RELATIONSHIP AMONG BINOMIAL COEFFICIENTS, BERNOULLI NUMBERS AND STIRLING NUMBERS

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ABSTRACT. We give relationships among the binomial coefficients, the Bernoulli numbers and the Stirling numbers. These relations are derived from the translation formulae in the linear discrete systems in Shin-Naito [8].

1. INTRODUCTION

In [5, 8] the solutions of periodic inhomogeneous linear differential equations have been represented as the form of the sum of exponential-like functions and periodic functions. Its proof is related to the translation formulae named in [8] (refer to Lemma 3.1). In particular, the translation formulae were obtained by comparing two representations of solutions corresponding to the matrices B and A in the linear discrete systems

$$x_{n+1} = Bx_n + b, B = e^{\tau A},\ \tau > 0. \tag{1.1}$$

In the present paper, as an application of the translation formulae, we will give relationships among the binomial coefficients, the Bernoulli numbers and the Stirling numbers (see [1], [4], [7]).

2. MAIN THEOREM

In order to state our results, we first introduce briefly some notations used in linear algebra and basic facts on the binomial theorem. For a complex $p \times p$ matrix H we denote by $\sigma(H)$ the set of all eigenvalues of H and by $G_H(\eta) = \mathrm{N}((H - \eta E)^{h_H(\eta)})$ the generalized eigenspace corresponding to $\eta \in \sigma(H)$, where E is the unit $p \times p$ matrix and $h_H(\eta)$ the geometric multiplicity of $\eta \in \sigma(H)$. $Q_\eta(H) : \mathbb{C}^p \to G_H(\eta)$ stands for the projection corresponding to the direct sum decomposition

$$\mathbb{C}^p = \bigoplus_{\eta\in\sigma(H)} G_H(\eta).$$

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Let $\mathbb{N} = \{1, 2, 3, \cdots\}$, $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$ and $\mathbb{Z} = \{0, \pm 1, \pm 2, \pm 3, \cdots\}$. If $x \in \mathbb{R}$ and $k \in \mathbb{N}_0$, then we define the well-known factorial function $(x)_k$ as

$$(x)_k = \begin{cases} 1, & (k = 0) \\ x(x-1)(x-2)\cdots(x-k+1) & (k \in \mathbb{N}). \end{cases}$$

In particular, if $x = n$ is a positive integer, then

$$\frac{(n)_k}{k!} = \binom{n}{k} := \frac{n!}{k!(n-k)!}, \quad (n)_k = 0 \ (k > n).$$

The Stirling numbers $\left[\begin{matrix} j \\ k \end{matrix} \right]$ of the first kind and the Stirling numbers $\left\{ \begin{matrix} k \\ j \end{matrix} \right\}$ of the second kind are introduced as the coefficients of the transform of bases of polynomials as follows:

$$(x)_j = \sum_{k=0}^{j} \left[\begin{matrix} j \\ k \end{matrix} \right] x^k, \quad j \in \mathbb{N}_0, \quad x^k = \sum_{j=0}^{k} \left\{ \begin{matrix} k \\ j \end{matrix} \right\} (x)_j, \quad k \in \mathbb{N}_0,$$

(cf.[7]). Note that the definition of the above Stirling number of the first kind is slightly different from the one in [2].

Let $B_0 = 1$, B_k, $k = 1, 2, \cdots$, be Bernoulli's numbers (refer to [7]). Now, we are in a position to state the main theorem in the present paper.

Theorem 2.1. *Let* $k \geq j$, k, $j \in \mathbb{N}_0$.

(1)

$$\frac{1}{j+1} \left\{ \begin{matrix} k \\ j \end{matrix} \right\} = \sum_{i=j}^{k} \frac{1}{i+1} \binom{k}{i} \left\{ \begin{matrix} i+1 \\ j+1 \end{matrix} \right\} B_{k-i}.$$

(2)

$$\frac{1}{k+1} \left[\begin{matrix} k+1 \\ j+1 \end{matrix} \right] = \frac{1}{j+1} \sum_{i=j}^{k} \binom{i}{j} \left[\begin{matrix} k \\ i \end{matrix} \right] B_{i-j}.$$

(3)

$$\frac{1}{k+1} \left[\begin{matrix} k+1 \\ j+1 \end{matrix} \right] = \frac{1}{j+1} \sum_{i=j}^{k} \binom{k}{i} \left[\begin{matrix} i \\ j \end{matrix} \right] \frac{(-1)^{(k-i)}(k-i)!}{k-i+1}.$$

Combining (2) with (3) in Theorem 2.1, we easily obtain the following result.

Corollary 2.2.

$$\sum_{i=j}^{k} \binom{i}{j} \left[\begin{matrix} k \\ i \end{matrix} \right] B_{i-j} = \sum_{i=j}^{k} \binom{k}{i} \left[\begin{matrix} i \\ j \end{matrix} \right] \frac{(-1)^{(k-i)}(k-i)!}{k-i+1}.$$

3. THE PROOF OF THE MAIN THEOREM

First, we state the translation formulae given in [8]. Now, we assume that B is nonsingular, that is, $B = e^{\tau A}$, $\tau > 0$ for some a complex $p \times p$ matrix A. By the spectral mapping theorem it is easy to see that $\sigma(B) = e^{\tau\sigma(A)}$ and

$$\sigma_\mu(A) := \{\lambda \in \sigma(A) \mid \mu = e^{\tau\lambda}\} \neq \emptyset$$

for $\mu \in \sigma(B)$. Set

$$A_{k,\lambda} = \frac{\tau^k}{k!}(A - \lambda E)^k \quad (\lambda \in \sigma(A)) \text{ and } B_{[k,\mu]} = \frac{1}{k!\mu^k}(B - \mu E)^k \quad (\mu \in \sigma(B)).$$

The following matrix $Y_\lambda(A)$ has been introduced to study of the representations of solutions to the linear discrete system (1.1) (see [8]) :

$$Y_\lambda(A) = \sum_{k=0}^{h_A(\lambda)-1} B_k A_{k,\lambda} \quad (\lambda \in i\omega\mathbb{Z} \cap \sigma(A)),$$

where $\omega = 2\pi/\tau$. Set $P_\lambda = Q_\lambda(A)$. Then $BP_\lambda = P_\lambda B$.

The following result is a part of the translation formulae in [8].

Lemma 3.1. [8] *Let $B = e^{\tau A}$, $\tau > 0$ and $\lambda \in \sigma_\mu(A)$.*

(1) *If $0 \leq k \leq h_B(\mu) - 1$, then*

$$B_{[k,\mu]}P_\lambda = \sum_{j=k}^{h_B(\mu)-1} \left\{ \begin{matrix} j \\ k \end{matrix} \right\} A_{j,\lambda}P_\lambda, \tag{3.1}$$

or equivalently, if $0 \leq j \leq h_B(\mu) - 1$, then

$$A_{j,\lambda}P_\lambda = \sum_{k=j}^{h_B(\mu)-1} \left[\begin{matrix} k \\ j \end{matrix} \right] B_{[k,\mu]}P_\lambda. \tag{3.2}$$

(2) *Let $\mu = 1$. If $0 \leq k \leq h_B(1) - 1$, then*

$$\frac{1}{k+1}B_{[k,1]}P_\lambda = \sum_{j=k}^{h_B(1)-1} \frac{1}{j+1} \left\{ \begin{matrix} j+1 \\ k+1 \end{matrix} \right\} A_{j,\lambda}Y_\lambda(A)P_\lambda, \tag{3.3}$$

or equivalently, if $0 \leq j \leq h_B(1) - 1$, then

$$\frac{1}{j+1}A_{j,\lambda}Y_\lambda(A)P_\lambda = \sum_{k=j}^{h_B(1)-1} \frac{1}{k+1} \left[\begin{matrix} k+1 \\ j+1 \end{matrix} \right] B_{[k,1]}P_\lambda. \tag{3.4}$$

The following result is also needed for the proof of main theorem.

Lemma 3.2. [8] *Let $\lambda \in i\omega\mathbb{Z} \cap \sigma(A)$. Then the following relations hold :*

(1)

$$A_{j,\lambda}Y_\lambda(A)P_\lambda = \sum_{i=j}^{h_B(1)-1} \binom{i}{j} B_{i-j}A_{i,\lambda}P_\lambda.$$

(2)

$$(B-E)Y_\lambda(A)P_\lambda = A_{1,\lambda}P_\lambda.$$

(3)

$$Y_\lambda(A)P_\lambda = \sum_{k=0}^{h_B(1)-1} \frac{(-1)^k}{k+1}(B-E)^k P_\lambda. \tag{3.5}$$

As an application of the translation formulae, we can give another proof of the well-known relation

$$\sum_{k=0}^{i-1} B_k \binom{i}{k} = 0,\ i>1. \tag{3.6}$$

Indeed, using (3.1) and Lemma 3.2, we have

$$\begin{aligned} A_{1,\lambda}P_\lambda &= (B-E)Y_\lambda(A)P_\lambda = \sum_{j=1}^{h_A(\lambda)-1} A_{j,\lambda}Y_\lambda(A)P_\lambda \\ &= \sum_{j=1}^{h_A(\lambda)-1}\sum_{i=j}^{h_A(\lambda)-1} B_{i-j}\binom{i}{j}A_{i,\lambda}P_\lambda \\ &= \sum_{i=1}^{h_A(\lambda)-1}\left(\sum_{j=1}^{i} B_{i-j}\binom{i}{i-j}\right)A_{i,\lambda}P_\lambda \\ &= \sum_{i=1}^{h_A(\lambda)-1}\left(\sum_{k=0}^{i-1} B_k\binom{i}{k}\right)A_{i,\lambda}P_\lambda. \end{aligned}$$

Comparing these coefficients, we can derive the equality (3.6).

The proof of Theorem 2.1 (1) Substituting (3.1) into the left side of the equality (3.3), we have

$$\frac{1}{j+1}\sum_{k=j}^{h_B(1)-1}\left\{ \begin{matrix} k \\ j \end{matrix} \right\} A_{k,\lambda}P_\lambda = \sum_{i=j}^{h_B(1)-1} \frac{1}{i+1}\left\{ \begin{matrix} i+1 \\ j+1 \end{matrix} \right\} A_{i,\lambda}Y_\lambda(A)P_\lambda.$$

On the other hand, it follows from Lemma 3.2 that

$$\begin{aligned}
&\sum_{i=j}^{h_B(1)-1} \frac{1}{i+1}\left\{ \begin{matrix} i+1 \\ j+1 \end{matrix} \right\} A_{i,\lambda} Y_\lambda(A) P_\lambda \\
&= \sum_{i=0}^{h_B(1)-1} \frac{1}{i+1}\left\{ \begin{matrix} i+1 \\ j+1 \end{matrix} \right\} \sum_{k=i}^{h_B(1)-1} \binom{k}{i} B_{k-i} A_{k,\lambda} P_\lambda \\
&= \sum_{k=0}^{h_B(1)-1} \left(\sum_{i=0}^{k} \frac{1}{i+1}\left\{ \begin{matrix} i+1 \\ j+1 \end{matrix} \right\} \binom{k}{i} B_{k-i} \right) A_{k,\lambda} P_\lambda \\
&= \sum_{k=j}^{h_B(1)-1} \left(\sum_{i=j}^{k} \frac{1}{i+1}\binom{k}{i}\left\{ \begin{matrix} i+1 \\ j+1 \end{matrix} \right\} B_{k-i} \right) A_{k,\lambda} P_\lambda .
\end{aligned}$$

Therefore, we obtain

$$\begin{aligned}
&\frac{1}{j+1} \sum_{k=j}^{h_B(1)-1} \left\{ \begin{matrix} k \\ j \end{matrix} \right\} A_{k,\lambda} P_\lambda \\
&= \sum_{k=j}^{h_B(1)-1} \left(\sum_{i=j}^{k} \frac{1}{i+1}\binom{k}{i}\left\{ \begin{matrix} i+1 \\ j+1 \end{matrix} \right\} B_{k-i} \right) A_{k,\lambda} P_\lambda ,
\end{aligned}$$

from which the coefficients in both sides coincide with each other.

(2) It follows from (3.4) and Lemma 3.2 that

$$\begin{aligned}
&\sum_{k=j}^{h_B(1)-1} \left[\begin{matrix} k+1 \\ j+1 \end{matrix} \right] \frac{1}{k+1} B_{[k,1]} P_\lambda \\
&= \frac{1}{j+1} A_{j,\lambda} Y_\lambda(A) P_\lambda = \frac{1}{j+1} \sum_{i=j}^{h_B(1)-1} \binom{i}{j} B_{i-j} A_{i,\lambda} P_\lambda .
\end{aligned}$$

Using (3.1) we have

$$\begin{aligned}
\sum_{i=j}^{h_B(1)-1} \binom{i}{j} B_{i-j} A_{i,\lambda} P_\lambda &= \sum_{i=j}^{h_B(1)-1} \binom{i}{j} B_{i-j} \sum_{k=i}^{h_B(1)-1} \left[\begin{matrix} k \\ i \end{matrix} \right] B_{[k,1]} P_\lambda \\
&= \sum_{k=0}^{h_B(1)-1} \left(\sum_{i=0}^{k} \binom{i}{j} \left[\begin{matrix} k \\ i \end{matrix} \right] B_{i-j} \right) B_{[k,1]} P_\lambda \\
&= \sum_{k=j}^{h_B(1)-1} \left(\sum_{i=j}^{k} \binom{i}{j} \left[\begin{matrix} k \\ i \end{matrix} \right] B_{i-j} \right) B_{[k,1]} P_\lambda .
\end{aligned}$$

Therefore,

$$\begin{aligned}
&\sum_{k=j}^{h_B(1)-1} \frac{1}{k+1} \begin{bmatrix} k+1 \\ j+1 \end{bmatrix} B_{[k,1]} P_\lambda \\
&= \frac{1}{j+1} \sum_{k=j}^{h_B(1)-1} \left(\sum_{i=j}^{k} \binom{i}{j} \begin{bmatrix} k \\ i \end{bmatrix} B_{i-j} \right) B_{[k,1]} P_\lambda .
\end{aligned}$$

Comparing these coefficients, we obtain the assertion (2).

(3) By (3.2) and (3.5) we have that for $\lambda \in i\omega\mathbb{Z} \cap \sigma(A)$,

$$\begin{aligned}
A_{j,\lambda} Y_\lambda(A) P_\lambda &= \left(\sum_{i=j}^{h_B(1)-1} \begin{bmatrix} i \\ j \end{bmatrix} B_{[i,1]} \right) \left(\sum_{m=0}^{h_B(1)-1} \frac{(-1)^m m!}{m+1} B_{[m,1]} \right) P_\lambda \\
&= \sum_{i=j}^{h_B(1)-1} \sum_{m=0}^{h_B(1)-1} \begin{bmatrix} i \\ j \end{bmatrix} \binom{m+i}{i} \frac{(-1)^m m!}{m+1} B_{[m+i,1]} P_\lambda \\
&= \sum_{i=j}^{h_B(1)-1} \sum_{k=i}^{h_B(1)-1} \begin{bmatrix} i \\ j \end{bmatrix} \binom{k}{i} \frac{(-1)^{(k-i)} (k-i)!}{k-i+1} B_{[k,1]} P_\lambda \\
&= \sum_{k=j}^{h_B(1)-1} \sum_{i=j}^{k} \binom{k}{i} \begin{bmatrix} i \\ j \end{bmatrix} \frac{(-1)^{(k-i)} (k-i)!}{k-i+1} B_{[k,1]} P_\lambda .
\end{aligned}$$

Thus, together with (3.4) we obtain

$$\begin{aligned}
&\sum_{k=j}^{h_B(1)-1} \frac{1}{k+1} \begin{bmatrix} k+1 \\ j+1 \end{bmatrix} B_{[k,1]} P_\lambda \\
&= \sum_{k=j}^{h_B(1)-1} \frac{1}{j+1} \sum_{i=j}^{k} \binom{k}{i} \begin{bmatrix} i \\ j \end{bmatrix} \frac{(-1)^{(k-i)} (k-i)!}{k-i+1} B_{[k,1]} P_\lambda .
\end{aligned}$$

Comparing these coefficients, we obtain the assertion (3). This completes the proof. □

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