SOME IDENTITIES OF BARNES-TYPE GENOCCHI POLYNOMIALS

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ABSTRACT. In this paper, we study some identities of Barnes-type Genocchi polynomials. We derive those identities by using the fermionic $p-$adic integral on $\mathbb{Z}_p$.

In [13], D.S. Kim and T.Kim established some identities of higher order Bernoulli and Euler polynomials arising from Bernoulli and Euler basis respectively. Using the idea developed in [13], we study various identities of special polynomials arising from Barnes-type Genocchi basis.

1. INTRODUCTION

As is known, the Genocchi polynomials of order r are defined by the generating function to be

$$(1) \qquad \left(\frac{2t}{e^t+1}\right)^r e^{xt} = \sum_{n=0}^{\infty} G_n^{(r)}(x)\frac{t^n}{n!}, \quad \text{(see [1 - 29])}.$$

When $x = 0$, $G_n^{(r)} = G_n^{(r)}(0)$ are called the Genocchi number of order r.

For $a_1, a_2, \cdots, a_r \neq 0 \in \mathbb{C}_p$, the Barnes-type Genocchi polynomials are defined by the generating function to be

$$(2) \qquad \prod_{i=1}^{r}\left(\frac{2t}{e^{a_i t}+1}\right)e^{xt} = \sum_{n=0}^{\infty} G_n(x|a_1, a_2, \cdots, a_r)\frac{t^n}{n!}.$$

When $x = 0$, $G_n(0|a_1, a_2, \cdots, a_r) = G_n(a_1, a_2, \cdots, a_r)$ are called Barnes Genocchi numbers (see [10, 12, 16, 21]).

Let p be a fixed odd prime number. Throughout this paper, $\mathbb{Z}_p$, $\mathbb{Q}_p$ and $\mathbb{C}_p$ will denote the ring of $p-$adic integers, the field of $p-$adic numbers and the completion of algebraic closure of $\mathbb{Q}_p$. The normalized $p-$adic norm is defined as $|p|_p = 1/p$. Let $UD(\mathbb{Z}_p)$ be the space of uniformly differentiable function on $\mathbb{Z}_p$. For $f \in UD(\mathbb{Z}_p)$, the bosonic $p-$adic integral on $\mathbb{Z}_p$ is

1991 *Mathematics Subject Classification.* 11B68; 11S40; 11S80.

Key words and phrases. Barnes-type Genocchi polynomial, Barnes-type Genocchi basis.

defined by

$$(3)\quad \begin{aligned} I_0(f) = \int_{\mathbb{Z}_p} f(x)d\mu_0(x) &= \lim_{N\to\infty} \sum_{x=0}^{p^N-1} f(x)\mu_0(x+p^N\mathbb{Z}_p) \\ &= \lim_{N\to\infty} \frac{1}{p^N} \sum_{x=0}^{p^N-1} f(x), \quad \text{(see [17, 19, 20])}. \end{aligned}$$

As is well known, the fermionic $p-$adic integral on $\mathbb{Z}_p$ is defined by Kim to be

(4)

$$\begin{aligned} I_{-1}(f) = \int_{\mathbb{Z}_p} f(x)d\mu_{-1}(x) &= \lim_{N\to\infty} \sum_{x=0}^{p^N-1} f(x)\mu_{-1}(x+p^N\mathbb{Z}_p) \\ &= \lim_{N\to\infty} \sum_{x=0}^{p^N-1} f(x)(-1)^x, \quad \text{(see [18, 19, 20])}. \end{aligned}$$

From (4), we can derive

$$(5)\quad I_{-1}(f_n) + (-1)^{n-1}I_{-1}(f) = 2\sum_{l=0}^{n-1}(-1)^{n-l-1}f(l),$$

where $f_n(x) = f(x+n)$, $(n \in \mathbb{N})$.

In particular, $n = 1$, we have

$$(6)\quad I_{-1}(f_1) + I_{-1}(f) = 2f(0), \quad \text{(see [18, 19, 20])}.$$

In section 2, we give a Witt's formula for the Barnes-type Genocchi polynomials by usig the fermionic $p-$adic integral on $\mathbb{Z}_p$. And we give some identities of Barnes-type Genocchi polynomials. And then, we give some identities of mixed-type Barnes-type Bernoulli and Barnes-type Genocchi polynomials.

In [13], D.S. Kim and T. Kim established some identities of higher order Bernoulli and Euler polynomials arising from Bernoulli and Euler basis respectively. Using the idea developed in [13], many authors have constructed interesting identities by using various polynomials basis respectively (see [5, 11, 13, 14, 15, 26]).

In section 3, by using the method of D.S. Kim and T. Kim (see [13]), we study some new identities and properties of special polynomials which are derived from the Barnes-type Genocchi basis.

2. Identities of Barnes-type Genocchi polynomials

Let $a_1, a_2, \cdots, a_r \neq 0 \in \mathbb{C}_p$. Then, by (6), we get

$$(7)\quad \begin{aligned} & t^r \int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} e^{(a_1x_1+\cdots+a_rx_r+x)t} d\mu_{-1}(x_1)\cdots d\mu_{-1}(x_r) \\ &= \prod_{i=1}^{r} \left(\frac{2t}{e^{a_it}+1}\right) e^{xt} \\ &= \sum_{n=0}^{\infty} G_n(x|a_1,\cdots,a_r)\frac{t^n}{n!}. \end{aligned}$$

From (7), we obtain the following result.

Theorem 1. *For $a_1, a_2, \cdots, a_r \neq 0 \in \mathbb{C}_p$ and $n \in \mathbb{Z} \geq 0$, we have*

$$\frac{G_{n+r}(x|a_1,\cdots,a_r)}{\binom{n+r}{r}r!} = \int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} (a_1x_1 + \cdots a_rx_r + x)^n d\mu_{-1}(x_1)\cdots d\mu_{-1}(x_r),$$

where $\binom{n+r}{r} = \frac{(n+r)!}{n!r!}$.

The right hand side of Theorem 1 is the Witt's formula for Barnes-type Euler polynomial $E_n(x|a_1,\cdots,a_r)$ (see [12, 20, 22]). Thus, we have the following theorem.

Theorem 2. *For $n, r \in \mathbb{N}$, we have*

$$E_n(x|a_1,\cdots,a_r) = \frac{1}{\binom{n+r}{n}r!} G_{n+r}(x|a_1,\cdots,a_r).$$

moreover,

$$E_n(a_1,\cdots,a_r) = \frac{1}{\binom{n+r}{n}r!} G_{n+r}(a_1,\cdots,a_r).$$

From, (2), we get the following:

$$(8)\quad \begin{aligned} \sum_{n=0}^{\infty} G_n(x|a_1,\cdots,a_r)\frac{t^n}{n!} &= t^r \int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} e^{(a_1x_1+\cdots+a_rx_r+x)t} \\ &\quad \times d\mu_{-1}(x_1)\cdots d\mu_{-1}(x_r) \\ &= \left(\sum_{n=0}^{\infty} G_n(a_1,\cdots,a_r)\frac{t^n}{n!}\right)\left(\sum_{n=0}^{\infty} \frac{x^nt^n}{n!}\right) \\ &= \sum_{n=0}^{\infty}\left(\sum_{k=0}^{n}\binom{n}{k} G_k(a_1,\cdots,a_r)x^{n-k}\right)\frac{t^n}{n!}. \end{aligned}$$

Thus we have the following result.

Theorem 3. *For $n \in \mathbb{Z} \geq 0$, we have*

$$G_n(x|a_1, \cdots, a_r) = \sum_{k=0}^{n} \binom{n}{k} G_k(a_1, \cdots, a_r) x^{n-k} = \sum_{k=0}^{n} \binom{n}{k} G_{n-k}(a_1, \cdots, a_r) x^k.$$

Note that

$$(9) \quad (a_1x_1 + \cdots + a_rx_r)^n = \sum_{l_1+\cdots+l_r=n} \binom{n}{l_1, \cdots, l_r} a_1^{l_1} x_1^{l_1} \cdots a_r^{l_r} x_r^{l_r} = \sum_{l_1+\cdots+l_r=n} \binom{n}{l_1, \cdots, l_r} \left(\prod_{i=1}^{r} a_i^{l_i}\right) x_1^{l_1} \cdots x_r^{l_r}.$$

By (9) and Theorem 3, we obtain the following corollary.

Corollary 4. *For $n \in \mathbb{N}$, we have*

$$G_n(a_1, \cdots, a_r) = \sum_{l_1+\cdots+l_r=n} \binom{n}{l_1, \cdots, l_r} (\prod_{i=1}^{r} a_i^{l_i}) G_{l_1} \cdots G_{l_r},$$

where $G_n = G_n(1)$ is the $n-$th Genocchi number.

From (4) and (5), we can derive the following equation:

$$t \int_{\mathbb{Z}_p} e^{(x+1)t} d\mu_{-1}(x) + t \int_{\mathbb{Z}_p} e^{xt} d\mu_{-1}(x) = 2t.$$

Thus, we have

$$t \int_{\mathbb{Z}_p} e^{(x+y)t} d\mu_{-1}(y) = \frac{2t}{e^t + 1} e^{xt} = \sum_{n=0}^{\infty} G_n(x) \frac{t^n}{n!},$$

where $G_n(x)$ is the $n-$th Genocchi polynomial.

From (5), we note that, for $n \in \mathbb{N}$,

(10)

$$t \int_{\mathbb{Z}_p} e^{a(x+n)t} d\mu_{-1}(x) + (-1)^{n-1} t \int_{\mathbb{Z}_p} e^{axt} d\mu_{-1}(x) = 2t \sum_{l=0}^{n-1} (-1)^{n-1-l} e^{alt}.$$

Let us assume that $n \in \mathbb{N}$ with $n \equiv 1 \pmod 2$. Then, by (10), we get

$$(11) \quad t \int_{\mathbb{Z}_p} e^{axt} d\mu_{-1}(x) = \frac{2t}{e^{ant} + 1} \sum_{l=0}^{n-1} (-1)^l e^{alt}.$$

Now, we consider the multivariate $p-$adic fermionic integral on $\mathbb{Z}_p$ related to the Barnes-type Genocchi numbers as follows:
(12)

$$\begin{aligned}
&t^r \int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} e^{(a_1x_1+\cdots+a_rx_r)t} d\mu_{-1}(x_1)\cdots d\mu_{-1}(x_r) \\
&= \left(\prod_{i=1}^{r} \frac{2t}{e^{a_int}+1}\right) \sum_{l_1=0}^{n-1} \cdots \sum_{l_r=0}^{n-1} (-1)^{l_1+\cdots+l_r} e^{(a_1l_1+\cdots+a_rl_r)t} \\
&= \sum_{l_1=0}^{n-1} \cdots \sum_{l_r=0}^{n-1} (-1)^{l_1+\cdots+l_r} \left(\prod_{i=1}^{r} \frac{2t}{e^{a_int}+1}\right) e^{\left(\frac{a_1l_1+\cdots+a_rl_r}{n}\right)nt} \\
&= \sum_{m=0}^{\infty} \sum_{l_1=0}^{n-1} \cdots \sum_{l_r=0}^{n-1} (-1)^{l_1+\cdots+l_r} G_m\left(\frac{a_1l_1+\cdots+a_rl_r}{n} \middle| a_1n,\cdots,a_rn\right) n^m \frac{t^m}{m!},
\end{aligned}$$

where $n \in \mathbb{N}$ with $n \equiv 1 \pmod 2$.

Thus, by (12), we get

$$\begin{aligned}
&\int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} (a_1x_1+\cdots+a_rx_r)^m d\mu_{-1}(x_1)\cdots d\mu_{-1}(x_r) \\
&= \frac{1}{\binom{m+r}{r} r!} \sum_{l_1=0}^{n-1} \cdots \sum_{l_r=0}^{n-1} (-1)^{l_1+\cdots+l_r} \\
&\quad \times G_{m+r}\left(\frac{a_1l_1+\cdots+a_rl_r}{n} \middle| a_1n,\cdots,a_rn\right) n^{m+r}, \qquad (13)
\end{aligned}$$

where $m \in \mathbb{Z} \geq 0$, $n \in \mathbb{N}$ with $n \equiv 1 \pmod 2$.

Therefore by Theorem 1 and (13), we obtain the following theorem.

Theorem 5. *For $m \in \mathbb{Z} \geq 0$, $n \in \mathbb{N}$ with $n \equiv 1 \pmod 2$, we have*

$$\begin{aligned}
&G_m(a_1,\cdots,a_r) \\
&= n^m \sum_{l_1=0}^{n-1} \cdots \sum_{l_r=0}^{n-1} (-1)^{l_1+\cdots+l_r} G_m\left(\frac{a_1l_1+\cdots+a_rl_r}{n} \middle| a_1n,\cdots,a_rn\right).
\end{aligned}$$

Moreover,

$$\begin{aligned}
&G_m(x|a_1,\cdots,a_r) \\
&= n^m \sum_{l_1=0}^{n-1} \cdots \sum_{l_r=0}^{n-1} (-1)^{l_1+\cdots+l_r} G_m\left(\frac{a_1l_1+\cdots+a_rl_r+x}{n} \middle| a_1n,\cdots,a_rn\right).
\end{aligned}$$

For $d \in \mathbb{N}$ with $d \equiv 1 \pmod 2$, we observe that

$$\int_{\mathbb{Z}_p} f(x) d\mu_{-1}(x) = \sum_{a=0}^{d-1} (-1)^a \int_{\mathbb{Z}_p} f(a+dx) d\mu_{-1}(x). \qquad (14)$$

From (14), we can derive the following equation:

(15)

$$t^r \int_{\mathbf{Z}_p} \cdots \int_{\mathbf{Z}_p} e^{(a_1x_1+\cdots+a_rx_r)t} d\mu_{-1}(x_1)\cdots d\mu_{-1}(x_r)$$

$$= \sum_{l_1=0}^{d-1} \cdots \sum_{l_r=0}^{d-1} (-1)^{l_1+\cdots+l_r} t^r \int_{\mathbf{Z}_p} \cdots \int_{\mathbf{Z}_p} e^{\{a_1l_1+\cdots+a_rl_r+(a_1x_1+\cdots+a_rx_r)d\}t}$$

$$\times\, d\mu_{-1}(x_1)\cdots d\mu_{-1}(x_r)$$

$$= \sum_{n=0}^{\infty} d^n \sum_{l_1=0}^{d-1} \cdots \sum_{l_r=0}^{d-1} (-1)^{l_1+\cdots+l_r} t^r \int_{\mathbf{Z}_p} \cdots \int_{\mathbf{Z}_p} \left(\frac{a_1l_1+\cdots+a_rl_r}{d} + a_1x_1 \right.$$

$$\left. + \cdots + a_rx_r \right)^n d\mu_{-1}(x_1)\cdots d\mu_{-1}(x_r) \frac{t^n}{n!}.$$

By (15), we get

$$\int_{\mathbf{Z}_p} \cdots \int_{\mathbf{Z}_p} (a_1x_1+\cdots+a_rx_r)^n d\mu_{-1}(x_1)\cdots d\mu_{-1}(x_r)$$

$$(16) \quad = d^n \sum_{l_1=0}^{d-1} \cdots \sum_{l_r=0}^{d-1} (-1)^{l_1+\cdots+l_r} \int_{\mathbf{Z}_p} \cdots \int_{\mathbf{Z}_p} \left(\frac{a_1l_1+\cdots+a_rl_r}{d} + a_1x_1 \right.$$

$$\left. + \cdots + a_rx_r \right)^n d\mu_{-1}(x_1)\cdots d\mu_{-1}(x_r).$$

Therefore, by (16), we obtain the following theorem.

Theorem 6. *For $d \in \mathbb{N}$ with $d \equiv 1 \pmod 2$, $n \geq 0$ we have*

$$G_n(a_1,\cdots,a_r)$$

$$= d^n \sum_{l_1=0}^{d-1} \cdots \sum_{l_r=0}^{d-1} (-1)^{l_1+\cdots+l_r} G_n\left(\frac{a_1l_1+\cdots+a_rl_r}{d} | a_1,\cdots,a_r \right).$$

Moreover,

$$G_n(x|a_1,\cdots,a_r)$$

$$= d^n \sum_{l_1=0}^{d-1} \cdots \sum_{l_r=0}^{d-1} (-1)^{l_1+\cdots+l_r} G_n\left(\frac{x+a_1l_1+\cdots+a_rl_r}{d} \middle| a_1,\cdots,a_r \right).$$

For $a_1, a_2, \cdots, a_r \in \mathbb{C}_p \backslash \{0\}$, the Barnes-type multiple Bernoulli polynomials are known as follows:

$$
\begin{aligned}
&\int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} e^{(a_1x_1+\cdots+a_rx_r+x)t} d\mu_0(x_1)\cdots d\mu_0(x_r) \\
(17) \qquad &= \left(\frac{t}{e^{a_1t}-1}\right) \times \cdots \times \left(\frac{t}{e^{a_rt}-1}\right) e^{xt} \\
&= \sum_{n=0}^{\infty} B_n(x|a_1, \cdots, a_r)\frac{t^n}{n!}.
\end{aligned}
$$

When $x = 0$, $B_n(a_1, \cdots, a_r) = B_n(0|a_1, \cdots, a_r)$ is called the $n-$th Barnes-type Bernoulli number.

Thus, we have

(18)

$$
\begin{aligned}
&\int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} (a_1x_1 + \cdots + a_rx_r + b_1y_1 + \cdots + b_sy_s)^n d\mu_{-1}(y_1) \cdots d\mu_{-1}(y_s) \\
&\times d\mu_0(x_1) \cdots d\mu_0(x_r) \\
&= \int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} \frac{G_{n+s}(a_1x_1 + \cdots + a_rx_r|b_1, \cdots, b_s)}{\binom{n+s}{s}s!} d\mu_0(x_1)\cdots d\mu_0(x_r) \\
&= \frac{1}{\binom{n+s}{s}s!} \sum_{l=0}^{n} \binom{n}{l} G_{n+s-l}(b_1, \cdots, b_s) \int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} (a_1x_1 + \cdots + a_rx_r)^l \\
&\times d\mu_0(x_1) \cdots d\mu_0(x_r) \\
&= \frac{1}{\binom{n+s}{s}s!} \sum_{l=0}^{n} \binom{n}{l} G_{n+s-l}(b_1, \cdots, b_s) B_l(a_1, \cdots, a_r).
\end{aligned}
$$

Now, we define mixed-type Barnes' Genocchi and Bernoulli numbers as follows:

$$
\begin{aligned}
&GB_n(b_1, \cdots, b_s; a_1, \cdots, a_r) \\
(19) \qquad &= \int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} (a_1x_1 + \cdots + a_rx_r + b_1y_1 + \cdots + b_sy_s)^n \\
&\times d\mu_{-1}(y_1) \cdots d\mu_{-1}(y_s) d\mu_0(x_1) \cdots d\mu_0(x_r),
\end{aligned}
$$

where $a_1, \cdots, a_r, b_1, \cdots, b_s \neq 0$.

By (18) and (19), we get

$$
GB_n(b_1, \cdots, b_s; a_1, \cdots, a_r) = \frac{1}{\binom{n+s}{s}s!} \sum_{l=0}^{n} \binom{n}{l} G_{n+s-l}(b_1, \cdots, b_s) B_l(a_1, \cdots, a_r).
$$

3. Some identities arising from Barnes-type Genocchi basis

Let $\mathbb{P}_n = \{p(x) \in \mathbb{Q}[x] | \deg p(x) \leq n\}$ be the $(n+1)-$dimensional vector space over $\mathbb{Q}$. Probably, $\{1, x, \cdots, x^n\}$ is the most natural basis for $\mathbb{P}_n$. But $\{G_r(x|a_1, \cdots, a_r), G_{r+1}(x|a_1, \cdots, a_r), \cdots, G_{n+r}(x|a_1, \cdots, a_r)\}$ is also a good basis for the space $\mathbb{P}_n$ for our purpose of arithmetical applications of Barnes-type Genocchi polynomials.

Let us take $p(x) \in \mathbb{P}_n$. Then $p(x)$ can be expressed as a $\mathbb{Q}-$linear combination of $G_r(x|a_1, \cdots, a_r), G_{r+1}(x|a_1, \cdots, a_r), \cdots, G_{n+r}(x|a_1, \cdots, a_r)$ as follows:

$$p(x) = \sum_{k=r}^{n+r} b_k G_k(x|a_1, \cdots, a_r) \tag{20}$$

Now, let us consider two linear operators $\tilde{\Delta}_y$ and D by

$$\begin{aligned} \tilde{\Delta}_y p(x) &= p(x+y) + p(x), \\ Dp(x) &= \frac{d}{dx} p(x) \end{aligned} \tag{21}$$

and they satisfy $D\tilde{\Delta}_y = \tilde{\Delta}_y D$.

Then, by (20) and (21), we set

$$\tilde{\Delta}_{a_1} p(x) = \sum_{k=r}^{n+r} b_k (G_k(x + a_1|a_1, \cdots, a_r) + G_k(x|a_1, \cdots, a_r)). \tag{22}$$

From (1), we note that

$$\begin{aligned} &\sum_{n=0}^{\infty} \{G_n(x + a_1|a_1, \cdots, a_r) + G_n(x|a_1, \cdots, a_r)\} \frac{t^n}{n!} \\ &= \prod_{i=1}^{r} \frac{2t}{e^{a_i t} + 1} e^{(x+a_1)t} + \prod_{i=1}^{r} \frac{2t}{e^{a_i t} + 1} e^{xt}. \end{aligned} \tag{23}$$

By (2), (3) and (23), we get

$$\tilde{\Delta}_{a_1} G_n(x|a_1, \cdots, a_r) = 2n G_n(x|a_2, \cdots, a_r). \tag{24}$$

Thus, by (20), we get

$$\tilde{\Delta}_{a_1} p(x) = \sum_{l=r}^{n+r} 2l b_l G_l(x|a_2, \cdots, a_r),$$

and

$$\tilde{\Delta}_{a_2} \tilde{\Delta}_{a_1} p(x) = \sum_{l=r}^{n+r} 2^2 l(l-1) b_l G_{l-1}(x|a_3, \cdots, a_r).$$

Continuing this process, we have

$$\tag{25} \begin{aligned} \tilde{\Delta}_{a_r} \cdots \tilde{\Delta}_{a_1} p(x) &= \sum_{l=r}^{n+r} 2^{r-1} l(l-1)\cdots(l-r+2) b_l \tilde{\Delta}_{a_r} G_{l-r+1}(x|a_r) \\ &= \sum_{l=r}^{n+r} 2^r (l)_r b_l x^{l-r}, \end{aligned}$$

where we use the falling fractorial notation, $(l)_r = l(l-1)\cdots(l-r+1)$.

Let us apply the operator D^j on (25). Then

$$\tag{26} \begin{aligned} D^j \tilde{\Delta}_{a_r} \cdots \tilde{\Delta}_{a_1} p(x) &= \sum_{l=r}^{n+r} 2^r b_l (l)_r D^j x^{l-r} \\ &= \sum_{l=r}^{n+r} 2^r b_l (l)_{r+j} x^{l-r-j} \end{aligned}$$

Let us take $x=0$ on (26). Then we get

$$\tag{27} D^j \tilde{\Delta}_{a_r} \cdots \tilde{\Delta}_{a_1} p(0) = 2^r b_{r+j} (r+j)!.$$

Thus

$$\tag{28} b_{r+j} = \frac{1}{2^r (r+j)!} D^j \tilde{\Delta}_{a_r} \cdots \tilde{\Delta}_{a_1} p(0)$$

Therefore, by (20) and (28), we procure the following theorem.

Theorem 7. *For $n, r \in \mathbb{N}$ and $p(x) \in \mathbb{P}_n$, then we have*

$$p(x) = \sum_{j=0}^{n} \frac{1}{2^r (r+j)!} D^j \tilde{\Delta}_{a_r} \cdots \tilde{\Delta}_{a_1} p(0) G_{r+j}(x|a_1, \cdots, a_r).$$

Let us take $p(x) = x^n \in \mathbb{P}_n$. Then we have

$$\tilde{\Delta}_{a_1} x^n = (x+a_1)^n + x^n$$

and

$$\tilde{\Delta}_{a_2} \tilde{\Delta}_{a_1} x^n = (x+a_1+a_2)^n + (x+a_1)^n + (x+a_2)^n + x^n.$$

Continuing this process, we have

$$\tag{29} \tilde{\Delta}_{a_r} \cdots \tilde{\Delta}_{a_1} x^n = (x+a_1+\cdots+a_r)^n + (x+a_1+\cdots+a_{r-1})^n + \cdots + (x+a_r)^n + x^n.$$

Let us apply the operator D^j on (29). Then

$$\tag{30} \begin{aligned} D^j \tilde{\Delta}_{a_r} \cdots \tilde{\Delta}_{a_1} x^n = (n)_j \{ &(x+a_1+\cdots+a_r)^{n-j} + (x+a_1+\cdots+a_{r-1})^{n-j} \\ &+ \cdots + (x+a_r)^{n-j} + x^{n-j} \}. \end{aligned}$$

Let us take $x = 0$ on (30). Then we get

$$D^j \tilde{\Delta}_{a_r} \cdots \tilde{\Delta}_{a_1} p(0) = (n)_j \{(a_1 + \cdots + a_r)^{n-j} + (a_1 + \cdots + a_{r-1})^{n-j} + \cdots + (a_r)^{n-j}\}. \tag{31}$$

Thus, (31) and by Theorem 7, we have

$$x^n = \sum_{j=0}^{n} \frac{(n)_j}{2^r (r+j)!} \{(a_1 + \cdots + a_r)^{n-j} + (a_1 + \cdots + a_{r-1})^{n-j} + \cdots + (a_r)^{n-j}\} G_{r+j}(x|a_1, \cdots, a_r). \tag{32}$$

Therefore, by (32), we obtain the following theorem.

Theorem 8. *For $n, r \in \mathbb{N}$, then we have*

$$x^n = \sum_{j=0}^{n} \frac{(n)_j}{2^r (r+j)!} \{(a_1 + \cdots + a_r)^{n-j} + (a_1 + \cdots + a_{r-1})^{n-j} + \cdots + (a_r)^{n-j}\} G_{r+j}(x|a_1, \cdots, a_r).$$

If we take $a_1 = a_2 = \cdots = a_r = 1$, then we have the following known result representing $p(x) = x^n$ by using higher order Genocchi basis (see [5, 11]).

Corollary 9. *For $n, r \in \mathbb{N}$, then we have*

$$x^n = \sum_{j=0}^{n} \frac{\sum_{i=0}^{r} \binom{r}{i} D^j p(i)}{2^r j!} G_{r+j}^{(r)}(x)$$
$$\frac{1}{2^r} \sum_{j=0}^{n} \sum_{i=0}^{r} \frac{1}{(r+j)!} \binom{r}{i} \binom{n}{j} i^{n-j} G_{r+j}^{(r)}(x).$$

Let $p(x) = E_n(x|a_1, \cdots, a_r) \in \mathbb{P}_n$, then by (21),we have

$$\begin{aligned} \tilde{\Delta}_{a_1} E_n(x|a_1, \cdots, a_r) &= E_n(x + a_1|a_1, \cdots, a_r) + E_n(x|a_1, \cdots, a_r) \\ &= 2E_n(x|a_2, \cdots, a_r) \end{aligned}$$

and

$$\tilde{\Delta}_{a_2} \tilde{\Delta}_{a_1} E_n(x|a_1, \cdots, a_r) = 2^2 E_n(x|a_3, \cdots, a_r).$$

Continuing this process, we have

$$\tilde{\Delta}_{a_r} \cdots \tilde{\Delta}_{a_1} E_n(x|a_1, \cdots, a_r) = 2^r x^n. \tag{33}$$

Let us apply the operator D^j on (33). Then

$$D^j \tilde{\Delta}_{a_r} \cdots \tilde{\Delta}_{a_1} E_n(x|a_1, \cdots, a_r) = 2^r (n)_j x^{n-j}. \tag{34}$$

Let us take $x = 0$ on (34) and by (28), we get

(35) $$b_j = \frac{2^r j!}{2^r j!}.$$

Thus, from (35) and Theorem 7,

(36) $$E_n(x|a_1, \cdots, a_r) = \sum_{j=r}^{n+r} G_j(x|a_1, \cdots, a_r).$$

Therefore, by (36), we obtain the following theorem.

Theorem 10. *For $n, r \in \mathbb{N}$, then we have*

$$E_n(x|a_1, \cdots, a_r) = \sum_{j=r}^{n+r} G_j(x|a_1, \cdots, a_r).$$

Thus, by the Theorem 2 and Theorem 10, we get the following identity.

$$\sum_{j=r}^{n+r-1} G_j(x|a_1, \cdots, a_r) = 0.$$

Let us take $p(x) = B_n(x|a_1, \cdots, a_r)$, then by (21),

$$\begin{aligned}\tilde{\Delta}_{a_1} B_n(x|a_1, \cdots, a_r) &= B_n(x + a_1|a_1, \cdots, a_r) + B_n(x|a_1, \cdots, a_r) \\ &= B_n(x|a_2, \cdots, a_r) + 2B_n(x|a_1, \cdots, a_r)\end{aligned}$$

and

$$\begin{aligned}\tilde{\Delta}_{a_2}\tilde{\Delta}_{a_1} B_n(x|a_1, \cdots, a_r) &= B_n(x|a_3, \cdots, a_r) + 2B_n(x|a_2, \cdots, a_r) \\ &\quad + 2B_n(x|a_2, \cdots, a_r) + 2^2 B_n(x|a_1, \cdots, a_r).\end{aligned}$$

Continuing this process, we have

(37) $$\tilde{\Delta}_{a_r} \cdots \tilde{\Delta}_{a_1} p(x) = \sum_{k=0}^{r} 2^{r-k} \binom{r}{k} (n)_k B_{n-k}(x|a_{k+1}, a_{k+2}, \cdots, a_r).$$

Let us apply the operator D^j on (37). Then

(38)
$$D^j \tilde{\Delta}_{a_r} \cdots \tilde{\Delta}_{a_1} p(x) = \sum_{k=0}^{r} 2^{r-k} \binom{r}{k} (n)_k (n-k)_j B_{n-k-j}(x|a_{k+1}, \cdots, a_r).$$

Let us take $x = 0$ on (38). Then we get

(39)
$$D^j \tilde{\Delta}_{a_r} \cdots \tilde{\Delta}_{a_1} p(0) = \sum_{k=0}^{r} 2^{r-k} \binom{r}{k} (n)_k (n-k)_j B_{n-k-j}(a_{k+1}, \cdots, a_r).$$

Therefore, by (39) and Theorem 7, we obtain the following theorem.

Theorem 11. *For $n, r \in \mathbb{N}$, then we have*

$$B_n(x|a_1,\cdots,a_r)$$
$$=\sum_{j=r}^{n+r}\frac{1}{2^r j!}\sum_{k=0}^{r}2^{r-k}\binom{r}{k}(n)_k(n-k)_j B_{n-k-j}(a_{k+1},\cdots,a_r)G_j(x|a_1,\cdots,a_r)$$
$$=\sum_{j=r}^{n+r}\sum_{k=0}^{r}\frac{1}{2^k}\binom{r}{k}\frac{n!}{j!(n-k)!}B_{n-k-j}(a_{k+1},\cdots,a_r)G_j(x|a_1,\cdots,a_r).$$

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