On Connected $m - K_2$–Residual Graphs *

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Abstract. P. Erdös, F. Harary and M. Klawe had studied K_n–residual graph, and came up with some conjectures and conclusions about $m - K_n$–residual graph. For connected $m - K_2$–residual graph, they constructed $m - K_2$–residual graph with order $3m + 2$ and proposed that $3m + 2$ is the minimum order which was not proved. In this paper, by operation property of set and some other methods, we proved that the minimum order of connected $m - K_2$–residual graph is $3m + 2$.

Key Words: residually graph; minimum order; extremal graph.

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1 Introduction

The definition of residual graph was put forward by P. Erdös, F. Harary and M. Klawe and they proved that C_5 is the only connected K_2–residual graph the minimum order of which is 5; for $n \neq 2$, the minimum order of connected K_n–residual graph is $2(n + 1)$; for $n \neq 2, 3, 4$, $K_{n+1} \times K_2$ is the only connected K_n–residual graph which has a minimum order. They also proved

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that for $m-K_n$−residual graph, $(m+1)K_n$ is the only $m-K_n$−residual graph the minimum order of which is $(m+1)n$. And at the same time, they proposed two conjectures about connected $m-K_n$−residual graph as follows:

Conjecture 1. *If $n \neq 2$, then every connected $m-K_n$−residual graph has at least $min\{2n(m+1),(n+m)(m+1)\}$ vertices.*

Conjecture 2. *For n large, there is a unique smallest connected $m-K_n$−residual graph.*

There has been quite a few results [2-9] on the study of residual graph, of which the reference[2] discussed about the weakly minimum order of complete residual graphs and proved that there exists K_n−residual graph whose order is $2(n+k)$ for any $n,k \in N^+$ and for $n=2,4$, the order of K_n−residual graph is $2n+3$ and for $n=6$, $C_5[K_3]$ is the only K_6−residual graph with a minimum odd order 15. The reference [3] discussed about odd order complete residual graphs, proved that for any odd n, there exist no odd order complete residual graphs and also it got the minimum order complete residual graph for $n \equiv 0(mod2)$ and constructed some special odd order complete residual graphs. The reference [4] solved problems about $2-K_n$−residual graph, proved the conclusion that $K_{n+m} \times K_{m+1}$ is an $m-K_n$−residual graph founded by Erdös et al for $n \geq 5$, $n \neq 6$, got the minimum order and the only extremal graph for $n=4$ and got two non-isomorphic $2-K_6$−residual graph. The reference [5] studied $3-K_n$−residual graph and proved the conclusion of Erdös et al for $n \geq 11$. As for studies on other kinds of residual graphs, the reference [6] solved problems about residual graphs composite by 3 dimension hyperplane graph and K_t and obtained its minimum order and the only extremal graph. In [7] and [8], it researched the minimum odd order and extremal graph of two special residual graphs. In [9] and [10], it showed some application of residual graphs.

The studies on connected $m-K_n$−residual graph is mainly about working out its minimum order, constructing its extremal graph and proving the uniqueness of its extremal graph, which is practically very difficult. Here we take two cases, the first case is that if $n=1$, $m=2$, then $2-K_1$−residual graph with order as figure 1 can be connected, however, the proving of minimum order and uniqueness of its extremal graph haven't been worked out. Another case: for $n=1$, $m=3$, two non-isomorphic $3-K_1$−residual graphs with order 8 as figure 2, 3 can be constructed, but it's difficult to find out all the isomorphic $3-K_1$−residual graphs.

For connected $m-K_2$−residual graph in reference[1], P.Erdös et al constructed connected $m-K_2$−residual graph whose order is $(3m+2)$ and proposed that $(3m+2)$ is the minimum order which was not proved. In this paper, by operation property of set and some other methods, we prove that the minimum order of $m-K_2$–residual graph is $3m+2$.

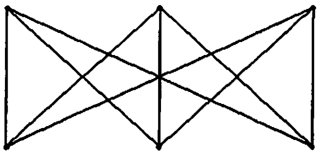

Fig1 : 2-K_1 -residual graph with order 6

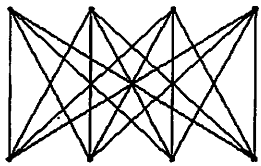

Fig2: 3-K_1 -residual graph with order 8

2 Preliminaries

We need to recall the following concepts and results for solving the $m-K_2$–Residual Graphs.

Definition 2.1 *[4] Assume $G=(V,E)$ is a simple graph with vertex set V and the edge set E, denote $E=E(G)$, $V=V(G)$, $\nu(G)=|V|$ is the order of G. If denote $U=\{u_1,u_2,\cdots,u_k\}\subset V$, is a nonempty subset of $V=V(G)$, denote by $<U>=<u_1,u_2,\cdots,u_k>$ the subgraph $G[U]$ induced by the subset U. In particular, for the sake of convenience, denote $G=<V>=<v_1,v_2,\cdots,v_n>$, where $V=V(G)=\{v_1,v_2,\cdots,v_n\}$.*

Definition 2.2 *Assume G is a simple graph, if $v\in V=V(G)$, then we say that v is in G, and simply denote by $v\in G$. If $v\in H$, H is a subgraph of G, denote by $H\subset G$, $N_H(v)=\{x\in H,\ x$ is adjacent to v, or $x=v\}$ represents the closed neighborhood of v in H. Generally, $N_G(v)$ is simply denoted as $N(v)$. If $F\subset G$, $N(F)=N_G(F)=\bigcup N(v), v\in F$ represents the closed neighborhood of F.*

Definition 2.3 *[11] Assume that F is a given graph. If for every vertice $u\in V(G)$, the graph obtained by removing the closed neighborhood of u from G is isomorphic to F, then G is said to be $F-$residual graph. We inductively define a multiply-F-residual graph by saying that G is $m-F-$residual graph if the removal of the closed neighborhood of any vertex of G results in an $(m-1)-F-$residual graph, where of course a $1-F-$residual graph is simply $F-$residual graph.*

Definition 2.4 *[11] Assume that $G_1=(V_1,E_1)$, $G_2=(V_2,E_2)$, $G=G_1[G_2]$ which is composite by G_1 and G_2 is defined as $V(G)=V_1\times V_2=\{v=(v_1,v_2)|(v_1\in V_1,v_2\in V_2\}$. Two vertices $u=(u_1,u_2)$, $v=(v_1,v_2)$ are adjacent to each other, if and only if, u_1 is adjacent to v_1 in G_1, or $u_1=v_1$ and u_2 is adjacent to v_2 in G_2.*

Definition 2.5 *Let* $G_1 = (V_1, E_1)$ *and* $G_2 = (V_2, E_2)$ *be two disjoint graphs, the join of two graphs* G_1 *and* G_2*, denoted by* $G = G_1 + G_2$*, is defined as* $V(G) = V_1 \cup V_2$*,* $E(G) = E_1 \cup E_2 \cup (V_1, V_2)$.

By Definition 2.5, we have

Lemma 2.6 *Let* $G = G_1 + G_2$*,* G *is a* $m - F-$*residual graph, if and only if, both* G_1 *and* G_2 *are* $m - F-$*residual graphs.*

Definition 2.7 *[4] Let* $X, Y \subset V(G)$*,* $X \cap Y = \emptyset$*.* X *is said to be adjacent to* Y*, if there exists* $x \in X$ *and* $y \in Y$*, such that* $xy \in E(G)$*, and vice versa. If* $xy \in E(G)$ *for all* $x \in X$ *and* $y \in Y$*, then* X *is said to be complete adjacent to* Y*, and vice versa. For example,* X *and* Y *are said to be nonadjacent if there are no edges between them.*

Definition 2.8 *[4] For any* $u \in G$*, define* $G_u = G - N(u)$*. For convenience, let* $< U >$ *represent the induced subgraph of* U *in* G*.*

Lemma 2.9 *[1] Assume* G *is a* $F-$*residual graph, then for any* $u \in G$*,* $d(u) = \nu(G) - \nu(F) - 1$.

Lemma 2.10 *[1] Every* $m - K_n-$*residual graph has at least* $(m+1)n$ *vertices, and* $(m+1)K_n$ *is the only and smallest* $m - K_n-$*residual graph with* $(m+1)n$ *vertices.*

Lemma 2.11 *[1]Assume* G *is a* K_2- *residual graph, then* C_5 *is the only connected* K_2- *residual graph with minimum order 5.*

3 On connected $m - K_2$–residual graph

Lemma 3.1 *If* G *is a* mK_2-*residual graph and* $G \neq (m+1)K_2$ *for* $m \geq 2$*, then* $\nu(G) \geq 4(m+1)$ *and* $K_{m+1,m+1}[K_2]$ *is the only extremal graph.*

Proof. For any $u \in G$, let $G_u = H_1 \cup H_2 \cup \cdots \cup H_m = mK_2, H_i \cong K_2, i = 1, 2, \cdots, m$. For $v \in H_m$, then $G_v = H_0 \cup H_1 \cup \cdots \cup H_{m-1} = mK_2, H_0 \cong K_2$, assume $w \in H_1$, then $G_w = H_0 \cup H_2 \cup \cdots \cup H_m = mK_2$. and $\langle H_0 \cup H_1 \cup H_2 \cup \cdots \cup H_m \rangle = G_1 \subset G$. By Lemma 2.10, $G_1 \cong (m+1)K_2$, hence $\nu(G_1) \geq 2(m+1)$. And since $G \neq (m+1)K_2$, let $X = G - G_1 \neq \emptyset$, then G_1 is completely adjacent to X. Let $G = \langle X \rangle + G_1$, by Lemma 2.6, $\langle X \rangle = G_2$ is mK_2-residual graph, and $\nu(G_2) \geq 2(m+1)$, hence

$$\nu(G) = \nu(G_1) + \nu(G_2) \geq 2(m+1) + 2(m+1) = 4(m+1).$$

And for $\nu(G) = 4(m+1)$, $G_1 \cong G_2 \cong (m+1)K_2$, hence $G \cong K_{m+1,m+1}[K_2]$.

In the following, we take the case $m = 3$ for an explicit explanation. For any $u = v_8 \in V(G)$, $G_u = G - N(u) \cong 3K_2$. Let $v = v_1 \in V(G)$, then

$G_v = G - N(v) \cong 3K_2$ and let $w = v_3 =\in V(G)$, then $G_w = G - N(w) \cong 3K_2$. Let $G_1 = 4K_2$ and $G_1 \subset G$. Let $V(G_1) = V'$, because of $G \neq 4K_2$, we have $V(G) - \{v_1, v_2, \cdots, v_8\} = X \neq \emptyset$ and X is completely adjacent to V'. Let $\langle X \rangle = G_2$, for any $x \in X$, $G - N(x) = G_2 - N_{G_2}(X) \cong 3K_2$, hence G_2 is $3K_2-$residual graph and $\nu(G_2) \geq 8$, then $G = \langle X \rangle + G_1$, $\nu(G) = \nu(G_1) + \nu(G_2) \geq 8 + 8 = 16$. We can construct G with order 16 as figure 4.

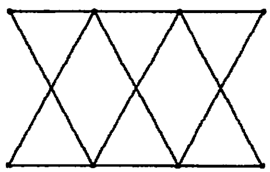

Fig3: 3-K_2 -residual graph with order 8

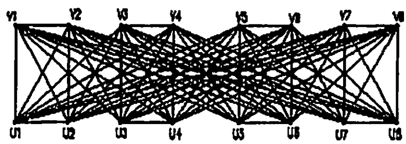

Fig4: 4 K_2-residual graph with order 16

Lemma 3.2 *Assume G is a connected $m - K_2-$residual graph. If $u \in G$, then for any $v \in G_u$, let $G_v = H(v) \cup F(v)$, $u \in H(v) \cong K_2$, $H(v)$ be non-adjacent to $F(v)$ and $F(v)$ be $(m-2) - K_2-$residual graph, $m \geq 2$, then $\nu(G) \geq min\{(4m+4, 4m+\delta(n)+1)\}$, where $\delta(n)$ is the minimum degree of vertices.*

Proof. Let $G_v = H_1 \cup F$, $u \in H_1 \cong K_2$, $N(H_1) \cap F = \emptyset$ and $X = N(H_1) - H_1$, then for any $w \in (G - N(H_1)) \subset G_u$,

$$G_w = H_1 \cup F_2,\ N(H_1) \cap F_2 = \emptyset,$$

hence $F_2 \subset G - N(H_1) = G_1$, then F_2 is $(m-2) - K_2-$residual graph, G_1 is $(m-1) - K_2-$residual graph, $\nu(G_1) \geq 2m$, $X \subset N(w)$ and G_1 is completely adjacent to X. In the following, the paper makes a discussion according to two cases:

Case 1 When H_1 is completely adjacent to X, $G = \langle X \rangle + \langle H_1 \cup G_1 \rangle$. By Lemma 2.6 and Lemma 2.10

$$\nu(G) \geq \nu(X) + \nu(H_1 \cup G_1) = 2(m+1)2 = 4m+4.$$

Case 2 When H_1 is not completely adjacent to X, if there exist a vertex $u' \in H_1$, which makes

$$X - N(u') = X_1 \neq \emptyset,\ G_2 = G - N(u') = \langle X_1 \cup G_1 \rangle = \langle X_1 \rangle + G_1,$$

then G_2 is a $(m-1) - K_2-$residual graph. By Lemma 2.6, Lemma 2.9 and Lemma 2.10,

$$\nu(G) = \nu(G_2) + d(u') + 1,\ \nu(G_2) \geq 2 \times 2(m+1-1) = 4m,$$

hence

$$\nu(G) \geq 4m + d(u') + 1 \geq 4m + \delta(n) + 1.$$

This completes the proof.

Lemma 3.3 *Assume G is a connected $m-K_2-$residual graph. If $m \geq 2$, $u \in G$, for every $v \in G_u$, G_v is disconnected, then $\nu(G) \geq \nu(F)+4(m-r)$, where $G_v = G-N(v)$, $G_u = G-N(u)$, $u \in F$ and F is a component of G_v for some $v \in G_u$, F is a $r-K_2-$ connected residual graph, $0 \leq r \leq m-2$.*

Proof. Assume for any $v \in G_u$, $G_v = G-N(v)$ is not connected. Let $A = \{H | H \subset G_v, v \in G_u, u \in G\}$, and $F \in A$(F is biggest connected subgraph) and let $G_v = F \cup G_1$, F be $r-K_2-$residual graph; G_1 be $(m-r-2)-K_2-$residual graph and $F \cong K_2$ for $r=0$, then for any $u \in G$, $G_u \cong mK_2$ and by Lemma 3.1, $\nu(G) \geq 4(m+1)$.

For any $w \in G-N(F) \subset G_u$, let $G_w = F_1 \cup G_2$. Because $F \subset G_w$, $F \subset F_1$. Since F is the biggest connected component, then $F_1 = F$ and $G_w = F \cup G_2$, $N(F) \cap G_2 = \emptyset$, hence $G_2 \subset G-N(F)$, G_2 is an $(m-r-2)-K_2-$residual graph, $G-N(F)$ is an $(m-r-1)-K_2-$residual graph, and $\nu(G-N(F)) \geq 2(m-r)$.

Let $N(F)-F = X$, then $G-N(F)$ is completely adjacent to X. In the following, the paper makes a discussion according to two cases:

Case 1 When F is completely adjacent to X,

$$G = \langle N(F) \cup (G-N(F)) \rangle = \langle X \cup F \cup (G-N(F)) \rangle$$

$$= \langle X \rangle + \langle F \cup (G-N(F)) \rangle = \langle X \rangle + \langle G-X \rangle$$

By Lemma 3.2, $|X| \geq 2(m+1)$, and by Lemma 3.1,

$$\nu(G) \geq 2(m+1)+\nu(F)+2(m-r) \geq \nu(F)+4(m-r).$$

In the following, we take the case $m=3$, $r=1$ for an explicit explanation. Let F be K_2-residual graph, let G_1 be K_2. X is completely adjacent to F and $G-N(F)$, $(G-N(F))$ is a K_2-residual graph. Then we can construct G as figure 5, $\nu(G) \geq \nu(F)+8 = \nu(F)+4(3-1)$.

Case 2 When F is not completely adjacent to X, we have two cases as follows.

When there are exactly $r+1$ independent sets, then $\{u_0, u_1, \cdots, u_r\} \subset F$ and $X - N(u_0, u_1, \cdots, u_r) = X_1 \neq \emptyset$, hence

$$G' = G - N(u_0, u_1, \cdots, u_r) = \langle X_1 \rangle + \langle G-N(F) \rangle,$$

then G' is $(m-r-1)-K_2-$residual graph, and $|X_1| \geq 2(m-r)$. By the proof of Case 1, $\nu(G) \geq \nu(F)+4(m-r)$.

Next, we take the case $m=3$, $r=1$ for an explicit explanation. Let F be K_2-residual graph and $(G-N(F))$ be K_2-residual graph. Then we can construct G as figure 6 and $\nu(G) \geq \nu(F)+8 = \nu(F)+4(3-1)$. In Fig.6, where $F = K_2 \cup K_2 \cup K_2 \cup K_2$ and the adjacent as show by the Fig.6. $X = K_2 \cup K_2$ and the adjacent as shown by the Fig.6. $G-N(F) = K_2 \cup K_2$ and the adjacent as shown by the Fig.6.

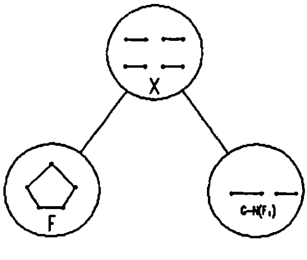

Fig5: 3-K_2-residual graph

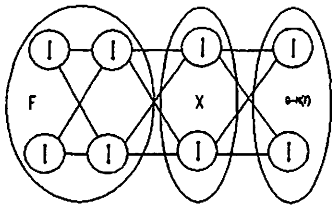

Fig6: 3-K_2-residual graph

When there are l independent sets, of which $l < r+1$ and for any $l+1$ independent sets, then $\{u_1, u_2, \cdots, u_l\} \subset F$. Let $X - N(u_1, u_2, \cdots, u_l) = X_2 \neq \emptyset$, but $X - N(u_1, u_2, \cdots, u_{l+1}) = \emptyset$. We have

$$G'' = G - N(u_1, u_2, \cdots, u_l) = \langle F \cup X \cup (G - N(F)) \rangle - N(u_1, u_2, \cdots, u_l)$$

$$= (F - N(u_1, u_2, \cdots, u_l)) \cup (X - N(u_1, u_2, \cdots, u_l)) \cup (G - N(F)$$

$$-N(u_1, u_2, \cdots, u_l)) = (F' \cup X_2 \cup (G - N(F))$$

$$= \langle X_2 \rangle + \langle F' \cup (G - N(F)) \rangle.$$

Where $F' = F - N(u_1, u_2, \cdots, u_l)$, G'' is $(m-l) - K_2$–residual graph, $\langle X_2 \rangle$ and $\langle F' \cup (G - N(F)) \rangle$ are $(m-l) - K_2$–residual graph. Then

$$\nu(G) = |N(u_1, u_2, \cdots, u_l)| + |X_2| + |F'| + |G - N(F)|$$

$$= |N(u_1, u_2, \cdots, u_l) \cup F'| + |X_2| + |G - N(F)|$$

$$\geq \nu(F) + 2(m - l + 1) + 2(m - r) > \nu(F) + 4(m - r)$$

And next, we take the case $m = 6$, $r = 3$, $l = 1$ for an explicit explanation. In order to illustrate it let $X = X_2$, for $u \in F$, then $X = X - N(u) = X_2 \neq \emptyset$, for any $w \in F$, but w does not belong to $N(u)$, then $X - N(u, w) = \emptyset$. And we can construct G as figure 7, then we have

$$G'' = G - N(u) = \langle F \cup X \cup (G - N(F)) \rangle - N(u)$$

$$= (F - N(u)) \cup (X - N(u)) \cup (G - N(F) - N(u))$$

$$= ((F - N(u)) \cup X_2 \cup (G - N(F)).$$

And

$$\nu(G) = |N(u)| + |G''|$$

$$= |N(u)| + |F - N(u)| + |X_2| + |G - N(F)|$$

$$= \nu(F) + |X_2| + |G - N(F)|$$

$$= \nu(F) + 12 + 6 > \nu(F) + 4(6 - 3).$$

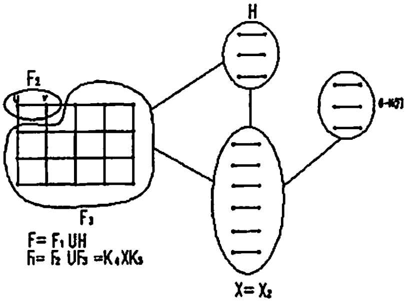

Fig7: 6–K_2–residual graph

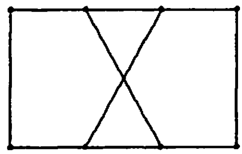

Fig8: 2–K_2–residual graph with order 8

In Fig.7, where $F_1 = K_4 \times K_5 = F_2 \cup F_3$, $F_2 =< u, v >$, $H = K_2 \cup K_2 \cup K_2$, $F = F_1 \cup H$, $X = X_2 = K_2 \cup K_2 \cup K_2 \cup K_2 \cup K_2 \cup K_2$, $G - N(F) = K_2 \cup K_2 \cup K_2$. In Fig.7, we have F_3 is completely adjacent to X and H, X is completely adjacent to and H and $G - N(F)$.

By Lemma 3.3 if $\nu(G) < \nu(F) + 4(m - r)$, there must exist a vertex $v \in G_u$, which makes $G_v = G - N(v)$ connected.

Lemma 3.4 *Assume G is a connected m–K_2–residual graph, then $\delta(G) \geq 2$.*

Proof. Because G is connected $m - K_2$–residual graph, for any $v \in G$, obviously $d(v) \geq 1$. Then we prove $d(v) \neq 1$: when $d(v) = 1$, assume v is adjacent to u and u is adjacent to w, $(u, w \in G)$ if removing w, then v is isolated point, which contradicts Definition 2.3, hence $d(v) > 1$, $\delta(G) \geq 2$. This completes the proof.

Theorem 3.5 *Assume G is a connected $m - K_2$–residual graph, then $\nu(G) \geq 3m + 2$ and the figure 10 is an $m - K_2$–residual graph with order $3m + 2$.*

Proof. The paper adopts mathematical induction to prove the theorem:

When $m = 1$, by Lemma 2.11, $\nu(G) \geq 3 \times 1 + 2 = 5$, $\nu(G) = 5$ is the minimum order, then the conclusion correct and C_5 is the only connected K_2–residual graph with minimum order 5.

When $m = 2$, let G be connected $2 - K_2$–residual graph. For any $v \in G$, if $G - N(v)$ is not connected, by Definition 2.3, then $G_v = G - N(v) \cong 2K_2$. By Lemma 3.1, $\nu(G) \geq 12$. Next we need to prove $\nu(G) \geq 8$, because $\nu(G) = 8 < 12$, by Lemma 3.3, there must exist a $v \in G_u$ which makes $G_v = G - N(v)$ and G_v connected. And because G_v is K_2–residual graph and $\nu(G_v) \geq 5$, by Lemma 3.4, $d(v) \geq 2$, and by Lemma 2.9, $\nu(G) = \nu(G_v) + d(v) + 1 \geq 5 + 2 + 1 = 8$, hence the conclusion is correct when

$m = 2$. $2 - K_2-$residual graph with order 8 as figure 8 can be constructed according to K_2-residual graph .

When $m = 3$, let G be connected $3 - K_2-$residual graph, next we need to prove $\nu(G) \geq 3 \times 3 + 2 = 11$: when $\nu(G) = 11$, $\nu(G)$ is the minimum order, for the reason that $\nu(F) + 4(m-r) = 5 + 4(3-1) = 13 > \nu(G) = 11$, of which F is K_2-residual graph, $r = 1$ and $\nu(F) \geq 5$. By Lemma 3.3, there must exist a $v \in G_u$ which makes $G_v = G - N(v)$ and G_v connected. And because G_v is $2 - K_2-$residual graph, and $\nu(G_v) \geq 8$, by Lemma 3.4, then $d(v) \geq 2$. And by Lemma 2.9, $\nu(G) = \nu(G_v) + d(v) + 1 \geq 8 + 2 + 1 = 11$, hence the conclusion is correct when $m = 3$. According to $2 - K_2-$residual graph, $3 - K_2-$residual graph with order 11 as figure 9 can be constructed.

Assume that the conclusion is correct when less than m. When m, let G be connected $m - K_2-$residual graph, because

$$\nu(F) + 4(m-r) = 3r + 2 + 4m - 4r = 3m + 2 + m - r > \nu(G) = 3m + 2,$$

of which F is $r - K_2-$residual graph($0 \leq r \leq m-2$), by induction hypothesis, then $\nu(F) \geq 3r + 2$. And by Lemma 3.3, for any $u \in G$, then exists $v \in G_u$ which makes $G_v = G - N(v)$ and G_v connected. Since G_v is $(m-1) - K_2-$residual graph, by induction hypothesis, $\nu(G_v) \geq 3(m-1) + 2$. By Lemma 3.4, $d(v) \geq 2$. And because

$$\nu(G) = \nu(G_v) + d(v) + 1 \geq 3(m-1) + 2 + 2 + 1 = 3m + 2,$$

the conclusion is correct and $\nu(G) \geq 3m + 2$. According to the constructions of $2 - K_2-$residual graph and $3 - K_2-$residual graph, we can construct $m - K_2-$residual graph with order $(3m + 2)$ as figure 10.

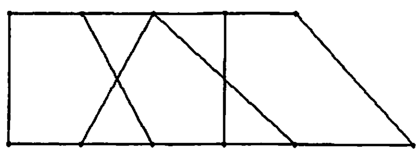

Fig9: 3-K2-residual graph with minimum order 11

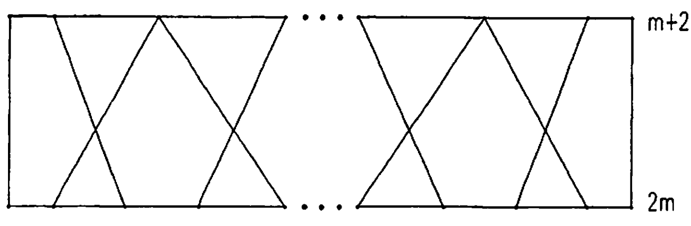

Fig10: m-K2 -residual graph with minimum order 3m+2

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