \in V_2, v \in V_3\}$. When the $M_1^*(G_n)$ and $M_2^*(G_n)$ are calculated for all edges in the graph G_n, the edges in two cases should be examined.

Case1. Let e_{uv} be any edge of set E_1. It is clear that $u \in V_1$ and $v \in V_2$. Then, we have $\varepsilon_{G_n}(u) + \varepsilon_{G_n}(v) = 5$ and $\varepsilon_{G_n}(u)\varepsilon_{G_n}(v) = 6$ for every $e_{uv} \in E_1$.

Case2. Let e_{uv} be any edge of set E_2. We know that $u \in V_2$ and $v \in V_3$. Hence, $\varepsilon_{G_n}(u) + \varepsilon_{G_n}(v) = 7$ and $\varepsilon_{G_n}(u)\varepsilon_{G_n}(v) = 12$ for every $e_{uv} \in E_2$.

By Cases1 and 2, we have:

$$M_1^*(G_n) = \sum_{e_{uv} \in E_1} [\varepsilon_{G_n}(u) + \varepsilon_{G_n}(v)] + \sum_{e_{uv} \in E_2} [\varepsilon_{G_n}(u) + \varepsilon_{G_n}(v)]$$
$$= \sum_{i=1}^{n} 5 + \sum_{i=1}^{2n} 7 = 19n. \quad \square$$

$$M_2^*(G_n) = \sum_{e_{uv} \in E_1} [\varepsilon_{G_n}(u)\varepsilon_{G_n}(v)] + \sum_{e_{uv} \in E_2} [\varepsilon_{G_n}(u)\varepsilon_{G_n}(v)]$$
$$= \sum_{i=1}^{n} 6 + \sum_{i=1}^{2n} 12 = 30n. \quad \square$$

$$M_1^{**}(G_n) = \sum_{v \in V_1} [\varepsilon_{G_n}(v)]^2 + \sum_{v \in V_2} [\varepsilon_{G_n}(v)]^2 + \sum_{v \in V_3} [\varepsilon_{G_n}(v)]^2$$
$$= 2^2 + \sum_{i=1}^{n} 3^2 + \sum_{i=1}^{n} 4^2 = 25n + 4. \quad \square$$

Theorem 2.2 *Let G_n be a gear graph. Then, $M_1^*(\overline{G_n}) = M_2^*(\overline{G_n}) = 8n^2 - 8n$ and $M_1^{**}(\overline{G_n}) = 8n + 4$.*

Proof. It is clear that in the graph $\overline{G_n}$, cardinality of vertices and edge set are $(2n+1)$ and $(2n^2 - 2n)$, respectively. Let $V(\overline{G_n}) = V_1 \cup V_2$, where V_1 includes vertices of *n-cycle* in W_n and V_2 includes vertices that are added to *n-cycle* in G_n and the center vertex c. Thus, the vertices of V_1 form a complete graph of order n in the graph $\overline{G_n}$. Similarly, vertices of V_2 form a complete graph of order $(n+1)$ in the graph $\overline{G_n}$. Furthermore, the graph $\overline{G_n}$ contains some edges joining graph K_{n+1} to the graph K_n. Let $u \in V_1$ and $v \in V_2$. If there exist $w_1 \in V_2$ and $e_{uw_1} \notin E(\overline{G_n})$ or $w_2 \in V_1$ and $e_{vw_2} \notin E(\overline{G_n})$, then $d(u,w_1)$ and $d(v,w_2)$ are equal to two for every $e_{uv} \in E(\overline{G_n})$. So, we obtain $\varepsilon_{\overline{G_n}}(v) = 2$ for every $v \in V(\overline{G_n})$.

Then, we have

$M_1^*(\overline{G_n}) = \sum_{e_{uv}\in E(\overline{G_n})}[\varepsilon_{\overline{G_n}}(u) + \varepsilon_{\overline{G_n}}(v)] = \sum_{i=1}^{2n^2-2n} 4 = 8n^2 - 8n.$ □

$M_2^*(\overline{G_n}) = \sum_{e_{uv}\in E(\overline{G_n})}[\varepsilon_{\overline{G_n}}(u)\varepsilon_{\overline{G_n}}(v)] = \sum_{i=1}^{2n^2-2n} 4 = 8n^2 - 8n.$ □

$M_1^{**}(\overline{G_n}) = \sum_{v\in V(\overline{G_n})}[\varepsilon_{\overline{G_n}}(v)]^2 = \sum_{i=1}^{2n+1} 2^2 = 8n + 4.$ □

Definition 2.3 [4] The line graph $L(G)$ of a graph G is a graph such that each vertex of $L(G)$ represents an edge of G, and any two vertices of $L(G)$ are adjacent if and only if their edges are incident, meaning they share a common end vertex, in the graph G.

Theorem 2.3 *Let G_n be a gear graph. Then, $M_1^*(L(G_n)) = 2n^2 + 20n$, $M_2^*(L(G_n)) = 2n^2 + 28n$ and $M_1^{**}(L(G_n)) = 22n$.*

Proof. The number of vertices and edges of the graph $L(G_n)$ are $(3n)$ and $((n^2 + 7n)/2)$, respectively. It is easy to see that the degree of vertices of the graph $L(G_n)$ are 3 or $(n + 1)$. We partition the vertices of the graph $L(G_n)$ into two subsets V_1 and V_2, as follows: $V_1 = \{v_i \in V(L(G_n)) | d_{L(G_n)}(v_i) = 3, 1 \le i \le 2n\}$ and $V_2 = \{v_i \in V(L(G_n)) | d_{L(G_n)}(v_i) = n + 1,\ 2n + 1 \le i \le 3n\}$.
To calculate the $\varepsilon_{L(G_n)}(v_i)$ of vertices of the graph $L(G_n)$, the vertices should be examined two cases.

Case1. Let $u, v, x, y \in V_1$ and $w \in V_2$. If $e_{uw} \in E(L(G_n))$, then $d(u, w) = 1$. Otherwise, $e_{uv} \notin E(L(G_n))$, then $d(u, w) = 2$. Because vertices of V_2 form a complete graph K_n. Let $x = N_{L(G_n)}(u)$ and $y = N_{L(G_n)}(v)$. It is clear that when $x \neq y$, *path* $uxyv$ is found. Hence, $\varepsilon_{L(G_n)}(v) = 3$ for every vertices of V_1.

Case2. Similar to *Case1*, value of $\varepsilon_{L(G_n)}(v)$ is equal to 2 for every vertices of V_2.

Furthermore, we partition the edges of $L(G_n)$ into three subsets E_1, E_2 and E_3, as follows: $E_1 = \{e_{uv} \in E(L(G_n)) | u, v \in V_1\}$, $E_2 = \{e_{uv} \in E(L(G_n)) | u \in V_1, v \in V_2\}$ and $E_3 = \{e_{uv} \in E(L(G_n)) | u, v \in V_2\}$. It is clear that cardinality of sets E_1, E_2 and E_3 are $(2n)$, $(2n)$ and $((n^2 - n)/2)$, respectively. Then, we have

$$M_1^*(L(G_n)) = \sum_{e_{uv}\in E_1}[\varepsilon_{L(G_n)}(u) + \varepsilon_{L(G_n)}(v)] + \sum_{e_{uv}\in E_2}[\varepsilon_{L(G_n)}(u) + \varepsilon_{L(G_n)}(v)] + \sum_{e_{uv}\in E_3}[\varepsilon_{L(G_n)}(u) + \varepsilon_{L(G_n)}(v)]$$
$$= \sum_{i=1}^{2n} 6 + \sum_{i=1}^{2n} 5 + \sum_{i=1}^{(n^2-n)/2} 4 = 2n^2 + 20n. \quad \square$$

$$M_2^*(L(G_n)) = \sum_{e_{uv}\in E_1}[\varepsilon_{L(G_n)}(u)\varepsilon_{L(G_n)}(v)] + \sum_{e_{uv}\in E_2}[\varepsilon_{L(G_n)}(u)\varepsilon_{L(G_n)}(v)] + \sum_{e_{uv}\in E_3}[\varepsilon_{L(G_n)}(u)\varepsilon_{L(G_n)}(v)]$$
$$= \sum_{i=1}^{2n} 9 + \sum_{i=1}^{2n} 6 + \sum_{i=1}^{(n^2-n)/2} 4 = 2n^2 + 28n. \ \square$$

$$M_1^{**}(L(G_n)) = \sum_{v\in V_1}[\varepsilon_{L(G_n)}(v)]^2 + \sum_{v\in V_2}[\varepsilon_{L(G_n)}(v)]^2$$
$$= \sum_{i=1}^{2n} 3^2 + \sum_{i=1}^{n} 2^2 = 22n. \ \square$$

Definition 2.4 [2] The (Cartesian) product $G_1 \times G_2$ of graphs G_1 and G_2 also has $V(G_1) \times V(G_2)$ as its vertex set, but here (u_1, u_2) is adjacent to (v_1, v_2) if either $u_1 = v_1$ and u_2 is adjacent to v_2 or $u_2 = v_2$ and u_1 is adjacent to v_1.

Theorem 2.4 *Let G_n be a gear graph. Then,* $M_1^*(K_2 \times G_n) = 68n + 6$, $M_2^*(K_2 \times G_n) = 145n + 9$ *and* $M_1^{**}(K_2 \times G_n) = 82n + 18$.

Proof. Since the definition of cartesian product, there are two gear graphs G_n are denoted by G_1 and G_2, where adding a perfect matching between corresponding vertices which have same label. So, we obtain $|V(K_2 \times G_n)| = 4n + 2$ and $|E(K_2 \times G_n)| = 8n + 1$. Let $V(K_2 \times G_n) = V_1 \cup V_2 \cup V_3 \cup V_4 \cup V_5 \cup V_6$, where:
$V_1 = \{v \in V(G_1) | \, d_{K_2\times G_n}(v) = n + 1\}$, $V_2 = \{v \in V(G_1) | \, d_{K_2\times G_n}(v) = 4\}$,
$V_3 = \{v \in V(G_1) | \, d_{K_2\times G_n}(v) = 3\}$, $V_4 = \{v \in V(G_2) | \, d_{K_2\times G_n}(v) = n + 1\}$,
$V_5 = \{v \in V(G_2) | \, d_{K_2\times G_n}(v) = 4\}$ and $V_6 = \{v \in V(G_2) | \, d_{K_2\times G_n}(v) = 3\}$.

Since the structure of $K_2 \times G_n$ and the Theorem 2.1, $\varepsilon_{K_2\times G_n}(v) = \varepsilon_{G_n}(v) + 1$. So, we obtain $\varepsilon_{K_2\times G_n}(v)=3$, $\varepsilon_{K_2\times G_n}(v)=4$ and $\varepsilon_{K_2\times G_n}(v)=5$ for every $v \in V_1$ or $v \in V_4$, $v \in V_2$ or $v \in V_5$ and $v \in V_3$ or $v \in V_6$, respectively. Then, we partition the edges of $K_2 \times G_n$ into five subsets E_1, E_2, E_3, E_4 and E_5, as follows:
$E_1 = \{e_{uv} \in E(K_2 \times G_n) | \, u \in V_1, v \in V_2\}$, $E_2 = \{e_{uv} \in E(K_2 \times G_n) | \, u \in V_2, v \in V_3\}$, $E_3 = \{e_{uv} \in E(K_2 \times G_n) | \, u \in V_4, v \in V_5\}$, $E_4 = \{e_{uv} \in E(K_2 \times G_n) | \, u \in V_5, v \in V_6\}$ and $E_5 = \{e_{uv} \in E(K_2 \times G_n) | \, u \in V_3, v \in V_6\}$.

Moreover, we have $|E_1| = n$, $|E_2| = 2n$, $|E_3| = n$, $|E_4| = 2n$ and $|E_5| = 2n + 1$. Thus, we have

$$M_1^*(K_2 \times G_n) = \sum_{e_{uv}\in E_1}[\varepsilon_{K_2\times G_n}(u) + \varepsilon_{K_2\times G_n}(v)] + \sum_{e_{uv}\in E_2}[\varepsilon_{K_2\times G_n}(u) + \varepsilon_{K_2\times G_n}(v)] + \sum_{e_{uv}\in E_3}[\varepsilon_{K_2\times G_n}(u) + \varepsilon_{K_2\times G_n}(v)] + \sum_{e_{uv}\in E_4}[\varepsilon_{K_2\times G_n}(u) + \varepsilon_{K_2\times G_n}(v)] + \sum_{e_{uv}\in E_5}[\varepsilon_{K_2\times G_n}(u) + \varepsilon_{K_2\times G_n}(v)]$$
$$= \sum_{i=1}^{n} 7 + \sum_{i=1}^{2n} 9 + \sum_{i=1}^{n} 7 + \sum_{i=1}^{2n} 9 + (6 + \sum_{i=1}^{n} 8 + \sum_{i=1}^{n} 10)$$
$$= 68n + 6. \ \square$$

$M_2^*(K_2 \times G_n) = \sum_{e_{uv}\in E_1}[\varepsilon_{K_2\times G_n}(u)\varepsilon_{K_2\times G_n}(v)] + \sum_{e_{uv}\in E_2}[\varepsilon_{K_2\times G_n}(u)\varepsilon_{K_2\times G_n}(v)] + \sum_{e_{uv}\in E_3}[\varepsilon_{K_2\times G_n}(u)\varepsilon_{K_2\times G_n}(v)] + \sum_{e_{uv}\in E_4}[\varepsilon_{K_2\times G_n}(u)\varepsilon_{K_2\times G_n}(v)] + \sum_{e_{uv}\in E_5}[\varepsilon_{K_2\times G_n}(u)\varepsilon_{K_2\times G_n}(v)]$

$= \sum_{i=1}^{n} 12 + \sum_{i=1}^{2n} 20 + \sum_{i=1}^{n} 12 + \sum_{i=1}^{2n} 20 + (9 + \sum_{i=1}^{n} 16 + \sum_{i=1}^{n} 25)$

$= 145n + 9.$ □

$M_1^{**}(K_2 \times G_n) = \sum_{v\in V_1}[\varepsilon_{K_2\times G_n}(v)]^2 + \sum_{v\in V_2}[\varepsilon_{K_2\times G_n}(v)]^2 + \sum_{v\in V_3}[\varepsilon_{K_2\times G_n}(v)]^2 + \sum_{v\in V_4}[\varepsilon_{K_2\times G_n}(v)]^2 + \sum_{v\in V_5}[\varepsilon_{K_2\times G_n}(v)]^2 + \sum_{v\in V_6}[\varepsilon_{K_2\times G_n}(v)]^2$

$= 9 + \sum_{i=1}^{n} 16 + \sum_{i=1}^{n} 25 + 9 + \sum_{i=1}^{n} 16 + \sum_{i=1}^{n} 25$

$= 82n + 18.$ □

Theorem 2.5 *Let G be a connected graph order n and size m that includes only one induced sub graph H as a star. Then,* $M_1^*(G) = 4m - n + 1$, $M_2^*(G) = 4m - 2n + 2$ *and* $M_1^{**}(G) = 4n - 3$.

Proof. Let c be a vertex whose degree is $(n-1)$. We have $\varepsilon_G(c) = 1$ by the definition of eccentricity. Furthermore, let $x, y \in V(G) - \{c\}$. It is easy to see that $d(x,y) = 2$. So, we obtain $\varepsilon_G(x) = 2$ for all vertices $x \in V(G) - \{c\}$. Then, we partition the edges set of the graph G into two subsets E_1 and E_2, as follows:
$E_1 = \{e_{cv} \in E(G) | d_G(c) = n - 1, v \in V(G) - \{c\}\}$ and $E_2 = \{e_{uv} \in E(G) | u, v \in V(G) - \{c\}\}$. It is clear that cardinality of sets E_1 and E_2 are $(n-1)$ and $(m - n + 1)$, respectively. Thus, we have

$M_1^*(G) = \sum_{e_{cv}\in E_1}[\varepsilon_G(c) + \varepsilon_G(v)] + \sum_{e_{uv}\in E_2}[\varepsilon_G(u) + \varepsilon_G(v)]$

$= \sum_{i=1}^{n-1} 3 + \sum_{i=1}^{m-n+1} 4 = 4m - n + 1.$ □

$M_2^*(G) = \sum_{e_{cv}\in E_1}[\varepsilon_G(c)\varepsilon_G(v)] + \sum_{e_{uv}\in E_2}[\varepsilon_G(u)\varepsilon_G(v)]$

$= \sum_{i=1}^{n-1} 2 + \sum_{i=1}^{m-n+1} 4 = 4m - 2n + 2.$ □

$M_1^{**}(G) = [\varepsilon_G(c)]^2 + \sum_{v\in V(G)-\{c\}}[\varepsilon_G(v)]^2$

$= 1 + \sum_{i=1}^{n-1} 2^2 = 4n - 3.$ □

Definition 2.5 [9] Friendship graph D_3^n is collection of n triangles with a common point. Another way of obtaining friendship graph is addition of K_1 and n copies of K_2.

Result 2.1 $M_1^*(D_3^n) = 10n$, $M_2^*(D_3^n) = 8n$ *and* $M_1^{**}(D_3^n) = 8n + 1$.

Definition 2.6 [16] The corona product GoH of two graphs G and H is defined as the graph obtained by taking one copy of G and $|V(G)|$ copies of H and joining the *i-th* vertex of G to every vertex in the $i-th$ copy of H. The $i-th$ copy of H is denoted by H_i, where $1 \le i \le n$.

Theorem 2.6 *Let G be a connected graph order n and size m that includes only one induced sub graph H as a star and let Q be any connected graph order r and size p. Then,* $M_1^{**}(GoQ) = 16nr + 9n - 7r - 5$.

Proof. The number of vertices of the graph GoQ is $(nr + n)$. Let the vertices set of the graph GoQ be $V(GoQ) = V_1 \cup V_2 \cup V_3 \cup V_4$, as follows:
$V_1 = \{v_i \in V(G) | \ d_G(v_i) = n - 1, \ i = 1\}, V_2 = V(G) \backslash V_1$,
$V_3 = \{v_i \in V(H_i) | \ x \in V_1 \ and \ e_{vx} \in V(GoQ)\}$ and $V_4 = V(G) \backslash \{V_1 \cup V_2 \cup V_3\}$.

It is clear that we have $|V_1| = 1$, $|V_2| = n - 1$, $|V_3| = r$ and $|V_4| = nr - r$. Let $u \in V_1$, $a \in V_2$, $b \in V_3$ and $c \in V_4$. Since $d_G(u) = n - 1$ and $d_{GoQ}(u) = n + r - 1$, we have $d(u,a) = 1$, $d(u,b) = 1$ and $d(u,c) = 2$. So, $\varepsilon_{GoQ}(u) = 2$ is obtained. Let H_1 be any copies of the graph H and every vertices of H_1 joining the vertex u, where vertices of V_3 are them. Furthermore, we have $d(b,a) = 2$ and $d(b,c) = 3$. So, $\varepsilon_{GoQ}(b) = 3$ for every $b \in V_3$ is obtained. Similarly, $\varepsilon_{GoQ}(a) = 3$ and $\varepsilon_{GoQ}(c) = 4$ are found for every $a \in V_2$ and $c \in V_4$, respectively.
Then, we have

$$\begin{aligned} M_1^{**}(GoQ) &= \textstyle\sum_{v \in V_1}[\varepsilon_{GoQ}(v)]^2 + \sum_{v \in V_2}[\varepsilon_{GoQ}(v)]^2 + \sum_{v \in V_3}[\varepsilon_{GoQ}(v)]^2 + \sum_{v \in V_4}[\varepsilon_{GoQ}(v)]^2 \\ &= 2^2 + \textstyle\sum_{i=1}^{n-1} 3^2 + \sum_{i=1}^{r} 3^2 + \sum_{i=1}^{nr-r} 4^2 \\ &= 16nr + 9n - 7r - 5. \quad \square \end{aligned}$$

Theorem 2.7 *Let G be a connected graph order n and size m that includes only one induced sub graph H as a star and let* C_r *be a cycle graph. Then,* $M_1^*(GoC_r) = 15nr + 6m - 4r - n + 1$ *and* $M_2^*(GoC_r) = 28nr + 9m - 13r - 3n + 3$.

Proof. Firstly, let the vertices set of the graph GoC_r be $V(GoC_r) = V_1 \cup V_2 \cup V_3 \cup V_4$. These sets are the same as set finding Theorem 2.6. Furthermore, eccentricity values of every vertices of sets V_1, V_2, V_3 and V_4 are equal to 2, 3, 3 and 4, respectively. Then, we partition the edges set of GoC_r into six subsets E_1, $E_2, E_3, E_4,$ E_5 and E_6, as follows:
$E_1 = \{e_{uv} \in E(GoC_r) | \ u \in V_1, v \in V_2\}$, $E_2 = \{e_{uv} \in E(GoC_r) | \ u, v \in V_2\}$,
$E_3 = \{e_{uv} \in E(GoC_r) | \ u \in V_1, v \in V_3\}$, $E_4 = \{e_{uv} \in E(GoC_r) | \ u, v \in V_3\}$,
$E_5 = \{e_{uv} \in E(GoC_r) | \ u \in V_2, v \in V_4\}$ and $E_6 = \{e_{uv} \in E(GoC_r) | \ u, v \in V_4\}$.

It is clear that we have $|E_1| = n - 1$, $|E_2| = m - n + 1$, $|E_3| = r$, $|E_4| = r$, $|E_5| = nr - r$ and $|E_6| = nr - r$.
Thus, we get

$$\begin{aligned} M_1^*(GoC_r) = &\textstyle\sum_{e_{uv} \in E_1}[\varepsilon_{GoC_r}(u) + \varepsilon_{GoC_r}(v)] + \sum_{e_{uv} \in E_2}[\varepsilon_{GoC_r}(u) + \varepsilon_{GoC_r}(v)] + \\ &\textstyle\sum_{e_{uv} \in E_3}[\varepsilon_{GoC_r}(u) + \varepsilon_{GoC_r}(v)] + \sum_{e_{uv} \in E_4}[\varepsilon_{GoC_r}(u) + \varepsilon_{GoC_r}(v)] + \\ &\textstyle\sum_{e_{uv} \in E_5}[\varepsilon_{GoC_r}(u) + \varepsilon_{GoC_r}(v)] + \sum_{e_{uv} \in E_6}[\varepsilon_{GoC_r}(u) + \varepsilon_{GoC_r}(v)] \\ = &\textstyle\sum_{i=1}^{n-1} 5 + \sum_{i=1}^{m-n+1} 6 + \sum_{i=1}^{r} 5 + \sum_{i=1}^{r} 6 + \sum_{i=1}^{nr-r} 7 + \sum_{i=1}^{nr-r} 8 \\ = &\ 15nr + 6m - 4r - n + 1. \quad \square \end{aligned}$$

$$M_2^*(GoC_r) = \sum_{e_{uv}\in E_1}\left[\varepsilon_{GoC_r}(u)\varepsilon_{GoC_r}(v)\right] + \sum_{e_{uv}\in E_2}\left[\varepsilon_{GoC_r}(u)\varepsilon_{GoC_r}(v)\right] + \sum_{e_{uv}\in E_3}\left[\varepsilon_{GoC_r}(u)\varepsilon_{GoC_r}(v)\right] + \sum_{e_{uv}\in E_4}\left[\varepsilon_{GoC_r}(u)\varepsilon_{GoC_r}(v)\right] + \sum_{e_{uv}\in E_5}\left[\varepsilon_{GoC_r}(u)\varepsilon_{GoC_r}(v)\right] + \sum_{e_{uv}\in E_6}\left[\varepsilon_{GoC_r}(u)\varepsilon_{GoC_r}(v)\right]$$
$$= \sum_{i=1}^{n-1} 6 + \sum_{i=1}^{m-n+1} 9 + \sum_{i=1}^{r} 6 + \sum_{i=1}^{r} 9 + \sum_{i=1}^{nr-r} 12 + \sum_{i=1}^{nr-r} 16$$
$$= 28nr + 9m - 13r - 3n + 3. \quad \square$$

Theorem 2.8 *Let G be a connected graph order n and size m that includes only one induced sub graph H as a star and let* P_r *be a path graph. Then,* $M_1^*(GoP_r) = 15nr + 6m - 4r - 9n + 3$ *and* $M_2^*(GoP_r) = 28nr + 9m - 13r - 19n + 10$.

Proof. The proof follows directly from the Theorem 2.7.

3. Algorithm for the Zagreb Eccentricity Indices of a Graph

In this section, we offer an algorithm computing the new Zagreb indices for given graph *G*. Firstly, algorithms finds *distance matrix* by using *adjacency matrix*. Then, eccentricity values of every vertices are found and, then new *Zagreb indices* are computed by the algorithm. The time complexity for this algorithm is $O(n^3)$. Code that written in Pascal Programming Language of algorithm is given graph *G* as below:

```
Program New Zagreb Indices;
Uses crt;
VAR
a,d,v,s,e : array[1..1000,1..1000] of longint;
i,j,k,m,top,b,h,f,enb,enk,n,s1,s2,s3: integer;
sum1,sum2,sum3 : integer;
tus:char;
label L,T;
BEGIN
clrscr;
m:=1;
writeln('Input number of vertices of graph');
readln (n);

for i:=1 to n do
   for j:=1 to n do
     begin
     writeln ('Input value of ',i,'.row','
    'j,'.column');
     readln (a[i,j]);
     if i=j then a[i,j]:=0;
     end;

for i:=1 to n do
   for j:=1 to n do
     v[i,j]:=a[i,j];
```

```
if (m=1) then
        begin;
        for i:=1 to n do
           for j:=1 to n do
              if a[i,j]=1 then d[i,j]:=a[i,j]
                          else if i=j then d[i,j]:=a[i,j]
                          else d[i,j]:=50000;
        end;

f:=0;
for i:=1 to n do
   for j:=1 to n do
      if d[i,j]=50000 then f:=f+1;
      if f=0 then goto T;

L:
if (m>1) then
     begin
     for i:=1 to n do
        for j:=1 to n do
        if (d[i,j]=50000) and (a[i,j]=1) then
       d[i,j]:=m;
     end;

if (m>1) then
        f:=0;
```

```
          begin
          for i:=1 to n do
             for j:=1 to n do
             if d[i,j]=50000 then f:=f+1;
            if f=0 then goto T;
          end;
for i:=1 to n do
begin
   for j:=1 to n do
   begin
   top:=0;
      for b:=1 to n do
      begin
      top := (top + (a[i,b]*v[b,j]));
      end;
      s[i,j]:=top;
    end;
end;
for i:=1 to n do
begin
     for j:=1 to n do
     begin
      if i=j then s[i,j]:=0;
      if s[i,j]=0 then  s[i,j]:=0
                else s[i,j]:=1;
      end;
end;
for i:=1 to n do
begin
     for j:=1 to n do
     begin
     if i=j then a[i,j]:=0
           else a[i,j]:=s[i,j];
     end;
 end;
 m:=m+1;
 goto L;
 T:
 writeln;writeln;
 writeln ( 'Distance Matrix of Graph');
 writeln;writeln;
 for i:=1 to n do
 begin
    for j:=1 to n do
    begin
       if d[i,j]=50000 then write('-':3)
                      else write(d[i,j]:3);
                      writeln;
       end;
end;
for i:=1 to n do
begin
   for j:=1 to n do
   begin
    if (j=1) then enb:= D[i,j];
    if ( D[i,j] >= enb ) then enb:=D[i,j];
    end;
 E[i,1]:=enb;
 end;
 writeln; writeln;
 writeln ('Eccentricity Values For All
         Vertices');
 writeln; writeln;
 for i:=1 to n do
    for j:=1 to 1 do
    writeln( E[i,j]:3 ) ;
 writeln; writeln;
 sum1:=0; sum2:=0; sum3:=0;
 for i:=1 to n do
 begin
 s1:=E[i,1]*E[i,1];
 sum1:=sum1+s1;
      for j:=1 to n do
      begin
       if ((i<j) and (v[i,j]=1)) then
                          begin
                          s2:=E[i,1]+E[j,1];
                          sum2:=sum2+s2;
                          s3:=E[i,1]*E[j,1];
                          sum3:=sum3+s3;
                           end;
       end;
 end;
 writeln ('Value of index M*1=',sum2);
 writeln ('Value of index M**1=',sum1);
 writeln ('Value of index M*2=',sum3);
 writeln ( 'Press the enter for exit');
 repeat
 tus:=readkey;
 until tus=#13;
 END.
```

References

[1] A. Aytaç, T. Turacı, Vertex Vulnerability Parameter of Gear Graphs, *International Journal of Foundations of Computer Science* 22(5) (2011) 1187–1195.

[2] D. B. West, *Introduction to Graph Theory*, Prentice Hall, Upper Saddle River, 1996.

[3] D. Vukicevic, A. Graovac, Note on the Comparison of the First and Second Normalized Zagreb Eccentricity Indices, *Acta Chim. Slov.* 57 (2010), 524-528.

[4] F. Harary, F. Buckley, *Distance in Graphs*, Addison-Wesley Publishing Company, 1989.

[5] H. Wiener, Structural determination of the parafing boiling points, *J. Am. Chem. Soc.* 69 (1947) 17-20.

[6] I. Gutman, N. Trinajstić, Graph Theory and molecular orbitals. Total π-electron energy of alternant hydrocarbons, *Chem. Phys. Lett.* 17 (1972) 535-538.

[7] I. Gutman, O.E. Polansky, *Mathematical Consepts in Organic Chemistry*, Springer Verlag, Berlin, 1986.

[8] J.B.Babujee, S. Ramakrishnan, Topological Indices and New Structures, *App. Mat. Sci.* 108(6) (2012) 5383-5401.

[9] J. A. Gallian, A dynamic survey of graph labeling, *Elect. Jour. Combin.* 19 (2012) #DS6

[10] K. C. Das, Dae-Won Lee, A. Graovac, Some properties of the Zagreb eccentricity indices, *Ars Mathematica Contemporanea* 6 (2013) 117-125.

[11] K. C. Das, I. Gutman, Some properties of the Second Zagreb Index, *MATCH Commun. Math. Comput. Chem.* 52 (2004) 103-112.

[12] K. C. Das, I. Gutman, B. Zhou, New upper bounds on Zagreb Indices, J. Math. Chem. 46 (2009) 514-521.

[13] M. Ghorbani, M.A. Hosseinzadeh, A new version of Zagreb Indices, *Filomat* 26(1) (2012) 93-100.

[14] N. Trinajstić, I. Gutman, Mathematical chemistry, *Croat. Chem. Acta* 75 (2002) 329-356.

[15] R. Xing, B. Zhou, N. Trinajstić, On Zagreb Eccentricity Indices, *Croat. Chem. Acta* 84 (4) (2011) 493-497.

[16] Z. Yarahmadi, A.R. Ashrafi, The Szeged, vertex PI, first and second Zagreb indices of corona product of graphs, *Filomat* 26(3) (2012) 467-472.