The radio number of standard caterpillars*

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Abstract

A *radio labelling* of a connected graph G of diameter d is a mapping f: $V(G) \rightarrow \{0, 1, 2, \cdots\}$ such that $d(u, v) + |f(u) - f(v)| \geq d + 1$ for each pair of distinct vertices u and v of G, where $d(u, v)$ is the distance between u and v. The *value* $\text{rn}(f)$ of a radio labelling f is the maximum label assigned by f to a vertex of G. The *radio number* $\text{rn}(G)$ of G is the minimum value of $\text{rn}(f)$ taken over all radio labellings f of G. A *caterpillar* C is a special tree that consists of a path $x_1x_2 \cdots x_m (m \geq 3)$, with some pendant vertices adjacent to the inner vertices $x_2, x_3, \cdots, x_{m-1}$. If $d(x_i) = t$ (the degree of x_i) for $i = 2, 3, \cdots, m-1$, then the caterpillar is called *standard*, and denoted by $C(m, t)$. In this paper, we determine the exact value of the radio number of $C(m, t)$ for all integers $m \geq 4$ and $t \geq 2$, and explicitly construct an optimal radio labelling. Our results show that the radio number and the construction of optimal radio labelling of paths are the special cases of $C(m, t)$ with $t = 2$.

Key words: Radio coloring; Radio number; Tree; Standard caterpillar; Path.

Mathematics Subject Classifications: 05C12, 05C15, 05C78.

1 Introduction

Motivated by FM Radio Channel Assignment Problem [6, 7, 8], radio k-*colorings* were introduced by Chartrand, Erwin, Harary and Zhang [2, 4]. For a connected graph G of diameter d, an integer k with $1 \leq k \leq d$, a

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radio k-coloring of G is a mapping $f\colon V(G) \to \{0,1,2,\cdots\}$ (color set) such that

$$d(u,v) + |f(u) - f(v)| \geq k + 1$$

for all distinct vertices u and v of G, where $d(u,v)$ is the distance between u and v. The *value* $\mathrm{rc}_k(f)$ of a radio k-coloring f of G is the maximum color assigned by f to a vertex of G. The *radio-k chromatic number* $\mathrm{rc}_k(G)$ of G is the minimum value of $rc_k(f)$ taken over all radio k-colorings f of G. A radio k-coloring f of G with $\mathrm{rc}_k(f)$=$\mathrm{rc}_k(G)$ is called an *optimal radio k-coloring* of G. We remark that for technical reasons we follow the definitions in [16], and thus the radio-k chromatic number $\mathrm{rc}_k(G)$ defined here is one less than that defined in [2].

Radio k-colorings generalize many graph colorings. It is easy to see that the radio 1-colorings and ordinary vertex colorings are synonymous, and the radio 2-colorings problem correspond to the well studied $L(2,1)$-colorings (see [11] and references therein). Moreover, radio d-colorings are referred to as *radio labelings* and the *radio d-chromatic number* is called the *radio number*, denoted by $\mathrm{rn}(G)$. Radio $(d-1)$-colorings and radio $(d-2)$-colorings are referred to as *antipodal colorings* and *nearly antipodal colorings*, radio $(d-1)$-chromatic number and radio $(d-2)$-chromatic number are called the *antipodal chromatic number* and the *nearly antipodal chromatic number*, denoted by $\mathrm{ac}(G)$ and $\mathrm{ac}'(G)$, respectively (see [1, 3, 5]).

In general, the research of radio k-colorings are mainly concentrated on the three largest values of k. At present, there are several results for $k \in \{d, d-1, d-2\}$, restricted to some basic families of graphs, such as paths, cycles, square of paths, square of cycles, and hypercubes [1, 3, 5, 9, 10, 12, 13, 14, 15, 16, 17, 18].

Determining the radio number of a graph is interesting problem with potential applications to FM Radio Channel Assignment. As determining the radio number for paths and cycles was a challenging task, some scholars believe that in general determining the radio number would be difficult even for trees. In [12] Liu investigated the radio number for trees, presented a lower bound for the radio number of trees and characterized the trees with radio number achieving this bound. Moreover, Liu generalized the results for path to spiders (trees with at most one vertex of degree more than two), leading to determining the exact value of the radio number in certain special cases. In [15] Li, Mak and Zhou determined the radio number for any complete *m-ary* tree with any height.

Here, we investigate the radio number for another special kind of tree graph, namely the *standard caterpillar*. A tree is said to be a *caterpillar* if it consists of a path $P_m = x_1x_2\cdots x_m$ ($m \geq 3$), called the *spine* of C, with some pendant edges known as *legs*, which are incident to the inner vertices $x_2, x_3, \cdots, x_{m-1}$. The degree $d(x)$ of a vertex x in a graph G is the number of edges incident with x. If the degrees of $x_2, x_3, \cdots, x_{m-1}$ in C are the same, then the caterpillar C is called *standard*, and denoted by $C(m,t)$ if $d(x_i) = t$ for all $i = 2, 3, \cdots, m-1$.

In [2], Chartrand et al. have showed that for a complete k-partite graph $K_{n_1,n_2,\cdots,n_k}$, $\mathrm{rn}(K_{n_1,n_2,\cdots,n_k}) = (\sum_{i=1}^{k} n_i) + (k-2)$. For $m = 3$, as $C(m,t) = C(3,t) = K_{1,t}$, it follows that $\mathrm{rn}(C(3,t)) = t+1$ by the above result. In this paper, We determine the radio number of the standard caterpillars $C(m,t)$ for all integers $m \geq 4$ and $t \geq 2$, and explicitly construct an optimal radio labelling. Our results show that the radio number and the construction of optimal radio labelling of paths are the special cases of $C(m,t)$ with $t = 2$.

2 Preliminaries

Let T be a tree rooted at vertex r. A vertex u is called a *descendant* of another vertex v (or v is an *ancestor* of u) if v is on the unique path of T from r to u. The subtree of T induced by r, a child u of r, and all descendants of u is referred to as a *branch* of T. For any vertex $x \in V(T)$ as the root, define the *level function* on $V(T)$ by

$$L_x(u) = d(x,u) \quad \text{for all } u \in V(T).$$

For any $u, v \in V(T)$, define

$$\phi_x(u,v) = \{\max L_x(t) | t \text{ is a common ancestor of } u \text{ and } v\}.$$

For any vertex x in a tree T, define the *weight of T rooted at x* by $w_x(T) = \sum_{u \in V(T)} L_x(u)$. Let the *weight of T* be the smallest weight among all possible roots x of T:

$$w(T) = \min\{w_x(T), x \in V(T)\}.$$

A vertex r^* of a tree T is called a *weight center* of T if $w_{r^*}(T) = w(T)$.

For a graph G, a maximal connected subgraph of G is called a component of G. By the above definitions, the following facts are obvious.

Lemma 1. [12] *Suppose r is a weight center of a tree T. Then each component of $T - r$ contains at most $|V(T)|/2$ vertices.*

Lemma 2. [12] *Each tree T has either one or two weight centers, and T has two weight centers, say r and r', if and only if rr' is an edge of T and $T - rr'$ consists of two equal-sized components.*

Lemma 3. [12] *Let T be a tree rooted at r. Then for any vertices u and v,*

(1) $d(u,v) = L_r(u) + L_r(v) - 2\phi_r(u,v)$;

(2) $\phi_r(u,v) = 0$ if and only if $r \in \{u,v\}$ or u, v belong to different branches.

For $C(m,t)$ with $m \geq 4$ and $t \geq 2$, denote the spine of $C(m,t)$ by $P_m = x_1x_2\cdots x_m$, and let the legs of $C(m,t)$ be $x_{i,1}, x_{i,2}, \cdots, x_{i,t-2}$ for $i = 2, 3, \cdots, m-1$ and $t \geq 2$ (see Figure 1 for $m = 2k(k \geq 2)$ and Figure 2 for $m = 2k+1(k \geq 2)$, respectively).

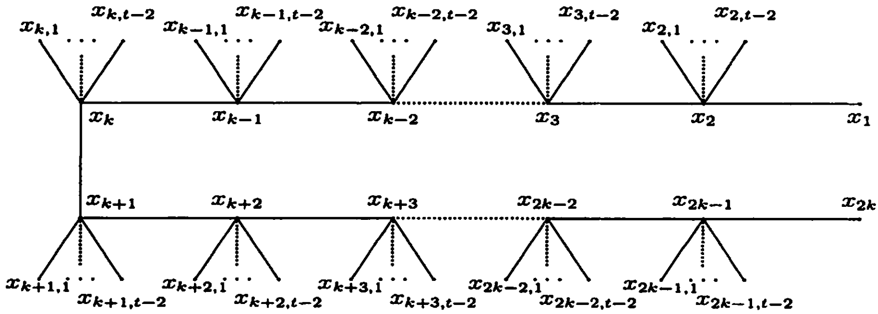

Figure 1: $C(m,t)$ with $m = 2k(k \geq 2)$.

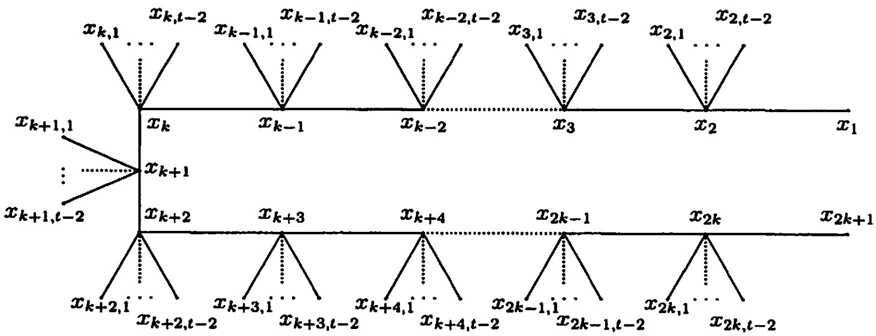

Figure 2: $C(m,t)$ with $m = 2k+1(k \geq 2)$.

By Lemma 1 and Lemma 2, it is easy to see that $r = x_k$ and $r = x_{k+1}$ are the weight centers of $C(2k,t)$, and $r = x_{k+1}$ is the weight center of $C(2k+1,t)$. Then, by the definition of weight of tree, we have

$$w(C(2k,t)) = (t-1)\Sigma_{i=1}^{k-1} i + 1 + (t-1)\Sigma_{i=2}^{k} i = (t-2)(k^2-1) + k^2; \quad (1)$$

$$w(C(2k+1,t)) = t + 2(t-1)\Sigma_{i=2}^{k} i = (t-2)(k^2+k-1) + k^2 + k. \quad (2)$$

3 Lower bounds of $\text{rn}(C(m,t))$

First of all, note that the diameter d of $C(m,t)$ is $m-1$. Let f be a radio labelling of $C(m,t)$. By the definition of radio labelling, f is injective, that is $f(u) \neq f(v)$ for distinct $u, v \in V(C(m,t))$. Hence, f induces a linear order $u_0, u_1, u_2, \cdots, u_{n-1}$ of $V(C(m,t))$, where $n = |V(C(m,t))| = m + (m-2)(t-2)$, with

$$0 = f(u_0) < f(u_1) < f(u_2) < \cdots < f(u_{n-1}). \tag{3}$$

Then the value of $\mathrm{rn}(f) = f(u_{n-1}) = \sum_{i=1}^{n-1}[f(u_i) - f(u_{i-1})]$.

Note $f(u_i) - f(u_{i-1}) \geq d + 1 - d(u_{i-1}, u_i)$ for $1 \leq i \leq n-1$ by the definition of radio label. We call

$$\varepsilon_i = f(u_i) - f(u_{i-1}) - [d + 1 - d(u_{i-1}, u_i)],\ 1 \leq i \leq n-1, \tag{4}$$

the *jump* of f from u_{i-1} to u_i.

For $C(m,t)$ with $m = 2k+1$ and $k \geq 2$ (see Figure 2), denote $V_0 = \{x_{k+1,1}, x_{k+1,2}, \cdots, x_{k+1,t-2}\}$, $V_1 = \{x_1, x_{2,1}, x_{2,2}, \cdots, x_{2,t-2}\}$, $V_2 = \{x_{2k+1}, x_{2k,1}, x_{2k,2}, \cdots, x_{2k,t-2}\}$.

Definition 1. *Let f be a radio labelling of $C(m,t)$ with $m = 2k+1$ and $k \geq 2$. Rename the vertices of $C(m,t)$ as $u_0, u_1, u_2, \cdots, u_{n-1}$ so that they satisfy* (3). *A vertex u_{i_0} ($1 \leq i_0 \leq n-2$) is called bad, if $u_{i_0} \in V_1 \cup V_2$, such that u_{i_0-1} and u_{i_0+1} belong to the same branch of $C(m,t)$, and none of u_{i_0-1} and u_{i_0+1} is the weight center $r = x_{k+1}$.*

Lemma 4. *Let f be a radio labelling of $C(m,t)$ with $m = 2k+1$ and $k \geq 2$. Rename the vertices of $C(m,t)$ as $u_0, u_1, u_2, \cdots, u_{n-1}$ so that they satisfy* (3) *and* (4). *If f satisfies the following conditions: for each $i = 1, 2, \cdots, n-1$, u_{i-1} and u_i belong to different branches, unless one of them is the weight center $r = x_{k+1}$, and $\{u_0, u_{n-1}\} = \{r, v\}$, where v is a vertex with $L_r(v) = 1$, then there must exist a bad vertex $u_{i_0} \in V_1 \cup V_2$, such that $\varepsilon_{i_0} + \varepsilon_{i_0+1} \geq 1$.*

Proof. Note that $|V_0| = t-2$ and $|V_1| = |V_2| = t-1$. Suppose u_i is not bad for all vertices $u_i \in V_1 \cup V_2$. Then u_{i-1} and u_{i+1} belong to different branches of $C(m,t)$, unless one of them is the weight center $r = x_{k+1}$. Since for each $i = 1, 2, \cdots, n-1$, u_{i-1} and u_i also belong to different branches (unless one of them is the weight centers r), and $\{u_0, u_{n-1}\} = \{r, v\}$, where v is a vertex with $L_r(v) = 1$, then at least one of u_{i-1} and u_{i+1} belong to $V_0 \cup \{r\}$ by the structure of $C(m,t)$. Denote $V_1 \cup V_2 = \{u_{i_1}, u_{i_2}, \cdots, u_{i_{2t-2}}\}$, where $i_1 < i_2 < \cdots < i_{2t-2}$. Without loss of generality, let $u_0 = r$. By the above discussion we know that for every pair u_{i_p} and $u_{i_{p+1}}$ ($p = 1, 2, \cdots, 2t-3$), there exists a vertex u_j with $i_p < j < i_{p+1}$ such that $u_j \in V_0$, otherwise it must be that $u_{i_{p+1}+1} \in V_0$. This implies that $|V_0| \geq |V_1 \cup V_2|/2 = (|V_1| + |V_2|)/2 = t-1$, contrary to $|V_0| = t-2$. Thus, there must exist a bad vertex, call it u_{i_0}, in $V_1 \cup V_2$.

In the following, we show $\varepsilon_{i_0} + \varepsilon_{i_0+1} \geq 1$. By (4) and Lemma 3,

$$
\begin{aligned}
&f(u_{i_0+1}) - f(u_{i_0-1}) = f(u_{i_0+1}) - f(u_{i_0}) + f(u_{i_0}) - f(u_{i_0-1}) \\
&= 2d + 2 - d(u_{i_0-1}, u_{i_0}) - d(u_{i_0}, u_{i_0+1}) + \varepsilon_{i_0} + \varepsilon_{i_0+1} \\
&= 2d + 2 - [L_r(u_{i_0-1}) + L_r(u_{i_0})] - [L_r(u_{i_0}) + L_r(u_{i_0+1})] + \varepsilon_{i_0} + \varepsilon_{i_0+1} \\
&= 2d + 2 - 2L_r(u_{i_0}) - [L_r(u_{i_0-1}) + L_r(u_{i_0+1})] + \varepsilon_{i_0} + \varepsilon_{i_0+1} \\
&= 2d + 2 - 2L_r(u_{i_0}) - d(u_{i_0-1}, u_{i_0+1}) - 2\phi_r(u_{i_0-1}, u_{i_0+1}) + \varepsilon_{i_0} + \varepsilon_{i_0+1}.
\end{aligned}
$$

As u_{i_0} is bad, then $d = 2L_r(u_{i_0})$ and $\phi_r(u_{i_0-1}, u_{i_0+1}) \geq 1$, we have

$$f(u_{i_0+1}) - f(u_{i_0-1}) \leq d - d(u_{i_0-1}, u_{i_0+1}) + \varepsilon_{i_0} + \varepsilon_{i_0+1}.$$

Suppose $\varepsilon_{i_0} + \varepsilon_{i_0+1} = 0$, then

$$f(u_{i_0+1}) - f(u_{i_0-1}) \leq d - d(u_{i_0-1}, u_{i_0+1}),$$

contrary to that f is a radio labelling of $C(m,t)$. Thus, we have $\varepsilon_{i_0} + \varepsilon_{i_0+1} \geq 1$. □

Theorem 1. *For all integers $m \geq 4$ and $t \geq 2$,*

$$\mathrm{rn}(C(m,t)) \geq \begin{cases} 2k^2 - 2k + 1 + 2(t-2)(k-1)^2, & \text{if } m = 2k, k \geq 2; \\ 2k^2 + 2 + (t-2)(2k^2 - 2k + 1), & \text{if } m = 2k+1, k \geq 2. \end{cases}$$

Proof. Let f be an optimal radio labelling of $C(m,t)$. Rename the vertices of $C(m,t)$ as $u_0, u_1, u_2, \cdots, u_{n-1}$ so that they satisfy (3) and (4). By Lemma 3, we have

$$
\begin{aligned}
&\mathrm{rn}(C(m,t)) = f(u_{n-1}) = \textstyle\sum_{i=1}^{n-1}[f(u_i) - f(u_{i-1})] \\
&= (n-1)(d+1) - \textstyle\sum_{i=1}^{n-1} d(u_{i-1}, u_i) + \sum_{i=1}^{n-1} \varepsilon_i \\
&= (n-1)(d+1) - \textstyle\sum_{i=1}^{n-1}[L_r(u_{i-1}) + L_r(u_i) - 2\phi_r(u_{i-1}, u_i)] + \sum_{i=1}^{n-1} \varepsilon_i \\
&= (n-1)(d+1) - 2\textstyle\sum_{i=0}^{n-1} L_r(u_i) + L_r(u_0) + L_r(u_{n-1}) \\
&\quad + 2\textstyle\sum_{i=1}^{n-1} \phi_r(u_{i-1}, u_i) + \sum_{i=1}^{n-1} \varepsilon_i \\
&= (n-1)(d+1) - 2\omega(C(m,t)) + L_r(u_0) + L_r(u_{n-1}) \\
&\quad + 2\textstyle\sum_{i=1}^{n-1} \phi_r(u_{i-1}, u_i) + \sum_{i=1}^{n-1} \varepsilon_i.
\end{aligned}
$$

Recall $n = m + (m-2)(t-2)$ and $d = m - 1$. We consider two cases according to the parity of m.

Case 1. $m = 2k, k \geq 2$.

In this case, $d = 2k - 1$ and $r = x_k$ or $r = x_{k+1}$. Note that $L_r(u_0) + L_r(u_{n-1}) + 2\sum_{i=1}^{n-1} \phi_r(u_{i-1}, u_i) + \sum_{i=1}^{n-1} \varepsilon_i \geq 1$. By (1), we have

$$
\begin{aligned}
&\mathrm{rn}(C(m,t)) = \mathrm{rn}(C(2k,t)) \\
&\geq [2k + (2k-2)(t-2) - 1] \cdot 2k - 2[(t-2)(k^2-1) + k^2] + 1 \\
&= 2k^2 - 2k + 1 + 2(t-2)(k-1)^2.
\end{aligned}
$$

Case 2. $m = 2k+1, k \geq 2$.

In this case, $d = 2k$ and $r = x_{k+1}$. By (2), we have

$$\begin{aligned}
&\mathrm{rn}(C(m,t)) = \mathrm{rn}(C(2k+1,t)) \\
&= [2k+1+(2k-1)(t-2)-1](2k+1) - 2[(t-2)(k^2+k-1)+k^2+k] \\
&\quad + L_r(u_0) + L_r(u_{n-1}) + 2\textstyle\sum_{i=1}^{n-1}\phi_r(u_{i-1},u_i) + \sum_{i=1}^{n-1}\varepsilon_i \\
&= 2k^2 + (t-2)(2k^2-2k+1) + L_r(u_0) + L_r(u_{n-1}) + 2\textstyle\sum_{i=1}^{n-1}\phi_r(u_{i-1},u_i) \\
&\quad + \textstyle\sum_{i=1}^{n-1}\varepsilon_i.
\end{aligned}$$

Note $L_r(u_0)+L_r(u_{n-1}) \geq 1$. If there exists an i_0 such that $\phi_r(u_{i_0-1}, u_{i_0}) \geq 1$, or $L_r(u_0) + L_r(u_{n-1}) \geq 2$, we have

$$\mathrm{rn}(C(m,t)) \geq 2k^2 + 2 + (t-2)(2k^2-2k+1).$$

Otherwise, for each $i = 1, 2, \cdots, n-1$, $\phi_r(u_{i-1}, u_i) = 0$ and $L_r(u_0) + L_r(u_{n-1}) = 1$, that is u_{i-1} and u_i belong to different branches (unless one of them is the weight center r), and $\{u_0, u_{n-1}\} = \{r, v\}$, where v is a vertex with $L_r(v) = 1$. Then by Lemma 4, there must exist a bad vertex $u_{i_0} \in V_1 \cup V_2$, such that $\varepsilon_{i_0} + \varepsilon_{i_0+1} \geq 1$. Thus, we also have

$$\mathrm{rn}(C(m,t)) \geq 2k^2 + 2 + (t-2)(2k^2-2k+1).$$

□

4 Upper bounds and the exact value of $\mathrm{rn}(C(m,t))$

Theorem 2. *For all integers $m \geq 4$ and $t \geq 2$,*

$$\mathrm{rn}(C(m,t)) \leq \begin{cases} 2k^2-2k+1+2(t-2)(k-1)^2, & \text{if } m = 2k, k \geq 2; \\ 2k^2+2+(t-2)(2k^2-2k+1), & \text{if } m = 2k+1, k \geq 2. \end{cases}$$

Proof. Case 1. $m = 2k, k \geq 2$.

In this case, $d = 2k-1$, and the distance condition of a radio label f is that

$$|f(u) - f(v)| + d(u,v) \geq d+1 = 2k \tag{5}$$

holds for every pair u and v of distinct vertices of $C(2k,t)$.

If $k = 2$, define a labelling f of $C(2k,t) = C(4,t)$ (see Figure 1 for $k=2$):

$$\begin{cases} f(x_1) = 2, \\ f(x_2) = 2t+1, \\ f(x_3) = 0, \\ f(x_4) = 3, \\ f(x_{2,j}) = 2j+2, & 1 \leq j \leq t-2, \\ f(x_{3,j}) = 2j+3, & 1 \leq j \leq t-2. \end{cases}$$

It is easy to see that the vertex $x_2 = x_k$ has the maximum label $f(x_2) = f(x_k) = 2t+1$, and (5) holds for every pair u and v of distinct vertices of $C(2k,t) = C(4,t)$. Thus,

$$\mathrm{rn}(C(2k,t)) = \mathrm{rn}(C(4,t)) \le 2t+1 = 2k^2 - 2k + 1 + 2(t-2)(k-1)^2.$$

If $k \ge 3$, define a labelling f of $C(2k,t)$ (see Figure 1 for $k \ge 3$):

$$\begin{cases} f(x_1) = k, \\ f(x_i) = 1 + (k-i)(2k+1) + 2(t-2)(k-1)^2, \quad 2 \le i \le k-1, \\ f(x_k) = 2k^2 - 2k + 1 + 2(t-2)(k-1)^2, \\ f(x_{k+1}) = 0, \\ f(x_{k+i}) = k + 2 + (k-i-1)(2k+1) + 2(t-2)(k-1)^2, \quad 2 \le i \le k-2, \\ f(x_{2k-1}) = k+2, \\ f(x_{2k}) = 2k^2 - 3k + 1 + 2(t-2)(k-1)^2, \\ f(x_{i,j}) = 2k + 2 + (k-i)(2k-3) + 2(j-1)(k-1)^2, \\ \qquad\qquad 2 \le i \le k, 1 \le j \le t-2, \\ f(x_{k+i,j}) = k + 4 + (k-i)(2k-3) + 2(j-1)(k-1)^2, \\ \qquad\qquad 1 \le i \le k-1, 1 \le j \le t-2. \end{cases}$$

It is easy to see that the vertex x_k has the maximum label

$$f(x_k) = 2k^2 - 2k + 1 + 2(t-2)(k-1)^2.$$

It suffices to show that (5) holds for every pair u and v of distinct vertices of $C(2k,t)$.

For convenience, we denote $X_0 = \{x_1, x_k, x_{k+1}, x_{2k-1}, x_{2k}\}$, $X = \{x_i | 2 \le i \le k-1\} \cup \{x_{k+i} | 2 \le i \le k-2\}$, $Y = \{x_{i,j} | 2 \le i \le k, 1 \le j \le t-2\}$ and $Z = \{x_{k+i,j} | 1 \le i \le k-1, 1 \le j \le t-2\}$. Then $V(C(2k,t)) = X_0 \cup X \cup Y \cup Z$.

In order to show that (5) holds for every pair u and v of distinct vertices of $C(2k,t)$, we consider seven subcases as follows.

Subcase 1.1. $|\{u,v\} \cap X_0| \ge 1$.

If $u, v \in X_0$, (5) can be verified easily. Let $u \in X_0$, $v \notin X_0$. Here we verify (5) for case $u = x_1$ only. For the cases $u = x_k, x_{k+1}, x_{2k-1}, x_{2k}$, (5) can be verified similarly.

Subcase 1.1.1. $u = x_1$, $v = x_i (2 \le i \le k-1)$.

$$\begin{aligned} &|f(x_1) - f(x_i)| + d(x_1, x_i) = f(x_i) - f(x_1) + (i-1) \\ &= 1 + (k-i)(2k+1) + 2(t-2)(k-1)^2 - k + i - 1 \ge (k-i)(2k) \ge 2k. \end{aligned}$$

Subcase 1.1.2. $u = x_1$, $v = x_{k+i} (2 \le i \le k-2)$.

$$\begin{aligned} &|f(x_1) - f(x_{k+i})| + d(x_1, x_{k+i}) = f(x_{k+i}) - f(x_1) + (k+i-1) \\ &= k + 2 + (k-i-1)(2k+1) + 2(t-2)(k-1)^2 - k + k + i - 1 \\ &\ge (k-i)(2k) > 2k. \end{aligned}$$

Subcase 1.1.3. $u = x_1$, $v = x_{i,j}(2 \le i \le k, 1 \le j \le t-2)$.

$$\begin{aligned}
&|f(x_1) - f(x_{i,j})| + d(x_1, x_{i,j}) = f(x_{i,j}) - f(x_1) + i \\
&= 2k + 2 + (k-i)(2k-3) + 2(j-1)(k-1)^2 - k + i \\
&\ge 2k + 2 + (k-i)(2k-4) > 2k.
\end{aligned}$$

Subcase 1.1.4. $u = x_1$, $v = x_{k+i,j}(1 \le i \le k-1, 1 \le j \le t-2)$.

$$\begin{aligned}
&|f(x_1) - f(x_{k+i,j})| + d(x_1, x_{k+i,j}) = f(x_{k+i,j}) - f(x_1) + (k+i) \\
&= k + 4 + (k-i)(2k-3) + 2(j-1)(k-1)^2 - k + (k+i) \\
&\ge 2k + 4 + (k-i)(2k-4) > 2k.
\end{aligned}$$

Subcase 1.2. $u, v \in X$.

We only need to verify that (5) holds for $u = x_i(2 \le i \le k-1)$, $v = x_{k+j}(2 \le j \le k-2)$ (the other cases can be checked easily). In fact,

$$\begin{aligned}
&|f(x_i) - f(x_{k+j})| + d(x_i, x_{k+j}) \\
&= |(j-i)(2k+1) + k| + (k+j-i) \\
&= \begin{cases} (j-i)(2k+2) + 2k \ge 2k, & i \le j; \\ (i-j)(2k) > 2k, & i > j. \end{cases}
\end{aligned}$$

Subcase 1.3. $u, v \in Y$.

Let $u = x_{i,j}(2 \le i \le k, 1 \le j \le t-2)$, $v = x_{p,q}(2 \le p \le k, 1 \le q \le t-2)$, $\{i,j\} \ne \{p,q\}$.

$$\begin{aligned}
&|f(x_{i,j}) - f(x_{p,q})| + d(x_{i,j}, x_{p,q}) \\
&= |(i-p)(2k-3) + 2(j-q)(k-1)^2| + d(x_{i,j}, x_{p,q}) \\
&\ge \begin{cases} 2(k-1)^2 + 2 = 2k(k-2) + 4 > 2k, & p = i, q \ne j; \\ (2k-3) + 3 = 2k, & p \ne i, q = j; \\ 2(k-1)^2 - (k-2)(2k-3) + 3 = 3k - 1 > 2k, & p \ne i, q \ne j. \end{cases}
\end{aligned}$$

Subcase 1.4. $u, v \in Z$.

(5) can be verified by an argument similar to that used in Subcase 1.3.

Subcase 1.5. $u \in X, v \in Y$.

Subcase 1.5.1. $u = x_p(2 \le p \le k-1)$, $v = x_{i,j}(2 \le i \le k, 1 \le j \le t-2)$.

$$\begin{aligned}
&|f(x_p) - f(x_{i,j})| + d(x_p, x_{i,j}) \\
&\ge f(x_p) - f(x_{i,j}) + 1 \ge f(x_{k-1}) - f(x_{2,t-2}) + 1 \\
&= 1 + (2k+1) + 2(k-1)^2 - [2k + 2 + (k-2)(2k-3)] + 1 = 3k - 3 \ge 2k.
\end{aligned}$$

Subcase 1.5.2. $u = x_{k+p}(2 \le p \le k-2)$, $v = x_{i,j}(2 \le i \le k, 1 \le j \le t-2)$.

$$\begin{aligned}
&|f(x_{k+p}) - f(x_{i,j})| + d(x_{k+p}, x_{i,j}) \\
&\ge f(x_{k+p}) - f(x_{i,j}) + 3 \ge f(x_{k+k-2}) - f(x_{2,t-2}) + 3 \\
&= k + 2 + (2k+1) + 2(k-1)^2 - [2k + 2 + (k-2)(2k-3)] + 3 = 4k > 2k.
\end{aligned}$$

Subcase 1.6. $u \in X, v \in Z$.

Subcase 1.6.1. $u = x_p (2 \le p \le k-1)$, $v = x_{k+i,j} (1 \le i \le k-1, 1 \le j \le t-2)$.

$$\begin{aligned}&|f(x_p) - f(x_{k+i,j})| + d(x_p, x_{k+i,j})\\ &\ge f(x_p) - f(x_{k+i,j}) + 3 \ge f(x_{k-1}) - f(x_{k+1,t-2}) + 3\\ &= 1 + (2k+1) + 2(k-1)^2 - [k+4+(k-1)(2k-3)] + 3 = 2k.\end{aligned}$$

Subcase 1.6.2. $u = x_{k+p} (2 \le p \le k-2)$, $v = x_{k+i,j} (1 \le i \le k-1, 1 \le j \le t-2)$.

$$\begin{aligned}&|f(x_{k+p}) - f(x_{k+i,j})| + d(x_{k+p}, x_{k+i,j})\\ &\ge f(x_{k+p}) - f(x_{k+i,j}) + 1 \ge f(x_{k+k-2}) - f(x_{k+1,t-2}) + 1\\ &= k+2+(2k+1) + 2(k-1)^2 - [k+4+(k-1)(2k-3)] + 1 = 3k-1\\ &> 2k.\end{aligned}$$

Subcase 1.7. $u \in Y, v \in Z$.

Let $u = x_{i,j} (2 \le i \le k, 1 \le j \le t-2)$, $v = x_{k+p,q} (1 \le p \le k-1, 1 \le q \le t-2)$.

$$\begin{aligned}&|f(x_{i,j}) - f(x_{k+p,q})| + d(x_{i,j}, x_{k+p,q})\\ &= |k-2+(p-i)(2k-3) + 2(j-q)(k-1)^2| + (k+p-i+2).\end{aligned}$$

We now consider the following cases, depending on how j and p are related.

Subcase 1.7.1. $j = q$.

$$\begin{aligned}&|f(x_{i,j}) - f(x_{k+p,q})| + d(x_{i,j}, x_{k+p,q})\\ &= |k-2+(p-i)(2k-3)| + (k+p-i+2)\\ &\ge \begin{cases} 2k + (p-i)(2k-2) \ge 2k, & p \ge i;\\ (i-p)(2k-4) + 4 \ge 2k-4+4 = 2k, & p < i.\end{cases}\end{aligned}$$

Subcase 1.7.2. $j > q$.

$$\begin{aligned}&|f(x_{i,j}) - f(x_{k+p,q})| + d(x_{i,j}, x_{k+p,q})\\ &\ge [2(j-q)(k-1)^2 + k - 2] - |p-i|(2k-3) + 3\\ &\ge 2(k-1)^2 + k - 2 - (k-1)(2k-3) + 3 = 2k.\end{aligned}$$

Subcase 1.7.3. $j < q$.

$$\begin{aligned}&|f(x_{i,j}) - f(x_{k+p,q})| + d(x_{i,j}, x_{k+p,q})\\ &= |2(q-j)(k-1)^2] - (k-2) - (p-i)(2k-3)| + (k+p-i+2)\\ &\ge \begin{cases} 2(k-1)^2 + (i-p)(2k-4) + 4 \ge 2k(k-2) + 6 > 2k, & p \le i;\\ 2(k-1)^2 - (k-2) - (k-3)(2k-3) + 3 = 4k-2 > 2k, & p > i.\end{cases}\end{aligned}$$

Combine all discussions above, (5) holds for every pair u and v of distinct vertices of $C(2k,t)$ for $k \geq 3$. Thus, for $k \geq 2$, we have

$$\mathrm{rn}(C(2k,t)) \leq 2k^2 - 2k + 1 + 2(t-2)(k-1)^2.$$

Case 2. $m = 2k+1, k \geq 2$.

In this case, $d = 2k$, and the distance condition of a radio label f is that

$$|f(u) - f(v)| + d(u,v) \geq d + 1 = 2k + 1 \qquad (6)$$

holds for every pair u and v of distinct vertices of $C(2k+1,t)$. Define a labelling g of $C(2k+1,t)$ (see Figure 2):

$$\begin{cases} g(x_1) = k+1, \\ g(x_i) = 2 + (k+1-i)(2k+1) + (t-2)(2k^2-2k+1), \ 2 \leq i \leq k, \\ g(x_{k+1}) = 0, \\ g(x_{k+i}) = k+3+(k+1-i)(2k+1)+(t-2)(2k^2-2k+1), \ 2 \leq i \leq k, \\ g(x_{2k+1}) = k+2, \\ g(x_{i,j}) = k+3+(i-1)(2k-1)+(j-1)(2k^2-2k+1), \\ \qquad\qquad 2 \leq i \leq k, 1 \leq j \leq t-2, \\ g(x_{k+i,j}) = 3 + i(2k-1) + (j-1)(2k^2-2k+1), \\ \qquad\qquad 1 \leq i \leq k, 1 \leq j \leq t-2. \end{cases}$$

It is easy to see that the vertex x_{k+2} has the maximum label

$$g(x_{k+2}) = 2k^2 + 2 + (t-2)(2k^2 - 2k + 1).$$

And (6) can be verified by an argument similar to that used in Case 1, we omit it. Thus,

$$\mathrm{rn}(C(2k+1,t)) \leq 2k^2 + 2 + (t-2)(2k^2 - 2k + 1).$$

□

Theorem 3. *For all integers $m \geq 4$ and $t \geq 2$,*

$$\mathrm{rn}(C(m,t)) = \begin{cases} 2k^2 - 2k + 1 + 2(t-2)(k-1)^2, & \text{if } m = 2k, k \geq 2; \\ 2k^2 + 2 + (t-2)(2k^2 - 2k + 1), & \text{if } m = 2k+1, k \geq 2. \end{cases}$$

Proof. This follows from Theorem 1 and Theorem 2 immediately. □

5 Some discussions and examples

As the radio labellings f of $C(2k,t)$ and g of $C(2k+1,t)$ constructed in the proof of Theorem 2 all attain the lower bounds in Theorem 1, f and g are optimal radio labellings of $C(2k,t)$ and $C(2k+1,t)$ respectively.

For clarity, as examples we give the optimal radio labellings of $C(8,4)$ and $C(7,5)$ as follows.

Example 1. *An optimal radio labelling f of $C(8,4)$.*

Here $m = 2k = 8$, $k = 4$ and $t = 4$, then $\mathrm{rn}(f) = \mathrm{rn}(C(8,4)) = [2k^2 - 2k + 1 + 2(t-2)(k-1)^2]_{k=4,t=4} = 61$ (see Figure 3).

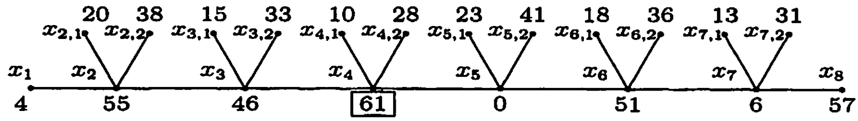

Figure 3: An optimal radio labelling f of $C(8,4)$.

Example 2. *An optimal radio labelling g of $C(7,5)$.*

Here $m = 2k+1 = 7$, $k = 3$ and $t = 5$, then $\mathrm{rn}(g) = \mathrm{rn}(C(7,5)) = [2k^2 + 2 + (t-2)(2k^2 - 2k + 1)]_{k=3,t=5} = 59$ (see Figure 4).

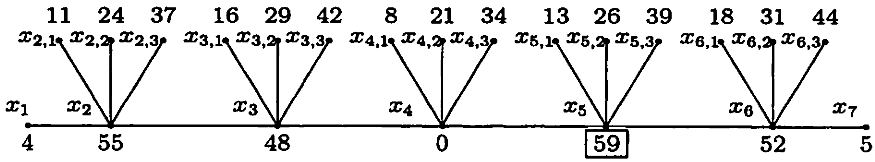

Figure 4: An optimal radio labelling g of $C(7,5)$.

Moreover, in [16], Liu and Zhu have determined the radio number of paths. They showed that

Theorem 4. *For all integers $m \geq 4$, denote a path with order m by P_m, then*

$$\mathrm{rn}(P_m) = \begin{cases} 2k^2 - 2k + 1, & \text{if } m = 2k, k \geq 2; \\ 2k^2 + 2, & \text{if } m = 2k+1, k \geq 2. \end{cases}$$

Note that if let $t = 2$, then $C(m,t) = P_m$, thus Theorem 4 is a direct corollary of Theorem 3, and the radio labellings f and g constructed in the proof of Theorem 2 with $t = 2$ are optimal radio labellings of P_{2k} and P_{2k+1} respectively.

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