All (64, 28, 12) difference sets and related structures

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Abstract.
There are 267 nonisomorphic groups of order 64. It was known that 259 of these groups admit (64, 28, 12) difference sets. In [4], the author found *all* (64, 28, 12) difference sets in 111 groups. In this paper we find *all* (64, 28, 12) difference sets in all the remaining groups of order 64 that admit (64, 28, 12) difference sets. Also, we find all nonisomorphic symmetric (64, 28, 12) designs that rise from these difference sets. We use these (64, 28, 12) difference sets to construct all (64,27,10,12) and (64,28,12,12) partial difference sets. Finally, we look at the corresponding strongly regular graphs with parameters (64,27,10,12) and (64,28,12,12).

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1 Introduction

A (v, k, λ) difference set is a subset D of size k in a group G of order v with the property that for every nonidentity g in G, there are exactly λ ordered pairs $(x, y) \in D \times D$ such that

$$xy^{-1} = g.$$

One may identify the set D with an element $\hat{D}$ in the group ring $\mathbb{Z}(G)$. In this case write

$$\hat{D} = \sum_{g \in D} g$$

and

$$\hat{D}^{(-1)} = \sum_{g \in D} g^{-1}.$$

We also write $\hat{G}$ for $\sum_{g\in G} g$. D is a difference set if the group ring element $\hat{D}$ satisfies the equation

$$\hat{D}\hat{D}^{(-1)} = (k-\lambda)1_G + \lambda\hat{G}. \tag{1}$$

If a group G has a difference set D then $\{gD : g \in G\}$ is the set of blocks of a symmetric (v, k, λ) design with point set G. On this design G acts by left multiplication as a sharply transitive automorphism group. Conversely, any symmetric design with a sharply transitive automorphism group acting on points may be constructed as the set of left translates of a difference set. A difference set is called cyclic (abelian, nonabelian) if the group G is cyclic (abelian, nonabelian). Difference sets were first introduced in cyclic groups in the study of projective plans, see [9] and [16]. Most of the progress in the study of difference sets has occurred in abelian groups; indeed the term "difference" comes from the abelian (additive) version of the formula in the definition.

For a basic introduction on difference sets and more details, the reader may consult [8], [18], [21].

Difference sets with parameters $(q^{d+1}(\frac{q^{d+1}-1}{q-1}+1), q^d\frac{q^{d+1}-1}{q-1}, q^d\frac{q^d-1}{q-1})$, where $q = p^f$ is a prime power, are known as McFarland difference sets. For further discussion on McFarland difference sets, see [11] and [23].

Difference sets with parameters $(4N^2, 2N^2-N, N^2-N)$ are known as Menon-Hadamard difference sets. More details on these difference sets can be found in [10] and [17].

The intersection between McFarland and Menon-Hadamard difference sets happens when $q = 2$. In that case we get the parameters $(2^{2d+2}, 2^{2d+1} - 2^d, 2^{2d}-2^d)$. When $d = 1$ we have the $(16, 6, 2)$ difference sets. There are 14 groups of order 16. Among them there are 12 groups that admit $(16, 6, 2)$ difference sets. Kibler found *all* the $(16, 6, 2)$ difference sets in these groups and there are three nonisomorphic symmetric $(16, 6, 2)$ designs that rise from these difference sets, see [20]. When $d = 2$ we have the $(64, 28, 12)$ difference sets. There are 267 nonisomorphic groups of order 64. These groups are listed in GAP (Groups, Algorithms and Programming), see [15], SmallGroups library as $[64, 1], \ldots, [64, 267]$. In [4], the author found all $(64, 28, 12)$ difference sets in 111 groups. In this paper we find all $(64, 28, 12)$ difference sets in all groups of order 64 that admit $(64, 28, 12)$ difference sets. Note that, similar work have been done for the $(96, 20, 4)$ difference sets and all $(96, 20, 4)$ difference sets have been found, see [1, 2, 3, 5, 12, 13, 14].

Next, we present previous results on $(64, 20, 4)$ difference sets.

2 Summary of previous results on $(64, 28, 12)$ difference sets.

Turyn showed that an abelian group of order 2^{2d+2} and exponent greater than 2^{d+2} does not admit a difference set, see [26]. Turyn's exponent bound rules out the existence of $(64, 28, 12)$ difference sets in [64,1] $\cong \mathbb{Z}_{64}$ and [64,50] $\cong \mathbb{Z}_{32} \times \mathbb{Z}_2$.

Dillon showed that the existence of difference sets in dihedral groups gives difference sets in cyclic groups, see [11]. This is sometimes called "Dillon's dihedral trick". Dillon's dihedral trick rules out the existence of $(64, 28, 12)$ difference sets in every group of order 64 that has $\mathbb{D}_{32}$ as a factor group, where $\mathbb{D}_{32}$ is the dihedral group of order 32. Groups that have $\mathbb{D}_{32}$ as a factor group are [64,i], where $i \in \{38, 47, 52, 53, 54, 186\}$. Indeed groups that ruled out using Turyn's exponent bound and Dillon's dihedral trick are the only groups that do not admit $(64, 28, 12)$ difference sets. Next we will present the existence results of $(64, 28, 12)$ difference sets in all other groups.

McFarland constructed $(q^{d+1}(\frac{q^{d+1}-1}{q-1} + 1), q^d \frac{q^{d+1}-1}{q-1}, q^d \frac{q^d-1}{q-1})$ difference sets in abelian groups which have an elementary abelian normal subgroup of order q^{d+1}, where $q = p^f$, p is a prime and d is a positive integer, see [23]. McFarland's construction gives $(64, 28, 12)$ difference sets in [64,55] $\cong \mathbb{Z}_4^3$, [64,83]$\cong \mathbb{Z}_8 \times \mathbb{Z}_4 \times \mathbb{Z}_2$, and [64,183]$\cong \mathbb{Z}_{16} \times \mathbb{Z}_2^2$, [64,192]$\cong \mathbb{Z}_4^2 \times \mathbb{Z}_2^2$, [64,246]$\cong \mathbb{Z}_8 \times \mathbb{Z}_2^3$, [64,260]$\cong \mathbb{Z}_4 \times \mathbb{Z}_2^4$, and [64,267]$\cong \mathbb{Z}_2^6$.

Dillon generalized McFarland's construction to work for a larger set of groups. He constructed McFarland difference sets in groups that have an elementary abelian normal subgroup of order q^{d+1} in its center, see [11]. Dillon's construction gives $(64, 28, 12)$ difference sets in [64,i], where $i \in \{$ 17, 21, 56, 57, 58, 59, 60, 61, 62, 63, 64, 65, 66, 67, 68, 69, 70, 71, 72, 73, 74, 75, 76, 77, 78, 79, 80, 81, 82, 84, 87, 95, 96, 103, 106, 107, 193, 194, 195, 196, 197, 202, 203, 204, 205, 207, 208, 209, 211, 212, 247, 250, 251, 252, 261, 262, 263 $\}$. Note that all of these groups are nonabelian and have $\mathbb{Z}_2^3$ in their center.

Arasu constructed $(64, 28, 12)$ difference sets in the last two abelian groups [64,26] $\cong \mathbb{Z}_{16} \times \mathbb{Z}_4$ and [64,2] $\cong \mathbb{Z}_8 \times \mathbb{Z}_8$, see [7].

In an unpublished work by Dillon, he was able to construct $(64, 28, 12)$ difference sets in 258 groups. This left the existence of $(64, 28, 12)$ difference sets undecided in the single group [64,51] $\cong \langle x, y : x^{32} = y^2 = 1,\ yxy = x^{17}\rangle$. This group is called the Modular group. The exponent of the Modular group is 32.

Liebler and Smith constructed $(64, 28, 12)$ difference set in the Modular group, see [19]. This was the first example which demonstrated that Turyn's exponent bound for abelian groups can be violated in the nonabelian case.

All of this work can be summarize as follows. Among the 267 groups of order 64, there are 259 groups that admit $(64, 28, 12)$ difference sets and there are 8 groups that do not admit $(64, 28, 12)$ difference sets. As we mention before the groups that do not admit $(64, 28, 12)$ difference sets are the ones that have $\mathbb{Z}_{32}$ or $\mathbb{D}_{32}$ as a factor group. In all of the previous results, no research have been done to find all $(64, 28, 12)$ difference sets in a certain group of order 64. In [4], the author used the software GAP to find all $(64, 28, 12)$ difference sets in 111 groups. In this paper, we use the software GAP to find all $(64, 28, 12)$ difference sets in the remaining groups that admit $(64, 28, 12)$ difference sets.

3 All $(64, 28, 12)$ difference sets and related structures

We will describe how we find all $(64, 28, 12)$ difference sets in all groups that admit $(64, 28, 12)$ difference sets.

A homomorphism f from G onto G' induces, by linearity, a homomorphism from $\mathbb{Z}[G]$ onto $\mathbb{Z}[G']$. If the kernel of f is the subgroup U, let T be a complete set of distinct representatives of cosets of U and, for $g \in T$, set $t_g := |gU \cap D|$. The multiset $\{t_g : g \in T\}$ is the collection of "intersection numbers" of D with respect to U. The image of $\hat{D}$ under the function f is $f(\hat{D}) = \sum_{g \in T} t_g f(g)$.

This group ring element satisfies the equation

$$f(\hat{D})f(\hat{D})^{(-1)} = (k - \lambda)1_{G'} + \lambda|U|\hat{G}' \tag{2}$$

in the group ring $\mathbb{Z}[G']$.

The contraction of D to a smaller homomorphic image often provides useful conditions on the existence of a difference set in the original group. Usually, we use the software GAP to construct the images of difference sets (find the intersection numbers) recursively. In the $(64, 28, 12)$ case, we use the intersection numbers of the group of order two to find them for groups of order four then we use intersection numbers of groups of order four to find them for groups of order eight. We proceed up till we find all the intersection numbers in groups of order 32. So, we get the 32-images of putative $(64, 28, 12)$ difference sets in all groups of order 32. In each one of the groups of order 64, we write programs in GAP that use the 32-images to construct all possible $(64, 28, 12)$ difference sets. So, we are able to construct *all* $(64, 28, 12)$ difference sets in the 259 groups of order 64 that admit $(64, 28, 12)$ difference sets.

We say $D_1 \in \mathbb{Z}(G)$, $D_2 \in \mathbb{Z}(G)$ are equivalent if there is an element $g \in G$ and an automorphism φ of G such that $D_1 = g\varphi(D_2)$. We say that two difference sets are inequivalent if either they are subsets of nonisomorphic groups or if they are subsets in a common group G but are not equivalent in G (as defined above.) Inequivalent difference sets *may* give rise to isomorphic designs. In 245 of the groups of order 64 that admit

(64, 28, 12) difference sets, we were able to check equivalence. In Table 1, we provide the number of all inequivalent (64, 28, 12) difference sets in each one of these groups. In the other 14 groups we could not check equivalence and this is because the size of the automorphism groups of these groups are large. We provide the number of (64, 28, 12) difference sets (not necessarily inequivalent) in these groups in Table 2. In each one of the 259 groups of order 64 that admit (64, 28, 12) difference sets, we construct the symmetric (64, 28, 12) designs that rise from these difference sets. In Tables 1 and 2, we provide the number of nonisomorphic symmetric (64, 28, 12) designs in each one of the 259 groups. Note that we used the "DESIGN" package for GAP, see [24], to determine the nonisomorphic symmetric (64, 28, 12) designs that rise from (64, 28, 12) difference sets. A difference set D is reversible if $D = D^{(-1)}$. There are 184 groups of order 64 that admit reversible (64, 28, 12) difference sets. In each one of these groups we find all reversible (64, 28, 12) difference sets. In Tables 1 and 2, we provide the number of reversible (64, 28, 12) difference sets.

We have a list of all (64, 28, 12) difference sets in each one of the groups of order 64 that admit (64, 28, 12) difference sets in the webpage in [6]. We provide the (64, 28, 12) difference sets that give nonisomorphic symmetric (64, 28, 12) designs in this webpage. Also, we list all reversible (64, 28, 12) difference sets in this webpage.

A (v, k, λ, μ) partial difference set is a subset T of size k in a group G of order v such that the multiset $\{xy^{-1} : x, y \in T \text{ and } x \neq y\}$ contains each nonidentity element of T exactly λ times and each nonidentity element of $G \backslash T$ exactly μ times. It is clear that a (v, k, λ) difference set is a (v, k, λ, λ) partial difference set.

A partial difference set T is called reversible if $T = T^{(-1)}$. A reversible partial difference set is called regular if it does not contain the identity element. Two partial difference sets T_1 and T_2 in a group G are equivalent if there is an automorphism φ of G such that $\varphi(T_1) = T_2$. The following results can be found in [22].

Proposition 1. *Suppose that T is a reversible (v, k, λ, μ) partial difference set that contains the identity element. Then $T\backslash\{e\}$ is a regular $(v, k-1, \lambda-2, \mu)$ partial difference set.*

Proposition 2. *Suppose that D is a (v, k, λ) difference set in a group G and $g \in G$. Then gD is a regular (v, k, λ, λ) partial difference set if and only if $g^{-1} \notin D$ and gD is a reversible set. Also $gD\backslash\{e\}$ is a regular $(v, k-1, \lambda-2, \lambda)$ partial difference set if and only if $g^{-1} \in D$ and gD is a reversible set.*

Regular partial difference sets are closely related to strongly regular graphs. We give the definition of strongly regular graphs and Cayley graphs.

Definition 1. *A (v,k,λ,μ) strongly regular graph is a graph with v vertices which is regular with valency k such that any pair of adjacent vertices have exactly λ common neighbours and any pair of nonadjacent vertices have exactly μ common neighbours.*

Definition 2. *For a group G and a subset T of G with $e \notin T$ and $T = T^{-1}$, the Cayley graph $\Gamma = Cay(G,T)$ is a graph whose vertex set G and two vertices x and y are adjacent if $xy^{-1} \in T$.*

We have the following known theorem that relate strongly regular graphs and partial difference sets, see [8].

Theorem 1. *A Cayley graph $Cay(G,T)$ is a (v,k,λ,μ) strongly regular graph if and only if T is a (v,k,λ,μ) regular partial difference set in G.*

There are 184 groups of order 64 that admit regular $(64,27,10,12)$ and $(64,28,12,12)$ partial difference sets. In Tables 1 and 2, we state the number of regular $(64,27,10,12)$ and $(64,28,12,12)$ partial difference sets in each one of these groups. We list all of these regular $(64,27,10,12)$ and $(64,28,12,12)$ partial difference sets in the webpage in [6]. We find all nonisomorphic strongly regular graphs provided by the regular (64,27,10,12) and $(64,28,12,12)$ partial difference sets. The regular $(64,27,10,12)$ partial difference sets give 596 nonisomorphic $(64,27,10,12)$ strongly regular graphs. The regular $(64,28,12,12)$ partial difference sets give 904 nonisomorphic $(64,28,12,12)$ strongly regular graphs. More details on these strongly regular graphs can be found in the webpage in [6]. Note that, we have used the "GRAPE" package for GAP, [25], to construct strongly regular graphs and to check which ones are nonisomorphic.

Table 1: In the first column we have the group number. In the second column we have the number of inequivalent $(64,28,12)$ difference sets in this group. In the third column we have the number of nonisomorphic symmetric $(64,28,12)$ designs in this group. In the fourth column we have the number of reversible $(64,28,12)$ difference sets in this group. In the fifth column we have the number of regular $(64,28,12,12)$ partial difference sets and in the sixth column we have the number of regular (64,27,10,12) partial difference sets.

Table 1 :

Group	# DSs	# SDs	# RDSs	# (64, 28, 12, 12) PDSs	# (64, 27, 10, 12) PDSs
[64, 2]	31	13	6	3	3
[64, 3]	71	15	12	7	5
[64, 4]	468	204	32	16	16

Table 1 : (continue)

[64,5]	708	318	32	16	16
[64,6]	584	166	0	0	0
[64,7]	1320	394	0	0	0
[64,8]	616	169	296	158	138
[64,9]	2200	1161	8	4	4
[64,10]	300	101	80	43	37
[64,11]	522	195	10	7	3
[64,12]	67	16	8	4	4
[64,13]	688	209	8	4	4
[64,14]	319	120	0	0	0
[64,15]	104	17	8	4	4
[64,16]	104	17	8	4	4
[64,17]	1012	227	0	0	0
[64,18]	652	464	32	16	16
[64,19]	176	147	0	0	0
[64,20]	1944	1124	112	56	56
[64,21]	968	193	0	0	0
[64,22]	600	396	36	19	17
[64,23]	882	523	112	56	56
[64,24]	1026	711	72	38	34
[64,25]	1180	844	32	16	16
[64,26]	32	24	0	0	0
[64,27]	24	16	0	0	0
[64,28]	148	92	22	15	7
[64,29]	284	143	0	0	0
[64,30]	388	185	28	17	11
[64,31]	448	249	32	16	16
[64,32]	642	410	266	141	125
[64,33]	962	586	38	21	17
[64,34]	228	124	159	85	74
[64,35]	684	451	73	43	30
[64,36]	306	170	61	34	27
[64,37]	706	387	59	33	26
[64,39]	440	273	0	0	0
[64,40]	168	90	64	36	28
[64,41]	204	119	64	36	28
[64,42]	60	43	12	6	6
[64,43]	340	156	64	36	28
[64,44]	52	34	0	0	0

Table 1 : (continue)

[64, 45]	136	88	20	12	8
[64, 46]	152	75	12	6	6
[64, 48]	56	56	0	0	0
[64, 49]	52	35	0	0	0
[64, 51]	112	96	0	0	0
[64, 56]	624	57	38	20	18
[64, 57]	590	338	39	20	19
[64, 58]	2438	799	389	208	181
[64, 59]	3776	2135	147	76	71
[64, 60]	376	76	324	171	153
[64, 61]	2156	577	197	101	96
[64, 62]	1096	320	204	110	94
[64, 63]	1321	591	78	40	38
[64, 64]	868	395	69	36	33
[64, 65]	645	368	18	9	9
[64, 66]	4136	936	331	169	162
[64, 67]	3432	424	1387	742	645
[64, 68]	10380	4128	551	286	265
[64, 69]	7164	1339	1289	672	617
[64, 70]	6236	2949	191	99	92
[64, 71]	1678	206	649	344	305
[64, 72]	3534	1753	156	87	69
[64, 73]	979	46	874	460	414
[64, 74]	1401	339	132	71	61
[64, 75]	3042	223	1170	611	559
[64, 76]	2273	1069	42	21	21
[64, 77]	2482	507	261	136	125
[64, 78]	3250	435	642	335	307
[64, 79]	5146	2039	195	101	94
[64, 80]	1271	283	121	63	58
[64, 81]	4678	1763	282	146	136
[64, 82]	470	209	41	22	19
[64, 83]	463	306	0	0	0
[64, 84]	583	411	12	6	6
[64, 85]	770	578	28	14	14
[64, 86]	1122	805	0	0	0
[64, 87]	944	365	16	8	8
[64, 88]	760	465	152	92	60
[64, 89]	1162	743	240	148	92

Table 1 : (continue)

[64, 90]	964	549	484	277	207
[64, 91]	948	649	316	193	123
[64, 92]	422	280	184	107	77
[64, 93]	1622	1100	212	128	84
[64, 94]	900	638	240	148	92
[64, 95]	480	173	72	39	33
[64, 96]	1900	884	0	0	0
[64, 97]	1196	693	364	220	144
[64, 98]	1232	742	392	229	163
[64, 99]	398	276	88	44	44
[64, 100]	1942	1265	324	198	126
[64, 101]	2024	1134	560	336	224
[64, 102]	1604	1006	448	280	168
[64, 103]	674	286	0	0	0
[64, 104]	616	435	16	8	8
[64, 105]	1062	711	24	12	12
[64, 106]	442	335	52	27	25
[64, 107]	314	209	0	0	0
[64, 108]	602	461	0	0	0
[64, 109]	1348	929	68	35	33
[64, 110]	656	450	16	8	8
[64, 111]	616	458	16	8	8
[64, 112]	1186	793	16	8	8
[64, 113]	990	615	40	20	20
[64, 114]	1244	707	52	29	23
[64, 115]	1624	1018	6	3	3
[64, 116]	2364	1142	120	66	54
[64, 117]	1348	728	86	47	39
[64, 118]	320	192	8	4	4
[64, 119]	2024	1010	152	80	72
[64, 120]	2672	1743	0	0	0
[64, 121]	1808	904	64	32	32
[64, 122]	2450	1466	40	22	18
[64, 123]	306	192	114	60	54
[64, 124]	1184	890	0	0	0
[64, 125]	1164	874	0	0	0
[64, 126]	572	335	6	3	3
[64, 127]	1728	982	6	3	3
[64, 128]	234	100	416	233	183
[64, 129]	1466	599	560	338	222

Table 1 : (continue)

[64, 130]	1024	364	520	313	207
[64, 131]	738	366	216	122	94
[64, 132]	3786	2091	0	0	0
[64, 133]	2512	992	80	48	32
[64, 134]	342	128	592	356	236
[64, 135]	540	236	448	280	168
[64, 136]	1050	615	0	0	0
[64, 137]	1608	1037	0	0	0
[64, 138]	450	219	494	298	196
[64, 139]	751	405	0	0	0
[64, 140]	174	119	92	49	43
[64, 141]	554	255	60	30	30
[64, 142]	1178	595	132	73	59
[64, 143]	3134	1801	0	0	0
[64, 144]	904	384	192	106	86
[64, 145]	2400	885	176	100	76
[64, 146]	1492	640	296	169	127
[64, 147]	236	124	20	11	9
[64, 148]	1900	1062	0	0	0
[64, 149]	1416	628	168	94	74
[64, 150]	288	175	212	118	94
[64, 151]	1888	972	80	48	32
[64, 152]	880	489	64	36	28
[64, 153]	180	111	32	16	16
[64, 154]	1588	1067	0	0	0
[64, 155]	578	376	20	10	10
[64, 156]	1758	1202	92	48	44
[64, 157]	670	475	84	44	40
[64, 158]	1778	1305	0	0	0
[64, 159]	1200	726	8	4	4
[64, 160]	3192	1786	64	32	32
[64, 161]	398	204	84	45	39
[64, 162]	542	343	256	139	117
[64, 163]	920	508	36	19	17
[64, 164]	1558	955	184	103	81
[64, 165]	1358	919	0	0	0
[64, 166]	2312	1121	144	80	64
[64, 167]	214	154	72	40	32
[64, 168]	1090	757	0	0	0
[64, 169]	796	445	96	54	42

Table 1 : (continue)

[64, 170]	788	463	92	49	43
[64, 171]	156	95	0	0	0
[64, 172]	1092	767	52	29	23
[64, 173]	192	100	52	29	23
[64, 174]	13	8	14	8	6
[64, 175]	623	475	0	0	0
[64, 176]	379	184	0	0	0
[64, 177]	136	64	80	44	36
[64, 178]	952	457	64	40	24
[64, 179]	388	353	22	12	10
[64, 180]	647	484	0	0	0
[64, 181]	256	225	0	0	0
[64, 182]	1220	841	8	4	4
[64, 183]	160	120	0	0	0
[64, 184]	444	263	32	16	16
[64, 185]	436	408	0	0	0
[64, 187]	188	141	64	36	28
[64, 188]	300	253	0	0	0
[64, 189]	104	70	0	0	0
[64, 190]	88	54	128	72	56
[64, 191]	424	238	0	0	0
[64, 195]	1179	632	150	85	65
[64, 196]	1620	539	227	117	110
[64, 197]	1889	1396	16	8	8
[64, 198]	1422	686	0	0	0
[64, 199]	608	356	16	8	8
[64, 200]	711	459	32	18	14
[64, 201]	1276	637	0	0	0
[64, 202]	374	76	820	448	372
[64, 203]	1054	217	530	281	249
[64, 204]	2446	1110	28	14	14
[64, 205]	1316	375	235	121	114
[64, 206]	2282	837	8	4	4
[64, 207]	781	285	172	95	77
[64, 208]	1774	1145	22	11	11
[64, 209]	1186	531	111	59	52
[64, 210]	5912	2101	32	16	16
[64, 212]	1477	1069	0	0	0
[64, 213]	1170	489	8	4	4

Table 1 : (continue)

[64, 214]	3170	1546	4	2	2
[64, 215]	575	294	184	96	88
[64, 216]	659	365	360	210	150
[64, 217]	2117	876	0	0	0
[64, 218]	1054	449	0	0	0
[64, 219]	3272	1388	144	76	68
[64, 220]	5312	1981	0	0	0
[64, 221]	1656	756	88	44	44
[64, 222]	4464	1737	64	40	24
[64, 223]	5632	1979	0	0	0
[64, 224]	639	417	0	0	0
[64, 225]	2682	1215	0	0	0
[64, 226]	526	276	329	173	156
[64, 227]	3060	1131	144	74	70
[64, 228]	2789	1069	0	0	0
[64, 229]	2090	767	0	0	0
[64, 230]	2059	891	0	0	0
[64, 231]	597	291	32	16	16
[64, 232]	4290	1728	88	46	42
[64, 233]	7252	2745	0	0	0
[64, 234]	4372	1646	96	52	44
[64, 235]	4317	1839	0	0	0
[64, 236]	1200	576	27	14	13
[64, 237]	3829	1439	0	0	0
[64, 238]	2597	1295	0	0	0
[64, 239]	569	401	0	0	0
[64, 240]	1186	574	24	12	12
[64, 241]	844	456	158	83	75
[64, 242]	218	166	37	19	18
[64, 243]	2058	888	0	0	0
[64, 244]	3469	1320	0	0	0
[64, 245]	325	180	9	6	3
[64, 247]	603	503	92	55	37
[64, 248]	995	834	0	0	0
[64, 249]	834	708	32	18	14
[64, 250]	41	39	8	4	4
[64, 251]	346	230	176	105	71
[64, 252]	825	716	0	0	0
[64, 253]	750	449	0	0	0
[64, 254]	474	261	644	386	258

Table 1 : (continue)

[64, 255]	2294	1457	0	0	0
[64, 256]	1048	691	0	0	0
[64, 257]	256	162	0	0	0
[64, 258]	1072	706	0	0	0
[64, 259]	1396	1058	0	0	0

Table 2: In the first column we have the group number. In the second column we have the number of (64, 28, 12) difference sets (not necessarily inequivalent) in this group. In the third column we have the number of nonisomorphic symmetric (64, 28, 12) designs in this group. In the fourth column we have the number of reversible (64, 28, 12) difference sets in this group. In the fifth column we have the number of regular (64, 28, 12, 12) partial difference sets and in the sixth column we have the number of regular (64,27,10,12) partial difference sets.

Table 2:

Group	# DSs	# SDs	# RDSs	# (64, 28, 12, 12) RDSs	# (64, 27, 10, 12) RDSs
[64, 55]	1121	169	11	6	5
[64, 192]	1400	131	32	18	14
[64, 193]	1231	67	141	75	66
[64, 194]	1790	229	8	4	4
[64, 211]	301	34	202	112	90
[64, 246]	743	113	0	0	0
[64, 260]	2574	18	6	3	3
[64, 261]	264	21	121	65	56
[64, 262]	5834	139	0	0	0
[64, 263]	1064	155	14	7	7
[64, 264]	209	105	0	0	0
[64, 265]	481	210	0	0	0
[64, 266]	157	137	0	0	0
[64, 267]	132	3	1607	891	716

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