INTERSECTIONS OF SETS EXPRESSIBLE AS UNIONS OF k STARSHAPED SETS

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ABSTRACT. Let $\mathcal{K}$ be a family of sets in $\mathbb{R}^d$ and let k be a fixed natural number. Assume that every countable subfamily of $\mathcal{K}$ has an intersection expressible as a union of k starshaped sets, each having a d-dimensional kernel. Then $S \equiv \cap\{K : K \text{ in } \mathcal{K}\}$ is nonempty and is expressible as a union of k such starshaped sets.

If members of $\mathcal{K}$ are compact and every finite subfamily of $\mathcal{K}$ has as its intersection a union of k starshaped sets, then S again is a union of k starshaped sets. An analogous result holds for unions of k convex sets.

Finally, dual results hold for unions of subfamilies of $\mathcal{K}$.

1. INTRODUCTION.

We begin with some definitions from [2]. Let S be a set in $\mathbb{R}^d$. For points x and y in S, we say x *sees* y (x is *visible* from y) *via* S if and only if the corresponding segment $[x, y]$ lies in S. Similarly, for subsets A, B of S, we say A *sees* B *via* S if and only if a *sees* b *via* S for all a in A, b in B. Set S is called *starshaped* if and only if for some point p of S, p sees each point of S via S, and the collection of all such points p is the (convex) kernel of S. Notice that a starshaped set cannot be empty.

A familiar theorem by Victor Klee [7] establishes the following Helly-type result for countable intersections of convex sets: Let $\mathcal{C}$ be a family of convex sets in $\mathbb{R}^d$. If every countable subfamily of $\mathcal{C}$ has a nonempty intersection, then $\cap\{C : C \text{ in } \mathcal{C}\}$ is nonempty as well. Moreover, a result in [2] yields this starshaped analogue: Let $\mathcal{S}$ be a family of sets in $\mathbb{R}^d$. If every countable subfamily of $\mathcal{S}$ has a starshaped intersection, then $\cap\{S : S \text{ in } \mathcal{S}\}$ is (nonempty and) starshaped. Here we obtain some related results for unions of k starshaped sets in $\mathbb{R}^d$.

Throughout the paper, *int* S, *ker* S, *cl* S, and *conv* S will denote the interior, kernel, closure, and convex hull, respectively, for set S. We follow the usual convention that a 0-dimensional neighborhood of point t and a

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0-neighborhood of t will be just the singleton set $\{t\}$. The reader may refer to Valentine [11], to Lay [9], to Danzer, Grünbaum, Klee [5], and to Eckhoff [6] for discussions on Helly-type theorems and starshaped sets, and to Nadler [10] for information on the Hausdorff metric.

2. The results.

The following easy proposition will be useful.

Proposition 1. Let $\mathcal{C}$ be a family of sets in any second countable topological space. If every countable intersection of members of $\mathcal{C}$ has a nonempty interior, then $\cap\{C : C \text{ in } \mathcal{C}\}$ has a nonempty interior as well.

Proof. The argument parallels the proof in [3, Proposition 1], adapted for an arbitrary second countable space.

Theorem 1. Let $\mathcal{K}'$ be a family of sets in $\mathbb{R}^d$ and let k be a fixed natural number. Assume that every countable subfamily of $\mathcal{K}'$ has an intersection expressible as a union of k starshaped sets, each having a d-dimensional kernel. Then $S \equiv \cap\{K : K \text{ in } \mathcal{K}'\}$ is nonempty and also is expressible as a union of k such starshaped sets.

Proof. Using Proposition 1, we see that S is nonempty. Let $\mathcal{K}$ denote the family of all countable intersections of members of $\mathcal{K}'$. Extending an approach used by Bobylev [1], for each K_α in $\mathcal{K}$, there exist associated k-tuples $(t_1, \ldots, t_k)$ such that each point of S sees via K_α a full d-dimensional neighborhood of at least one $t_i, 1 \le i \le k$. Let M_α denote the set of all such k-tuples $(t_1, \ldots, t_k)$ for K_α. Certainly if K_α is a union of k starshaped sets $T_1, \ldots, T_k$ satisfying our hypothesis, then $(\mathit{int}\,\mathit{ker}\,T_1) \times \ldots \times (\mathit{int}\,\mathit{ker}\,T_k)$ lies in M_α. Hence M_α has nonempty interior. Let $\mathcal{M}$ denote the collection of all the M_α sets.

A standard argument will show that countable subfamilies of $\mathcal{M}$ have nonempty interior in the product space $\mathbb{R}^d \times \ldots \times \mathbb{R}^d$: For any countable collection $\{M_n : n \ge 1\}$ in $\mathcal{M}$ and corresponding collection $\{K_n : n \ge 1\}$ in $\mathcal{K}$, let $K_0 = \cap\{K_n : n \ge 1\}$ with M_0 the associated member of $\mathcal{M}$. It is easy to see that $M_0 \subseteq \cap\{M_n : n \ge 1\}$. Since M_0 has nonempty interior in the product space $\mathbb{R}^d \times \ldots \times \mathbb{R}^d, \cap\{M_n : n \ge 1\}$ has this property as well.

Hence we may use Proposition 1 to conclude that $\cap\{M_\alpha : M_\alpha \text{ in } \mathcal{M}\}$ has nonempty interior, too. Select a subset $N_1 \times \ldots \times N_k$ of this intersection, where N_i is nonempty, open, and convex in $\mathbb{R}^d, 1 \le i \le k$. We assert that S is expressible as a union of k starshaped sets, each of whose kernels contains some $N_i, 1 \le i \le k$. That is, for each s in S, s sees via S all points in at least one $N_i, 1 \le i \le k$. Suppose on the contrary that the result fails. Then for each $i, 1 \le i \le k$, there is some t_i in N_i with $[s, t_i] \not\subseteq S$. That is, for at least one K_i in $\mathcal{K}, [s, t_i] \not\subseteq K_i$. It follows that s cannot see via $K_1 \cap \ldots \cap K_k$ any $t_i, 1 \le i \le k$. Of course, $K_0 \equiv K_1 \cap \ldots \cap K_k$ belongs to $\mathcal{K}$. For M_0

the corresponding member of $\mathcal{M}, (t_1, \ldots, t_k) \notin M_0$, contradicting the fact that $(t_1, \ldots, t_k)$ belongs to $N_1 \times \ldots \times N_k$ and hence to every member of $\mathcal{M}$. Our supposition must be false, and S is a union of k starshaped sets, say $S_1, \ldots, S_k$, with $N_i \subseteq \mathit{ker}\, S_i, 1 \leq i \leq k$. Since each convex set N_i is fully d-dimensional, this finishes the proof.

As in [2], we have an associated dual result, using unions instead of intersections.

Theorem 1′. Let $\mathcal{K}'$ be a family of sets in $\mathbb{R}^d$ and let k be a fixed natural number. Assume that, for every countable subfamily of $\mathcal{K}'$, the corresponding union is expressible as a union of k starshaped sets, each having a d-dimensional kernel. Then $T \equiv \cup\{K : K \,\text{in}\, \mathcal{K}'\}$ also is expressible as a union of k such starshaped sets.

Proof. Let $\mathcal{K}$ denote the family of all countable intersections of members of $\mathcal{K}'$. Adapting an approach in [2, Theorem 2], for each K_α in $\mathcal{K}$, define $M_\alpha = \{(t_1, \ldots, t_k)$: each point of K_α sees via T a full d-dimensional neighborhood of at least one $t_i, 1 \leq i \leq k\}$. Let $\mathcal{M}$ represent the family of all the M_α sets.

It is not hard to show that countable intersections of members of $\mathcal{M}$ have nonempty interior in the product space $\mathbb{R}^d \times \ldots \times \mathbb{R}^d$: For any countable collection $\{M_n : n \geq 1\}$ in $\mathcal{M}$ and corresponding $\{K_n : n \geq 1\}$ in $\mathcal{K}, \cup\{K_n : n \geq 1\}$ is a union of k starshaped sets, say $S_1, \ldots, S_k$, satisfying our hypothesis. Then for every $(s_1, \ldots, s_k)$ in $(\mathit{int}\, \mathit{ker}\, S_1) \times \ldots \times (\mathit{int}\, \mathit{ker}\, S_k)$ and for every $n \geq 1$, each point of K_n sees via $\cup\{K_n : n \geq 1\}$ and hence via T a full d-dimensional neighborhood of at least one $s_i, 1 \leq i \leq k$. Therefore, $(s_1, \ldots, s_k) \in M_n$ for every $n \geq 1$. That is, $(\mathit{int}\, \mathit{ker}\, S_1) \times \ldots \times (\mathit{int}\, \mathit{ker}\, S_k) \subseteq \cap\{M_n : n \geq 1\}$, and $\cap\{M_n : n \geq 1\}$ has nonempty interior, the desired result.

We may use Proposition 1 to conclude that $\cap\{M_\alpha : M_\alpha \,\text{in}\, \mathcal{M}\}$ has nonempty interior as well. As in the proof of Theorem 1, select $N_1 \times \ldots \times N_k$ in this intersection, where N_i is nonempty, open, and convex in $\mathbb{R}^d, 1 \leq i \leq k$. We assert that each point of T sees some N_i via $T, 1 \leq i \leq k$: Suppose on the contrary that the result fails for some t in T, where t belongs to K_0 in $\mathcal{K}$, with M_0 the associated member of $\mathcal{M}$. Then for each $i, 1 \leq i \leq k$, there is some s_i in N_i with $[t, s_i] \not\subseteq T$. Certainly$(s_1, \ldots, s_k)$ cannot belong to M_0, contradicting our choice of $(s_1, \ldots, s_k)$ in $N_1 \times \ldots \times N_k \subseteq \cap\{M_\alpha : M_\alpha \,\text{in}\, \mathcal{M}\}$. Our supposition is false, and T is a union of k starshaped sets, say $T_1, \ldots, T_k$, with $N_i \subseteq \mathit{ker}\, T_i, 1 \leq i \leq k$. The sets $T_i, \ldots, T_k$ satisfy Theorem1′.

Example 1 from [2] shows that *countable* cannot be replaced by *finite* in Theorem 1, even when the sets are closed and the associated intersection is nonempty. Similarly, [2, Example 2] illustrates a similar situation for bounded sets having nonempty intersection. Analogous examples in [2, Examples 3 and 4] reveal that countable cannot be replaced by finite in the dual case either.

However, when the sets are compact, we have the following result.

Theorem 2. Let $\mathcal{K}'$ be a family of compact sets in $\mathbb{R}^d$. Let k be a fixed natural number, let $r_1, \ldots, r_k$ be fixed integers with $d \geq r_1 \geq \ldots \geq r_k \geq 0$, and let $\epsilon_1, \ldots, \epsilon_k$ be fixed nonnegative numbers. Assume that every finite subfamily of $\mathcal{K}'$ has as its intersection a union of k starshaped sets such that, for an appropriate labeling, the kernel of the i^{th} set contains an r_i-dimensional ϵ_i-neighborhood, $1 \leq i \leq k$. Then $S \equiv \cap\{K : K \text{ in } \mathcal{K}\}$ is a union of k such starshaped sets as well.

Proof. Observe that $\mathcal{K}'$ is a family of compact sets having the finite intersection property and hence S is nonempty. Let K denote the family of all finite intersections of members of $\mathcal{K}'$. Again we begin by adapting an approach used by Bobylev [1]. For each K_α in $\mathcal{K}$, define the corresponding set M_α to be the collection of all k-tuples $(T_1, \ldots, T_k)$ satisfying these properties: For $1 \leq i \leq k, T_i$ is the closure of an r_i-dimensional ϵ_i-neighborhood in K_α. Further, for each s in S, there is some $i, 1 \leq i \leq k$, such that s sees via K_α all points of T_i. Since $S \subseteq K_\alpha$ and K_α is a union of k starshaped sets satisfying our hypothesis, clearly $M_\alpha \neq \emptyset$.

We will show that each set M_α is compact relative to the Hausdorff metric: Certainly each M_α is bounded, since each of its points lies in $K_\alpha \times \ldots \times K_\alpha$, a product of k compact sets. To see that M_α is closed, let $\{(T_{1n}, \ldots, T_{kn}) : n \geq 1\}$ be a sequence in M_α converging to $(T_1', \ldots, T_k')$ relative to the Hausdorff metric. By an argument in [4, Lemma], for each $i, 1 \leq i \leq k, T_i'$ is also the closure of an r_i-dimensional ϵ_i-neighborhood. Let $s \in S$ to show that s sees via K_α some $T_i', 1 \leq i \leq k$. For each n, s sees via K_α some $T_{in}, 1 \leq i \leq k$. Hence for some particular $i_0, 1 \leq i_0 \leq k$, s sees infinitely many sets $T_{i_0 n}$. Passing to a subsequence if necessary, assume that s sees $T_{i_0 m}$ via K_α for all $m \geq 1$. Since $\{T_{i_0 m} : m \geq 1\}$ converges to T_{i_0}', by a standard argument, s sees T_{i_0}' via K_α. We conclude that $(T_1', \ldots, T_k')$ belongs to M_α and therefore M_α is closed, hence compact.

Let $\mathcal{M}$ denote the family of compact sets M_α. It is easy to show that $\mathcal{M}$ has the finite intersection property: For $M_1, \ldots, M_j$ in $\mathcal{M}$ and for corresponding $K_1, \ldots, K_j$ in $\mathcal{K}, K_1 \cap \ldots \cap K_j \equiv K_0$ belongs to $\mathcal{K}$. The associated nonempty set M_0 is a subset of $M_1 \cap \ldots \cap M_j$, so $M_1 \cap \ldots \cap M_j \neq \emptyset$.

Therefore, $\mathcal{M}$ is a family of compact sets having the finite intersection property, and it follows that $\cap\{M_\alpha : M_\alpha \text{ in } \mathcal{M}\} \neq \emptyset$. Select $(T_1, \ldots, T_k)$ in this intersection. Clearly each T_i lies in $\cap\{K_\alpha : K_\alpha \text{ in } \mathcal{K}\} = S, 1 \leq i \leq k$. We assert that each point of s sees via S some set $T_i, 1 \leq i \leq k$. The proof parallels an argument in Theorem 1: Suppose on the contrary that the result fails for some s in S. This implies that for each $i, 1 \leq i \leq k$, there is at least one corresponding K_i in $\mathcal{K}$ such that s fails to see via K_i some point of T_i. Then $K_1 \cap \ldots \cap K_k \equiv K_0$ belongs to $\mathcal{K}$, yet s cannot see any T_i via K_0, contradicting our choice of $(T_1, \ldots, T_k)$ in $\cap\{M_\alpha : M_\alpha \text{ in } \mathcal{M}\}$. Our supposition is false, each point of S sees via S some $T_i, 1 \leq i \leq k$, and S is a union of k starshaped sets, each having appropriate kernel. This finishes the proof of Theorem 2.

Corollary. Let $\mathcal{K}$ be a family of compact sets in $\mathbb{R}^d$, and let k be a fixed natural number. If every finite subfamily of $\mathcal{K}$ has as its intersection a union of k starshaped sets, then $\cap\{K : K \text{ in } \mathcal{K}\}$ is a union of k starshaped sets as well.

We have the following dual to Theorem 2.

Theorem 2′. Let $\mathcal{K}'$ be a family of sets in $\mathbb{R}^d$, with $T \equiv \cup\{K' : K' \text{ in } \mathcal{K}'\}$ compact. Let k be a fixed natural number, let $r_1, \ldots, r_k$ be fixed integers with $d \geq r_1 \geq \ldots \geq r_k \geq 0$, and let $\epsilon_1, \ldots, \epsilon_k$ be fixed nonnegative numbers. Assume that every finite subfamily of $\mathcal{K}'$ has as its union a union of k starshaped sets such that, for an appropriate labeling, the kernel of the i^{th} set contains an r_i-dimensional ϵ_i-neighborhood, $1 \leq i \leq k$. Then T is a union of k such starshaped sets as well.

Proof. Since the hypothesis above must also hold for the family $\{cl\, K' : K' \text{ in } \mathcal{K}'\}$, without loss of generality, we assume that each member of $\mathcal{K}'$ is closed, hence compact. Let $\mathcal{K}$ represent the family of all finite unions of members of $\mathcal{K}'$. Adapting earlier arguments, for each K_α in $\mathcal{K}$, let M_α denote the family of all k-tuples $(T_1, \ldots, T_k)$ satisfying these properties: Each T_i is the closure of an r_i-dimensional ϵ_i-neighborhood in T. Moreover, for each point s of K_α, there is some $i, 1 \leq i \leq k$, such that s sees via T all points of T_i. Let $\mathcal{M}$ denote the family of sets M_α.

Since T is compact, an argument in the proof of Theorem 2 shows that each set M_α is compact relative to the Hausdorff metric. Further, an argument in the proof of Theorem 1′ may be modified to prove that $\mathcal{M}$ has the finite intersection property. Hence $\cap\{M_\alpha : M_\alpha \text{ in } \mathcal{M}\} \neq \emptyset$. For $(T_1, \ldots, T_k)$ in this intersection, each point of T sees some T_i via T and hence T is a union of k appropriate starshaped sets, finishing the proof.

The previously mentioned examples demonstrate that Theorems 2 and 2′ fail without the requirement that T be compact.

We have the following analogues of Theorems 2 and 2′ for convex sets.

Theorem 3. Let $\mathcal{K}'$ be a family of compact sets in $\mathbb{R}^d$, and let k be a fixed natural number. If every finite subfamily of $\mathcal{K}'$ has as its intersection a nonempty union of k convex sets, then $S \equiv \cap\{K' : K' \text{ in } \mathcal{K}\}$ is a nonempty union of k convex sets as well.

Proof. The argument is similar to the previous proof. Notice that S is compact and nonempty. Let $\mathcal{K}$ denote the family of all finite intersections of members of $\mathcal{K}'$. For each set K_α in $\mathcal{K}$, let M_α represent the family of all

k-tuples $(T_1, \ldots, T_k)$ satisfying these properties: Each T_i is compact and nonempty, $1 \le i \le k, \cup\{T_i : 1 \le i \le k\} = S$, and $conv\, T_i \subseteq K_\alpha, 1 \le i \le k$. Observe that if K_α is a union of the k compact sets $C_1, \ldots, C_k$ and $S \cap C_i \neq \emptyset, 1 \le i \le k$, then $(S \cap C_1, \ldots, S \cap C_k)$ belongs to M_α. In case some $S \cap C_i$ is empty, a nonempty $S \cap C_j$ may be used instead to obtain a member of M_α. Hence $M_\alpha \neq \emptyset$.

Standard arguments show that each set M_α is compact relative to the Hausdorff metric in the product $K_\alpha \times \ldots \times K_\alpha \subseteq \mathbb{R}^d \times \ldots \times \mathbb{R}^d$ and that $\mathcal{M} \equiv \{M_\alpha : K_\alpha \text{ in } \mathcal{K}\}$ has the finite intersection property. Therefore, $\cap\{M_\alpha : M_\alpha \text{ in } \mathcal{M}\} \neq \emptyset$. For $(T_1, \ldots, T_k)$ in this intersection, $S = \cup\{T_i : 1 \le i \le k\} \subseteq \cup\{conv\, T_i : 1 \le i \le k\} \subseteq K_\alpha$ for every K_α in $\mathcal{K}$. Thus $S \subseteq \cup\{conv\, T_i : 1 \le i \le k\} \subseteq \cap\{K_\alpha : K_\alpha \text{ in } \mathcal{K}\} = S$, and S is the union of the k convex sets $conv\, T_i, 1 \le i \le k$, finishing the proof.

Theorem 3′. Let $\mathcal{K}'$ be a family of sets in $\mathbb{R}^d$, with $T \equiv \cup\{K' : K' \text{ in } \mathcal{K}'\}$ compact. Let k be a fixed natural number, let $r_1, \ldots, r_k$ be fixed integers with $d \ge r_1 \ge \ldots \ge r_k \ge 0$, and let $\epsilon_1, \ldots, \epsilon_k$ be fixed nonnegative numbers. Assume that every finite subfamily of $\mathcal{K}'$ has as its union a union of k convex sets such that, for an appropriate labeling, the i^{th} set contains an r_i-dimensional ϵ_i-neighborhood, $1 \le i \le k$. Then T is a union of k such convex sets as well.

Proof. Because the proof resembles earlier arguments, we give just an outline here. Without loss of generality, assume that each member of $\mathcal{K}'$ is compact, and let $\mathcal{K}$ represent the family of all finite unions of members of $\mathcal{K}'$. For each K_α in $\mathcal{K}$, let M_α denote the collection of all k-tuples $(C_1, \ldots, C_k)$ satisfying these properties: For $1 \le i \le k, C_i$ is compact and convex and contains a r_i-dimensional ϵ_i-neighborhood. Moreover, $K_\alpha \subseteq \cup\{C_i : 1 \le i \le k\} \subseteq T$. Again let $\mathcal{M}$ represent the family of sets M_α.

Standard arguments show that $\mathcal{M}$ is a family of compact sets relative to the Hausdorff metric in the product space $T \times \ldots \times T \subseteq \mathbb{R}^d \times \ldots \times \mathbb{R}^d$. Furthermore, $\mathcal{M}$ has the finite intersection property and hence $\cap\{M_\alpha : M_\alpha \text{ in } \mathcal{M}\} \neq \emptyset$. For $(C_1, \ldots, C_k)$ in this intersection and for every K_α in $\mathcal{K}, K_\alpha \subseteq \cup\{C_i : 1 \le i \le k\} \subseteq T$. Thus $T \equiv \cup\{K_\alpha : K_\alpha \text{ in } \mathcal{K}\} \subseteq \cup\{C_i : 1 \le i \le k\} \subseteq T$ and T is the union of the convex sets $C_i, 1 \le i \le k$, the desired result.

In conclusion, it is interesting to compare Theorem 3′ to a theorem by Lawrence, Hare, and Kenelly [8, Theorem 2] that characterizes arbitrary sets expressible as unions of k convex sets.

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