Continued fractions and the derangement polynomials of types A and B *

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Abstract

In this paper, we give the continued fraction expansions of the ordinary generating functions of the derangement polynomials of types A and B in a unified manner. Our proof is based on their exponential generating functions and the theory of exponential Riordan arrays.

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1 Introduction

Let S_n denote the symmetric group on the set $[n] = \{1, 2, 3, \ldots, n\}$. If $\sigma \in S_n$, then we write $\sigma = \sigma_1\sigma_2\cdots\sigma_n$, to mean that $\sigma(i) = \sigma_i$ for $i = 1, 2, \ldots, n$. The permutation $\sigma \in S_n$ is called a *derangement* if $\sigma_i \neq i$ for $i = 1, 2, \ldots, n$ (i.e., σ has no fixed points). We denote the set of all derangements of S_n by D_n. An element $i \in [n]$ is called an *excedance* of the permutation $\sigma \in S_n$ if $\sigma_i > i$. Denote $e(\sigma)$ by the number of excedances of σ. Brenti [3] defined the derangement polynomials of type A by

$$d_n(q) = \sum_{\sigma \in D_n} q^{e(\sigma)},$$

for $n \geqslant 1$ and $d_0(q) = 1$. Brenti [3] obtained that $d_n(q)$ is symmetric and unimodal. And he further proposed the conjecture that the derangement

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polynomials of type A $d_n(q)$ has only real zeros for $n \geqslant 1$. This conjecture has been settled by Zhang [14], Canfield as mentioned in [4] and Liu and Wang [11] independently, based on the following recurrence relation

$$d_n(q) = (n-1)qd_{n-1}(q) + q(1-q)d'_{n-1}(q) + (n-1)qd_{n-2}(q),$$

for $n \geqslant 2$ (see [14] for instance). In 2011, Chen and Xia [7] presented that for $n \geqslant 6$, the derangement polynomials of type A $d_n(q)$ are strictly ratio monotone, which implies the spiral property and the log-concavity, except for the last term when n is even.

Following Björner and Brenti [2], we regard B_n as the group of all signed permutations on the set $[n]$, i.e., the group of all bijections σ of the set $[\pm n]$ in itself such that $\sigma(-i) = -\sigma(i)$, for all $i \in [\pm n]$, with composition as group operation. If $\sigma \in B_n$, then we also write $\sigma = \sigma_1\sigma_2\cdots\sigma_n$, to mean that $\sigma(i) = \sigma_i$ for $i = 1, 2, \ldots, n$. A derangement of type B on $[n]$ is a signed permutation $\sigma = \sigma_1\sigma_2\cdots\sigma_n$ such that $\sigma_i \neq i$, for all $i \in [n]$. Let D_n^B denote the set of all derangements in B_n. Brenti [5] introduced the definition of excedances of type B. Given $\sigma \in B_n$ and $i \in [n]$, we say that i is a type B excedance of σ if $\sigma_i = -i$ or $\sigma_{|\sigma_i|} > \sigma_i$. We denote $e_B(\sigma)$ by the number of type B excedances of σ. Following Brenti's definition of excedances of type B, Chen, Tang and Zhao [6] gave a type B analogue of the derangement polynomials defined by

$$d_n^B(q) = \sum_{\sigma \in D_n^B} q^{e_B(\sigma)} = \sum_{k=0}^{n} d_{n,k}q^k,$$

for $n \geqslant 1$, where $d_{n,k}$ is the number of derangements in D_n^B with exactly k excedances of type B. For $n = 0$, set $d_0^B(q) = 1$. Chen, Tang and Zhao [6] derived some basic properties of the derangement polynomials of type B, such as the generating function formular, the Sturm sequence property, the asymptotic normal distribution, and the spiral property. In this paper, we get the continued fraction expressions of the ordinary generating functions of the derangement polynomials of types A and B in a unified manner.

This paper is organized as follows. In section 2, using the theory of exponential Riordan arrays and orthogonal polynomials, we give the continued fraction of the ordinary generating function of the polynomial sequence, whose exponential generating function generalizes the exponential generating function of the derangement polynomials of types A and B. As applications, we obtain the continued fraction expressions of the derangement polynomials of types A and B in a unified manner in section 3. Finally, in the Appendix, we can obtain a quick introduction to the exponential Riordan arrays and the orthogonal polynomials used in this paper.

where $\bar{f}(x)$ is the compositional inverse of $f(x)$.

By the direct calculation, we have

$$f'(x) = \frac{(1-q)^2 e^{d(1-q)x}}{(1-qe^{d(1-q)x})^2}.$$

Note that the compositional inverse of $f(x)$ satisfies

$$f(\bar{f}(x)) = \frac{e^{d(1-q)\bar{f}} - 1}{d(1-qe^{d(1-q)\bar{f}})} = x.$$

Then we have

$$\bar{f}(x) = \frac{1}{d(1-q)} \ln\left(\frac{1+dx}{1+dqx}\right).$$

Hence

$$r(x) = f'(\bar{f}(x)) = (1+dx)(1+dqx) = 1 + d(1+q)x + d^2qx^2.$$

On the other hand, set

$$G(x) = \left(\frac{(1-q)e^{a(1-q)x}}{1-qe^{d(1-q)x}}\right)^b.$$

Then we have

$$G'(x) = b\left(\frac{(1-q)e^{a(1-q)x}}{1-qe^{d(1-q)x}}\right)^{b-1} \frac{(1-q)^2 e^{a(1-q)x}(a+(d-a)qe^{d(1-q)x})}{(1-qe^{d(1-q)x})^2}.$$

So

$$\frac{G'(x)}{G(x)} = \frac{b(1-q)(a+(d-a)qe^{d(1-q)x})}{1-qe^{d(1-q)x}}.$$

$$\begin{aligned}\frac{G'(\bar{f}(x))}{G(\bar{f}(x))} &= \frac{b(1-q)(a+(d-a)qe^{d(1-q)\bar{f}})}{1-qe^{d(1-q)\bar{f}}} \\ &= ab(1+dqx) + b(d-a)q(1+dx) \\ &= b(a+(d-a)q) + bd^2qx.\end{aligned}$$

Note that

$$G(x) = e^{abx}g(x).$$

Hence we can get that

$$G'(x) = abe^{abx}g(x) + e^{abx}g'(x).$$

2 Main results

In this section, we give the continued fraction expression of the ordinary generating function of the polynomial sequence $\{T_n(q)\}_{n\geqslant 0}$, whose exponential generating function generalizes the exponential generating functions of the derangement polynomials of types A and B.

Theorem 2.1. *Suppose that the exponential generating function of the polynomial sequence $\{T_n(q)\}_{n\geqslant 0}$ has the following simple expression*

$$g(x)=\sum_{n\geqslant 0}\frac{T_n(q)}{n!}x^n=\left(\frac{(1-q)e^{-aqx}}{1-qe^{d(1-q)x}}\right)^b, \tag{2.1}$$

for $a,b,d\in \mathrm{R}$. Then the ordinary generating function of $\{T_n(q)\}_{n\geqslant 0}$ can be given by the continued fraction

$$h(x)=\sum_{n\geqslant 0}T_n(q)x^n=\cfrac{1}{1-s_0(q)x-\cfrac{t_1(q)x^2}{1-s_1(q)x-\cfrac{t_2(q)x^2}{1-s_2(q)x-\cdots}}}, \tag{2.2}$$

where

$$s_i(q)=di+(di+bd-ab)q \quad \text{and} \quad t_{i+1}(q)=d^2(i+1)(i+b)q \tag{2.3}$$

for $i\geqslant 0$.

In order to prove this theorem, we need three lemmas. Using the theory of the exponential Riordan arrays, the first lemma presents that the production matrix P of the exponential Riordan array $L=[g(x),f(x)]$, where $g(x)$ is the exponential generating function of $\{T_n(q)\}_{n\geqslant 0}$ given by (2.1), is tri-diagonal.

Lemma 2.1. *The production matrix P of the exponential Riordan array*

$$L=[g(x),f(x)]=\left[\left(\frac{(1-q)e^{-aqx}}{1-qe^{d(1-q)x}}\right)^b,\frac{e^{d(1-q)x}-1}{d[1-qe^{d(1-q)x}]}\right],$$

for $a,b,d\in \mathrm{R}$, is tri-diagonal.

Proof. In order to get the production matrix P, it suffices to calculate $r(x)$ and $c(x)$. Recall that

$$r(x)=f'(\bar{f}(x)),\quad c(x)=\frac{g'(\bar{f}(x))}{g(\bar{f}(x))},$$

Thus

$$\frac{G'(x)}{G(x)} = ab + \frac{g'(x)}{g(x)}.$$

Now we can get that

$$c(x) \;=\; \frac{g'(\bar{f}(x))}{g(\bar{f}(x))} = \frac{G'(\bar{f}(x))}{G(\bar{f}(x))} - ab = b(d-a)q + bd^2qx.$$

Thus the production matrix P of L is tri-diagonal, where

$$P = \begin{pmatrix} s_0(q) & 1 & 0 & 0 & 0 & 0\cdots \\ t_1(q) & s_1(q) & 1 & 0 & 0 & 0\cdots \\ 0 & t_2(q) & s_2(q) & 1 & 0 & 0\cdots \\ 0 & 0 & t_3(q) & s_3(q) & 1 & 0\cdots \\ 0 & 0 & 0 & t_4(q) & s_4(q) & 1\cdots \\ 0 & 0 & 0 & 0 & t_5(q) & s_5(q)\cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots\ddots \end{pmatrix}, \tag{2.4}$$

with $s_i(q)$ and $t_{i+1}(q)$ given by (2.3).

□

The second lemma constructs a family of orthogonal polynomials related to the production matrix P of the exponential Riordan array $L = [g(x), f(x)]$.

Lemma 2.2. *Suppose that the production matrix P of an exponential Riordan array L is tri-diagonal as above (2.4). Then we can construct a family of orthogonal polynomials $Q_n(x)$ defined by*

$$Q_n(x) = (x - s_{n-1}(q))Q_{n-1}(x) - t_{n-1}(q)Q_{n-2}(x), \tag{2.5}$$

with $Q_0(x) = 1$ and $Q_1(x) = x - s_0(q)$, where coefficients $s_{n-1}(q)$ and $t_{n-1}(q)$ are given by the expression (2.3) for $n \geqslant 1$.

Proof. In order to construct the family of orthogonal polynomials $Q_n(x)$, it suffices to get the coefficient matrix A of $Q_n(x)$ such that

$$\begin{pmatrix} Q_0(x) \\ Q_1(x) \\ Q_2(x) \\ Q_3(x) \\ \vdots \end{pmatrix} = A \begin{pmatrix} 1 \\ x \\ x^2 \\ x^3 \\ \vdots \end{pmatrix}. \tag{2.6}$$

So by the condition and the Favard's Theorem 4.1 in Appendix, we need to get that the orthogonal polynomials $Q_n(x)$ satisfies the following

$$P\begin{pmatrix} Q_0(x) \\ Q_1(x) \\ Q_2(x) \\ Q_3(x) \\ \vdots \end{pmatrix} = \begin{pmatrix} s_0(q) & 1 & 0 & 0 & 0 & 0\cdots \\ t_1(q) & s_1(q) & 1 & 0 & 0 & 0\cdots \\ 0 & t_2(q) & s_2(q) & 1 & 0 & 0\cdots \\ 0 & 0 & t_3(q) & s_3(q) & 1 & 0\cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots\ddots \end{pmatrix} \begin{pmatrix} Q_0(x) \\ Q_1(x) \\ Q_2(x) \\ Q_3(x) \\ \vdots \end{pmatrix}$$
$$= \begin{pmatrix} xQ_0(x) \\ xQ_1(x) \\ xQ_2(x) \\ xQ_3(x) \\ \vdots \end{pmatrix}.$$

After arrangement, we want to prove that the coefficient matrix A satisfies

$$PA\begin{pmatrix} 1 \\ x \\ x^2 \\ x^3 \\ \vdots \end{pmatrix} = A\begin{pmatrix} x \\ x^2 \\ x^3 \\ x^4 \\ \vdots \end{pmatrix} = A\bar{I}\begin{pmatrix} 1 \\ x \\ x^2 \\ x^3 \\ \vdots \end{pmatrix}, \tag{2.7}$$

where $\bar{I} = (\delta_{i+1,j})_{i,j\geqslant 0}$.

Since the polynomials sequence $\{x^k\}_{k\geqslant 0}$ is linearly independence. So the coefficient matrices of the first and last polynomials in (2.7) are equal, i.e., $PA = A\bar{I}$. Since

$$P = L^{-1}\bar{L}, \bar{I} = \bar{L}L^{-1}.$$

So we need to prove that the coefficient matrix A will satisfy

$$L^{-1}\bar{L}A = A\bar{L}L^{-1}.$$

Obviously we can get that $A = L^{-1}$ is a coefficient matrix of the orthogonal polynomials $Q_n(x)$ satisfying (2.5). The proof of the lemma is complete. □

Remark 2.1. Lemma 2.2 has been proved by Barry [1]. However our proof is more natural and based on the algebraic method.

The last lemma, obtained by Barry [1], gave the connection between the production matrix and the moments sequence of orthogonal polynomials.

Lemma 2.3 ([1]). *Let L, $T_n(q)$ and $Q_n(x)$ be as above. Then we have $\{T_n(q)\}_{n\geqslant 0}$ is the moments sequence of the associated family of orthogonal polynomials $Q_n(x)$.*

Now we can obtain that the ordinary generating function of $\{T_n(q)\}_{n\geqslant 0}$ is given by the continued fraction (2.2) from Theorem 4.2, which proves Theorem 2.1.

3 The derangement polynomials of types A and B

In this section, we can give the continued fraction expressions of the derangement polynomials of types A and B in a unified manner.

For the derangement polynomials of type A, it is known that the exponential generating function is

$$\sum_{n\geqslant 0} d_n(q)\frac{x^n}{n!} = \frac{(1-q)e^{-qx}}{1-qe^{x(1-q)}}, \tag{3.1}$$

(see Brenti [3] and Chow and Gessel [8]). So when $a = b = d = 1$, we have $T_n(q) = d_n(q)$. From Theorem 2.1, the ordinary generating function of $d_n(q)$ can be given by the continued fraction

$$\sum_{n\geqslant 0} d_n(q)x^n = \cfrac{1}{1-\cfrac{qx^2}{1-(1+q)x-\cfrac{4qx^2}{1-2(1+q)x-\cfrac{9qx^2}{1-3(1+q)x-\cdots}}}},$$

with $s_i(q) = i(1+q)$ and $t_{i+1}(q) = (i+1)^2q$ for $i \geqslant 0$.

For the derangement polynomials of type B, Chen, Tang and Zhao [6] obtained that the exponential generating function has the following expression

$$\sum_{n\geqslant 0} d_n^B(q)\frac{x^n}{n!} = \frac{(1-q)e^{-qx}}{1-qe^{2x(1-q)}}. \tag{3.2}$$

Hence when $a = b = 1, d = 2$, we have $T_n(q) = d_n^B(q)$. Now from Theorem 2.1, we get the ordinary generating function of the derangement polynomials of type B is given by

$$\sum_{n\geqslant 0} d_n^B(q)x^n = \cfrac{1}{1-(1+q)x-\cfrac{4qx^2}{1-(2+3q)x-\cfrac{16qx^2}{1-(4+5q)x-\cdots}}}.$$

Here $s_i(q) = 2i + (2i+1)q$ and $t_{i+1}(q) = 4(i+1)^2q$ for $i \geqslant 0$.

4 Appendix

The *exponential Riordan array* [1, 9, 10] denoted by $L = [g(x), f(x)]$, is an infinite lower triangular matrix whose exponential generating function of the kth column is $g(x)(xf(x))^k/k!$ for $k = 0, 1, 2, \ldots$, where $g(0) \neq 0 \neq f(0)$. An exponential Riordan array $L = (l_{i,j})_{i,j\geqslant 0}$ can also be characterized by two sequences $\{c_n\}_{n\geqslant 0}$ and $\{r_n\}_{n\geqslant 0}$ such that

$$l_{0,0} = 1, \quad l_{i+1,0} = \sum_{j\geqslant 0} j! c_j l_{i,j}, \quad l_{i+1,j} = \frac{1}{j!} \sum_{k\geqslant j-1} k!(c_{k-j} + j r_{k-j+1}) l_{i,j},$$

for $i, j \geqslant 0$ (see [9] for instance). Call $\{c_n\}_{n\geqslant 0}$ and $\{r_n\}_{n\geqslant 0}$ the $c-$ and $r-$ sequences of L respectively. Associated to each exponential Riordan array $L = [g(x), f(x)]$, there is a matrix $P = (p_{i,j})_{i,j\geqslant 0}$, called the *production matrix*, whose bivariate generating function is given by

$$e^{xy}[c(x) + r(x)y],$$

where

$$c(x) = \frac{g'(\bar{f}(x))}{g(\bar{f}(x))} := \sum_{n\geqslant 0} c_n x^n, r(x) = f'(\bar{f}(x)) := \sum_{n\geqslant 0} r_n x^n.$$

Deutsch *et al.* [9] obtained that elements of the production matrix $P = (p_{i,j})_{i,j\geqslant 0}$ is given by

$$p_{i,j} = \frac{i!}{j!}(c_{i-j} + j r_{i-j+1}).$$

Assume that $c_{-1} = 0$. Note that

$$P = L^{-1}\bar{L}, \bar{I} = \bar{L}L^{-1},$$

where $\bar{L}$ is obtained from L with the first row removed and $\bar{I} = (\delta_{i+1,j})_{i,j\geqslant 0}$, where $\delta_{i,j}$ is the usual Kronecker symbol.

The following well-known results establish the relationship among the orthogonal polynomials, three-term recurrences, recurrence coefficients and the continued fraction of the generating function of the moments sequence. The first result is the well-known "Favard's Theorem".

Theorem 4.1 ([12, Théoréme 9 on p. I-4], or [13, Theorem 50.1]). *Let $\{p_n(x)\}_{n\geqslant 0}$ be a sequence of monic polynomials with degree $n = 0, 1, 2, \ldots$ respectively. Then the sequence $\{p_n(x)\}_{n\geqslant 0}$ is (formally) orthogonal if and only if there exist sequences $\{\alpha_n\}_{n\geqslant 0}$ and $\{\beta_n\}_{n\geqslant 1}$ with $\beta_n \neq 0$ such that the three-term recurrence*

$$p_{n+1}(x) = (x - \alpha_n)p_n(x) - \beta_n p_{n-1}(x)$$

holds, for $n \geqslant 1$, with initial conditions $p_0(x) = 1$ and $p_1(x) = x - \alpha_0$.

Theorem 4.2 ([12, Proposition 1 (7) on p. V-5], or [13, Theorem 51.1]). *Let $\{p_n(x)\}_{n\geqslant 0}$ be a sequence of monic polynomials, which is orthogonal with respect to some linear functional $\mathcal{L}$. For $n \geqslant 1$, let*

$$p_{n+1}(x) = (x - \alpha_n)p_n(x) - \beta_n p_{n-1}(x),$$

be the corresponding three-term recurrence which is guaranted by Favard's theorem. Then the generating function

$$h(x) = \sum_{k=0}^{\infty} \mu_k x^k$$

for the moments $\mu_k = \mathcal{L}(x^k)$ satisfies

$$h(x) = \cfrac{\mu_0}{1 - \alpha_0 x - \cfrac{\beta_1 x^2}{1 - \alpha_1 x - \cfrac{\beta_2 x^2}{1 - \alpha_2 x - \cfrac{\beta_3 x^2}{1 - \alpha_3 x - \cdots}}}}.$$

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