Non-commuting graphs of AC-groups are End-regular*

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Abstract: A graph is called End-regular if its endomorphism monoid is regular. Which graphs are End-regular? It is an open question and difficult to obtain a general answer. In the present paper, we investigate the End-regularity of graphs which are obtained by adding or deleting vertices from End-regular graphs. As an application, we show that the non-commuting graphs of AC-groups are End-regular.

Keywords: Endomorphism monoid; Regularity; Non-commuting graph; Finite group
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1 Introduction

All graphs considered in this paper are finite undirected graphs without loops and multiple edges. For a graph Γ, we denote the vertex set and the edge set of Γ by $V(\Gamma)$ and $E(\Gamma)$, respectively. For two vertices x and y in Γ, by $x \sim y$ we mean that x and y are adjacent. The neighbour of x in Γ, denoted by $N_\Gamma(x)$ or simply $N(x)$ if no ambiguity caused, is the set of all vertices adjacent to x in Γ. Two vertices are called twin vertices if they share the same neighbour. Recall that a subgraph Δ of Γ is called an induced subgraph if it satisfies that $x \sim y$ in Δ if and only if $x \sim y$ in Γ for any $x, y \in V(\Delta)$. Let S be a subset of $V(\Gamma)$, we denote by $\Gamma - S$ the induced subgraph of Γ by deleting all vertices in S together with all edges

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that contain a deleted vertex. In particular, if $S = \{x\}$, then we simply write $\Gamma - x$.

Let Γ and Δ be two graphs. A (graph) homomorphism from Γ to Δ is mapping from $V(\Gamma)$ to $V(\Delta)$ which preserves adjacency. Moreover, if a graph homomorphism is a bijection and its inverse is also a graph homomorphism, then we say that it is an isomorphism. A homomorphism (resp. an isomorphism) from Γ to itself is called an endomorphism (resp. automorphism) of Γ. The set of all endomorphisms (resp. automorphisms) of Γ, denoted by End(Γ) (resp. Aut(Γ)) forms a monoid (resp. group) with composition as its multiplicity. A subgraph K of Γ is called a core of Γ if End(K) = Aut(K) and there is a homomorphism from Γ to K. A core of a graph is an induced subgraph and unique up to isomorphism. In particular, if Γ is a core of itself, then we say that Γ is unretractive. Between endomorphisms and automorphisms, there are some special endomorphisms. Here we lists two of them needed in this paper, and for other endomorphisms such as locally strong endomorphism, quasi-strong endomorphism, see [2]. Let $f \in$ End(Γ) and $x \in V(\Gamma)$, denote by $f^{-1}(f(x))$ the set of all pre-images of x in Γ. We say that f is half-strong if, for each $f(x) \sim f(y)$ in Γ, there exist some $u \in f^{-1}(f(x))$ and $v \in f^{-1}(f(y))$ with $u \sim v$, and that f is strong if $f(x) \sim f(y)$ in Γ implies $u \sim v$ for any $u \in f^{-1}(f(x))$ and any $v \in f^{-1}(f(y))$. The set of all strong endomorphisms of Γ, denoted by SEnd(Γ), is a submonoid of End(Γ), but the set of all half-strong endomorphisms of Γ is not in general. It has been shown in [2] that SEnd(Γ) is trivial if and only if Aut(Γ) is trivial.

Let $f \in$ End(Γ). The endomorphism image of Γ under f, denoted by I_f, is a subgraph of Γ whose vertex set is $f(V(\Gamma))$ and two vertices $f(x)$ and $f(y)$ are adjacent if and only if there exist $u \in f^{-1}(f(x))$ and $v \in f^{-1}(f(y))$ such that $u \sim v$. By ρ_f we denote the equivalence relation on $V(\Gamma)$ induced by f, that is, $(x, y) \in \rho_f$ if and only if $f(x) = f(y)$ for $x, y \in V(\Gamma)$. The factor graph of Γ under ρ_f, denoted by Γ/ρ_f, is a graph whose vertex set is the set of equivalence classes of ρ_f and two vertices $[x]$ and $[y]$ are adjacent if and only if there exist $u \in [x]$ and $v \in [y]$ such that $u \sim v$. It is shown in [5] that the graph homomorphism $\bar{f}$ induced by f which is defined by $\bar{f}([x]) = f(x)$ is an isomorphism from the factor graph Γ/ρ_f to the endomorphism image I_f.

Let S be a semigroup. An element $a \in S$ is called regular if there exists some $x \in S$ such that $axa = a$, here, x is called a pseudo-inverse of a in S. The semigroup S is called regular if all of its elements are regular. In this sense, a graph Γ is called End-regular if its endomorphism monoid End(Γ) is regular. On one hand, the structure of the endomorphism monoid of a graph has a close connection with the structure of the graph, especially the vertex chromatic number of the graph. On the other hand, regular semigroups play a central role in the structural regularity of semigroups. So it is meaningful to study the structural regularity of a graph's endomorphism monoid and to find various kinds of graphs who possess regular endomorphism monoids. Along the second line, some useful results are obtained. In [5], a necessary and sufficient condition for an endomorphism of a graph being regular was given by means of idempotents. In [10], the author classified connected bipartite End-regular graphs precisely. The complements of cycles C_n and of paths P_n are proved to be End-regular and end-orthodox in [6] and [4], respectively. Recall that a semigroup is called orthodox if it is regular and the set of all idempotent elements in it forms a semigroup under the same operation, and that a graph is called end-orthodox if its endomorphism monoid is orthodox. Meanwhile, some mathematicians paid attention to the regularity of some new graphs which are generated from old ones via binary graph operations. In [7], End-regular split graphs are considered. The join of two trees, of two connected bipartite graphs and of two unicyclic graphs with a regular endomorphism monoid are characterized explicitly in [3], [6] and [9], respectively.

The present paper is a continuation of the discussion of the End-regularity of graphs. It is shown in Proposition 2.5 below that End-regularity is retained when deleting a twin vertex together with edges adjacent to it from an End-regular graph. However, this statement is no longer true if we add a vertex as a twin vertex to an End-regular graph, it is not enough to require the original graph to be End-regular. Now, which graphs would retain their End-regularity after adding a vertex as a twin vertex? Part of the answer is provided in Theorem 2.7. As an application, we will prove in Theorem 3.3 that the non-commuting graphs of AC-groups are End-regular.

2 Endomorphism-regularity of graphs

Lemma 2.1. *[5] Let Γ be a graph, and let f be a graph endomorphism of Γ. Then f is regular if and only if there exist some idempotents τ, π of*

End(Γ) *such that* $I_f = I_\tau$ *and* $\rho_f = \rho_\pi$.

Lemma 2.2. *[7] Let* Γ *be a graph, and let* f *be a graph endomorphism of* Γ*. Then*
(1) f *is half-strong if and only if the endomorphic image* I_f *is an induced subgraph of* Γ.
(2) If f *is regular, then* f *is half-strong.*

Lemma 2.3. *Let* Γ *be a graph, and let* f *be a half-strong graph endomorphism of* Γ*. If* $f(I_f) = I_f$*, then* f *is regular.*

Proof. Since f is half-strong, it follows that I_f is an induced subgraph of Γ by Lemma 2.2. If $f(I_f) = I_f$, then the restriction $f|_{I_f}$ of f on I_f is an isomorphism. Let α be the inverse of $f|_{I_f}$. For each $x \in V(\Gamma)$, there exists a unique $y \in V(I_f)$ such that $f(x) = f(y)$, it follows that $\alpha f(x) = \alpha f(y) = y$ and $\alpha f \alpha f(x) = \alpha f(y) = y$, hence αf is an idempotent of End(Γ). It is clear that $I_f = I_{\alpha f}$. If $f(x) = f(y)$, then $\alpha f(x) = \alpha f(y)$. Conversely, if $\alpha f(x) = \alpha f(y)$, then $f(x) = f(y)$ for α is an isomorphism of I_f. Hence we have $\rho_f = \rho_{\alpha f}$. Therefore, f is regular by Lemma 2.1. □

Lemma 2.4. *[10] Let* Γ *be a connected bipartite graph. Then* Γ *is End-regular if and only if* Γ *is one of the following graphs:*
(1) Complete bipartite graphs;
(2) Trees of diameter 3;
(3) Cycles C_6 *and* C_8*;*
(4) Path of length 4.

Proposition 2.5. *Let* Γ *be a graph, and let* x, y *be two vertices of* Γ *such that* $N(x) = N(y)$*. If* Γ *is End-regular, then* $\Gamma - x$ *is End-regular.*

Proof. Let f be an arbitrary endomorphism of $\Gamma - x$. Define a mapping $\widetilde{f} : V(\Gamma) \to V(\Gamma)$ by $\widetilde{f}(u) = f(u)$ if $u \neq x$, and $\widetilde{f}(u) = f(y)$ if $u = x$, that is, $\widetilde{f}(x) = \widetilde{f}(y) = f(y)$. It is easy to check that $\widetilde{f} \in$ End(Γ). Since $\widetilde{f}$ is regular, there exists some $\widetilde{\alpha} \in$ End(Γ) such that $\widetilde{f}\widetilde{\alpha}\widetilde{f} = \widetilde{f}$. If $\widetilde{\alpha}(u) \neq x$ for any $u \in V(\Gamma - x)$, then the restriction $\widetilde{\alpha}|_{\Gamma - x}$ of $\widetilde{\alpha}$ on $\Gamma - x$ is an endomorphism of $\Gamma - x$. Hence $f\widetilde{\alpha}|_{\Gamma - x} f = f$.

Now assume that there exists some $z \in V(\Gamma - x)$ such that $\widetilde{\alpha}(z) = x$. Define a mapping $\widetilde{\beta} : V(\Gamma) \to V(\Gamma)$ by $\widetilde{\beta}(u) = \widetilde{\alpha}(u)$ if $\widetilde{\alpha}(u) \neq x$ and $\widetilde{\beta}(u) = y$ if $\widetilde{\alpha}(u) = x$. For each $u \sim v$, we have $\widetilde{\alpha}(u) \sim \widetilde{\alpha}(v)$. If

$\widetilde{\alpha}(u) \neq x$ and $\widetilde{\alpha}(v) \neq x$, then $\widetilde{\beta}(u) = \widetilde{\alpha}(u) \sim \widetilde{\alpha}(v) = \widetilde{\beta}(v)$. If $\widetilde{\alpha}(u) = x$, then $\widetilde{\alpha}(v) \neq x$ and $x \sim \widetilde{\alpha}(v)$, so $y \sim \widetilde{\alpha}(v)$ for $N(x) = N(y)$, it follows that $\widetilde{\beta}(u) \sim \widetilde{\beta}(v)$. Similarly, we have $\widetilde{\beta}(u) \sim \widetilde{\beta}(v)$ if $\widetilde{\alpha}(v) = x$. Hence $\widetilde{\beta} \in \mathrm{End}(\Gamma)$.

Next, we show that $\widetilde{f}\widetilde{\beta}\widetilde{f} = \widetilde{f}$. Indeed, we may distinguish the following four cases. If $u \neq x$ and $\widetilde{\alpha}\widetilde{f}(u) \neq x$, then $\widetilde{f}\widetilde{\beta}\widetilde{f}(u) = \widetilde{f}\widetilde{\alpha}\widetilde{f}(u) = \widetilde{f}(u)$. If $u \neq x$ and $\widetilde{\alpha}\widetilde{f}(u) = x$, then $\widetilde{f}\widetilde{\beta}\widetilde{f}(u) = \widetilde{f}(y) = \widetilde{f}(x) = \widetilde{f}\widetilde{\alpha}\widetilde{f}(u) = \widetilde{f}(u)$. If $u = x$ and $\widetilde{\alpha}\widetilde{f}(u) \neq x$, then $\widetilde{f}\widetilde{\beta}\widetilde{f}(u) = \widetilde{f}\widetilde{\beta}\widetilde{f}(y) = \widetilde{f}\widetilde{\alpha}\widetilde{f}(y) = \widetilde{f}(y) = \widetilde{f}(u)$. If $u = x$ and $\widetilde{\alpha}\widetilde{f}(u) = x$, then $\widetilde{f}\widetilde{\beta}\widetilde{f}(u) = \widetilde{f}(y) = \widetilde{f}(u)$. Hence, $\widetilde{\beta}$ is a pseudo-inverse of $\widetilde{f}$. Furthermore, note that $\widetilde{\beta}(u) \neq x$ for any $u \in \Gamma - x$, then the restriction $\widetilde{\beta}|_{\Gamma-x}$ of $\widetilde{\beta}$ on $\Gamma - x$ is an endomorphism of $\Gamma - x$. Hence $f\widetilde{\beta}|_{\Gamma-x}f = f$. □

The proposition above shows that End-regularity will be retained when deleting a twin vertex from an End-regular graph. However, we should note that the inverse of Proposition 2.5 is not true. That is, let Γ be a graph, and let x, y be two vertices of Γ such that $N(x) = N(y)$. In general, we can not deduce Γ is End-regular if $\Gamma - x$ is End-regular. As an example, we consider the path P_5 and the graph Γ by adding a pendent vertex x to P_5 such that one of pendent vertex in P_5 and x are twin points. Then $P_5 = \Gamma - x$. It is clear that both Γ and P_5 are bipartite graphs. By Lemma 2.4, we known that P_5 is End-regular but Γ is not. Neverthless, if we strengthen the condition, say, $\Gamma - x$ is a core of Γ rather than End-regular, then Γ is End-regular. In fact, we have a more general result.

Lemma 2.6. *Let Γ be a graph, and let f be a graph endomorphism of Γ. If f is strong, then f is regular.*

Proof. Let $[x_1], [x_2], \cdots, [x_m]$ be all the equivalence classes of ρ_f, and let $x_1, x_2, \cdots, x_m$ be representatives of $[x_1], [x_2], \cdots, [x_m]$, respectively. For each $x \in V(\Gamma)$, there exists a unique x_i such that $x \in [x_i]$. Define a mapping $\pi : V(\Gamma) \to V(\Gamma)$ by assigning to each vertex in Γ the representative of its equivalence class. Since f is strong, vertices in the same equivalence class have the same neighbour, so it is clear that π is an endomorphism of Γ. Moreover, from the definition we can see that π is an idempotent of $\mathrm{End}(\Gamma)$ with $\rho_f = \rho_\pi$.

Next, in order to prove f being regular by using Lemma 2.1, we will construct an idempotent τ of $\mathrm{End}(\Gamma)$ such that $I_f = I_\tau$. To do this, we need to re-choose special representatives for some equivalence classes of ρ_f as follows. For each equivalence class $[x_i]$, if it contains at least one image of f, then, in stead of x_i, we choose one of images in it arbitrarily as its representative, otherwise, we keep x_i as its representative. Thus, without lose of generality, we can assume that $f(u_1), f(u_2), \cdots, f(u_s), x_{s+1}, \cdots, x_m$ are representatives of $[x_1], [x_2], \cdots, [x_m]$, respectively, and that $f(u_{s+1}), f(u_{s+2}), \cdots, f(u_m)$ are other images which are not chosen as representatives. If $s = m$, then every equivalence class contains a unique image of f and $V(I_f)$ is just the set of all representatives for Γ/ρ_f is isomorphic to I_f. In this case, we define $\tau : V(\Gamma) \to V(\Gamma)$ by assigning to each vertex in Γ the representative of its equivalence class. Like π, we can also deduce that τ is an idempotent of End(Γ) with $I_f = I_\pi$. Now we assume that $s < m$. Since f is strong, the induced subgraph, denoted by Δ, whose vertex set consists of all representatives as mentioned above is isomorphic to Γ/ρ_f, so Δ and I_f are isomorphic with $f(u_1), f(u_2), \cdots, f(u_s)$ as their common vertices. For each $f(u_i)$, $s+1 \leq i \leq m$, there exists a unique $f(u_j)$ with $1 \leq j \leq s$ such that $f(u_i) \in [f(u_j)]$, thus we have $N(f(u_i)) = N(f(u_j))$. Hence, for each x_i, $s+1 \leq i \leq m$, there exists some $f(u_k)$ with $1 \leq k \leq s$ such that $N(x_i) = N(f(u_k))$. In this case, we define $\tau : V(\Gamma) \to V(\Gamma)$ as follows. For any vertex x in Γ, if $x \in [x_i]$ with $s+1 \leq i \leq m$, then $\tau(x) = f(u_k)$, where $N(f(u_k)) = N(x_i)$; if $x \in [f(u_i)]$ with $1 \leq i \leq s$ and x is not an image, then $\tau(x) = f(u_i)$; if $x = f(u_i)$ with $1 \leq i \leq m$, then $\tau(x) = x$. Finally, it is a routine to check that τ is an idempotent of End(Γ) such that $I_f = I_\tau$. This complete the proof. □

Theorem 2.7. *Let Γ be a graph, and let K be a core of Γ. If for each $x \in V(\Gamma) \setminus V(K)$ there exists some $y \in V(K)$ such that $N(x) = N(y)$, then Γ is end-regular.*

Proof. Let f be an arbitrary endomorphism of Γ, we need to show that f is strong. Let $f(x)$ and $f(y)$ are adjacent in Γ and let a, b be any pre-image of $f(x), f(y)$, respectively. Assume α is a graph homomorphism from Γ to K. Then αf is a surjective homomorphism from Γ to K satisfying $\alpha f(a)$ and $\alpha f(b)$ are adjacent. Since K is an induced subgraph of Γ, by Lemma 2.2, αf is half-strong. Then, it follows that there exist $u, v \in V(\Gamma)$ such that $\alpha f(u) = \alpha f(a)$, $\alpha f(v) = \alpha f(b)$ and $u \sim v$. Next, we show that $N(u) = N(a)$ and $N(v) = N(b)$. Without loss of generality, suppose, to the contrary, that $N(u) \neq N(a)$. Then there exists some core K' of Γ such that $u, a \in V(K')$. Let β be a graph isomorphism from K to K', then $\beta\alpha f(u) = \beta\alpha f(a)$. Note that $\beta\alpha f|_{K'}$ is an automorphism of K', so $u = a$. This is a contradiction. Recall that $u \sim v$, we have $a \sim b$, hence f is strong. It follows from Lemma 2.6 that f is regular. Therefore, Γ is end-regular. □

Corollary 2.8. *Let Γ be a graph, and let x, y be two vertices of Γ such that $N(x) = N(y)$. If $\Gamma - x$ is a core, then Γ is end-regular.*

3 Non-commuting graphs of AC-groups

Definition 3.1. *[1] Let G be a non-abelian group, and let $Z(G)$ be the center of G. The non-commuting graph Γ_G associated with G is a graph whose vertex set is $G\backslash Z(G)$ and two distinct vertices x, y are adjacent if $xy \neq yx$.*

The non-commuting graph Γ_G associated with G was first considered by Paul Erdös in 1975. Recently, researches on Γ_G mainly focused on the effect of graph theoretical properties of Γ_G on the group theoretical properties of G, see [1] for example. So far, no result can be found on the study of end-regularity of non-commuting graphs. In this section, as an application of Theorem 2.7, we show that the non-commuting graph associated with an AC-group is end-regular. Recall that a group is called an AC-group if the centralizer of every non-central element is abelian. For example, dihedral groups are AC-groups.

Lemma 3.2. *Let G be a finite non-abelian group, and let Γ_G be the non-commuting graph associated with G. Then G is an AC-group if and only if, for any vertices x, y in Γ_G, either $x \sim y$ or $N(x) = N(y)$.*

Proof. The result follows from Proposition 3.1 of [8]. □

Theorem 3.3. *If G is an AC-group, then the non-commuting graph Γ_G associated with G is end-regular.*

Proof. We define a relation R on $\Gamma_G \times \Gamma_G$ by xRy if and only if $N(x) = N(y)$. It is clear that R is an equivalence relation on Γ_G. Take one representative from each equivalence class, the set consisting of all these representatives is a subset of $V(\Gamma_G)$. Let X be the induced subgraph with this set as its vertex set. Note that for any two distinct vertices x, y in X, we have $N(x) \neq N(y)$ in Γ_G for x and y are chosen from distinct equivalence classes, hence X is a clique by Corollary 3.2. Furthermore, it is easy to see that, if each vertex in Γ_G is assigned to the representative of its own equivalence class, then we obtain a graph homomorphism from Γ_G to X, hence X is a core of Γ_G. Now, for each vertex x in Γ_G, there exists a unique vertex y in X such that $N(x) = N(y)$, hence Γ_G is end-regular by Theorem 2.7. □

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