Neighbor Sum Distinguishing Total Choosability of Graphs with Larger Maximum Average Degree

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Abstract: Let $G = (V,E)$ be a graph and ϕ: $V \cup E \to \{1,2,\dots,\alpha\}$ be a proper α-total coloring of G. Let $f(v)$ denote the sum of the color on vertex v and the colors on the edges incident with v. A neighbor sum distinguishing α-total coloring of G is a proper α-total coloring of G such that for each edge $uv \in E(G)$, $f(u) \neq f(v)$. Pilśniak and Woźniak first introduced this coloring and conjectured that such coloring exists for any simple graph G with maximum degree $\Delta(G)$ if $\alpha \geq \Delta(G)+3$. The maximum average degree of G is the maximum of the average degree of its non-empty subgraphs, which is denoted by mad(G). In this paper, by using the Combinatorial Nullstellensatz and the discharging method, we prove that this conjecture holds for graphs with larger maximum average degree in their list versions. More precisely, we prove that if G is a graph with $\Delta(G) \geq 11$ and mad$(G) < 5$, then $\mathrm{ch}''_{\Sigma}(G) \leq \Delta(G)+3$ (where $\mathrm{ch}''_{\Sigma}(G)$ is the neighbor sum distinguishing total choosability of G)

Keywords: neighbor sum distinguishing total coloring, Combinatorial Nullstellensatz, neighbor sum distinguishing total choosability

1 Introduction

Let G be a graph, we use $V(G)$, $E(G)$, $\delta(G)$ and $\Delta(G)$ to denote the vertex set, edge set, minimum degree and maximum degree of G respectively. Let $d(x)$ denote the degree of a vertex (or face) x in G. An l-, l^--, or l^+-vertex (or face) is a vertex (or face) of degree l, at most l or at least l respectively. The *average degree* of a graph G is $\frac{\sum_{v\in V(G)} d(v)}{|V(G)|}$, we denote it by ad$(G)$. The *maximum average degree* of G is the maximum of the average degree of its non-empty subgraphs and is denoted by mad(G). For a planar graph G two faces are adjacent if they have at least one common edge, and if they have at least one common vertex we call the two faces are intersecting.

Let ϕ: $E(G) \cup V(G) \to \{1,2,\dots,\alpha\}$ be a proper total coloring of G. By $f(v)$, we denote the sum of colors taken on the edges incident with v and color on the vertex v, i.e., $f(v) = \sum_{e\ni v}\phi(e) + \phi(v)$. We call the coloring ϕ such that $f(v) \neq f(u)$ for each edge $uv \in E(G)$ *a neighbor sum distinguishing α-total coloring* (abbrevd. α-tnsd-coloring). The smallest number α is the *neighbor sum distinguishing total chromatic number*, denoted by

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$\mathrm{tndi}_{\Sigma}(G)$. Pilśniak and Woźniak proposed the following conjecture.

Conjecture 1 [7] *If G is a graph with at least two vertices, then* $\mathrm{tndi}_{\Sigma}(G) \leq \Delta(G) + 3$.

In [7] Pilśniak et al. proved that Conjecture 1 holds for complete graphs, cycles, and bipartite graphs. Dong et al. proved the following result.

Theorem 1 [3] *Let G be a graph with at least two vertices. If* $\mathrm{mad}(G) < 3$, *then* $\mathrm{tndi}_{\Sigma}(G) \leq k + 2$, *where* $k = \max\{\Delta(G), 5\}$.

For planar graph, Li et al. [5] proved the following theorem.

Theorem 2 [5] *Let G be a planar graph with $\Delta(G) \geq 13$, then* $\mathrm{tndi}_{\Sigma}(G) \leq \Delta(G) + 3$.

For a given graph G, let $(L_x)_{x \in V(G) \cup E(G)}$ be a set of lists of real numbers, each of size α. The smallest α for which for any specified collection of such lists there exists a neighbor sum distinguishing total coloring using colors from L_x for each $x \in V(G) \cup E(G)$ is the *neighbor sum distinguishing total choosability of G*, and denoted by $\mathrm{ch}''_{\Sigma}(G)$. In [2], Ding et al. proved that if $\Delta(G) = 3$ and $\mathrm{mad}(G) < \frac{20}{7}$, then $\mathrm{ch}''_{\Sigma}(G) \leq 6$. More references see [4, 6, 9, 10, 11, 12, 13]. In [8], Qu et al. proved the following result.

Theorem 3 [8] *For any graph G, if there exists a pair $(k, m) \in \{(6, 4), (5, \frac{18}{5}), (4, \frac{16}{5})\}$ such that G satisfies $\Delta(G) \geq k$ and* $\mathrm{mad}(G) < m$, *then* $\mathrm{ch}''_{\Sigma}(G) \leq \Delta(G) + 3$.

In this paper, we will prove the following results.

Theorem 4 *Let G be a graph with $\Delta(G) \geq 11$ and* $\mathrm{mad}(G) < 5$, *then* $\mathrm{ch}''_{\Sigma}(G) \leq \Delta(G) + 3$.

Corollary 1 *Let G be a planar graph without adjacent triangles with $\Delta(G) \geq 11$, then* $\mathrm{ch}''_{\Sigma}(G) \leq \Delta(G) + 3$.

Obviously, it holds that $\mathrm{tndi}_{\Sigma}(G) \leq \mathrm{ch}''_{\Sigma}(G)$, thus any upper bound proven for $\mathrm{ch}''_{\Sigma}(G)$ is valid for $\mathrm{tndi}_{\Sigma}(G)$, so we have the following corollaries.

Corollary 2 *Let G be a graph with $\Delta(G) \geq 11$ and* $\mathrm{mad}(G) < 5$, *then* $\mathrm{tndi}_{\Sigma}(G) \leq \Delta(G) + 3$.

Corollary 3 *Let G be a planar graph without adjacent triangles with $\Delta(G) \geq 11$, then* $\mathrm{tndi}_{\Sigma}(G) \leq \Delta(G) + 3$.

2 Preliminaries

Let $P(x_1, x_2, \ldots, x_n)$ be a polynomial in n variables, where $n \geq 1$. By $c_P(x_1^{k_1} x_2^{k_2} \cdots x_n^{k_n})$, we denote the coefficient of the monomial $x_1^{k_1} x_2^{k_2} \cdots x_n^{k_n}$ in $P(x_1, x_2, \ldots, x_n)$, where k_i $(1 \leq i \leq n)$ is a non-negative integer. Let $F(x), A(x)$ denote the forbidden colors set and available colors set of x in certain case when we color x, respectively, where $x \in V(G) \cup E(G)$.

Lemma 1 (Alon[1], Combinatorial Nullstellensatz). *Let $\mathbb{F}$ be an arbitrary* *field, and let $P = P(x_1, x_2, \ldots, x_n)$ be a polynomial in $\mathbb{F}[x_1, x_2, \ldots, x_n]$. Suppose the degree $\deg(P)$ of P equals $\sum_{i=1}^{n} k_i$, where each k_i is a non-negative integer, and suppose $c_P(x_1^{k_1} x_2^{k_2} \ldots x_n^{k_n}) \neq 0$. Then if $S_1, S_2, \ldots, S_n$ are subsets of F with $|S_i| > k_i$, there are $s_i \in S_i$ $(i = 1, 2, \ldots, n)$ so that $P(s_1, s_2, \ldots, s_n) \neq 0$.*

Lemma 2 [2] *Let $P(x_1, x_2, \ldots, x_k) = \prod_{1 \le i < j \le k} (x_i - x_j)(\sum_{t=1}^{k} x_t)^b$ be a polynomial in k variables, where $k \geq 2$ and b $(\neq 1)$ is a non-negative integer, then $c_P(x_1^{k-1} x_2^{k+b-2} x_3^{k-3} \cdots x_{k-2}^2 x_{k-1}) \neq 0$.*

Lemma 3 *Let $P(x_1, x_2, \ldots, x_k) = \prod_{1 \le i < j \le k} (x_i - x_j)(\sum_{t=1}^{k} x_t)$ be a polynomial in k variables, where $k \geq 2$, then $c_P(x_1^k x_2^{k-2} x_3^{k-3} \cdots x_{k-2}^2 x_{k-1}) \neq 0$.*

Proof. Let $P_0 = \prod_{1 \le i < j \le k} (x_i - x_j)$, then it can be easily verified that $c_P(x_1^k x_2^{k-2} \ldots x_{k-1}) = c_{P_0}(x_1^{k-1} x_2^{k-2} \cdots x_{k-1}) = 1 \neq 0$.

Lemma 4 *Let G be a planar graph without adjacent triangles, then $\mathrm{mad}(G) < \frac{24}{5}$.*

Proof. Let G be a planar graph without adjacent triangles, H is a subgraph of G, then H is a planar graph without adjacent triangles. We can easily verify the number of 3-faces in H is at most $\lfloor \frac{|E(H)|}{3} \rfloor$. Let $|F(H)|$ denote the number of faces of H. Then $|F(H)| \leq \frac{2|E(H)| - 3\lfloor \frac{|E(H)|}{3} \rfloor}{4} + \lfloor \frac{|E(H)|}{3} \rfloor = \frac{|E(H)|}{2} + \frac{1}{4} \lfloor \frac{|E(H)|}{3} \rfloor \leq \frac{7}{12}|E(H)|$. We recall the Euler's formula and have: $|V(H)| - |E(H)| + |F(H)| = 2$. Then we have $\frac{2|E(H)|}{|V(H)|} = \frac{2|E(H)|}{2 + |E(H)| - |F(H)|} \leq \frac{2|E(H)|}{2 + |E(H)| - \frac{7}{12}|E(H)|} = \frac{2}{\frac{2}{|E(H)|} + \frac{5}{12}} < \frac{24}{5}$ for every subgraph H of G.

3 The proof of Theorem 4

Our proof proceeds by reduction and absurdum. Let connected graph G be a counterexample to Theorem 4 such that $|V(G)| + |E(G)|$ is as small as possible. Obviously, $\Delta(G) \geq 11$. Let $(L_x)_{x \in V \cup E}$ be any given set of lists of real numbers, each of size $\Delta(G) + 3$. Let $\alpha = \Delta(G) + 3$, by the choice of G, any proper subgraph G' of G has a α-tnsd-coloring with numbers in L_x for each $x \in V \cup E$. For any α-tnsd-coloring ϕ of G', in the proof we will extend the coloring ϕ to the desired coloring ϕ' of G to get a contradiction. Let $f(v)$, $f'(v)$ denote the sum of the color on vertex v and the colors of the edges incident with v in the coloring ϕ, ϕ', respectively. Obviously, the desired coloring ϕ' of G will satisfy the following conditions:

(1) $\phi'(u) \neq \phi'(v)$ for every pair u, v of adjacent vertices;

(2) $\phi'(v) \neq \phi'(e)$ for every vertex v and every edge e incident with v;

(3) $\phi'(e) \neq \phi'(e')$ for every pair e, e' of adjacent edges;

(4) $f'(u) \neq f'(v)$ for every $uv \in E(G)$.

Let H be the graph obtained by removing all the 1-vertices and 2-vertices of G. Clearly, when $\Delta(G) \geq 11$, if $d(v) \leq 4$, then we can recolor v easily if necessary to get a coloring as desired, so in the following proof we will omit the colorings of 4^--vertices, that is when we color a vertex (or an edge), the forbidden colors set of it doesn't include the colors on its adjacent (or incident) 4^--vertices.

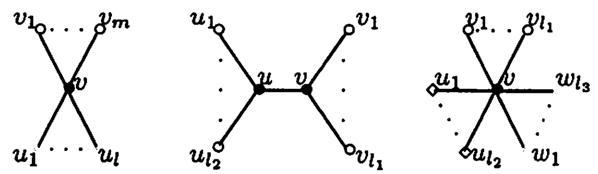

Figure 1

In Figure 1, the neighbors of black vertices are shown in the figure, the degree of grey vertices are one or two, the degree of $'\circ'$ are at least three, and $'\diamond'$ represents vertices whose degrees dependent on the particular case.

Claim 1 $\delta(H) \geq 3$.

Proof. If not, then G contains a subgraph isomorphic to configuration F_1 in Figure 1, where $0 \leq m \leq 2$ and $l \geq 1$. Let $G' = G - \{u_1v, u_2v, \ldots, u_lv\}$, then G' has an α-tnsd-coloring ϕ. Now we prove that we can get a desired coloring ϕ' of G. Clearly, based on the coloring conditions (2) and (3), $|F(u_iv)| \leq m+2$, then $|A(u_iv)| \geq \alpha - (m+2) \geq l+1$, where $1 \leq i \leq l$. Associate with $u_1v, u_2v, \ldots, u_lv$ a variable $x_1, x_2, \ldots, x_l$, respectively. According to the coloring conditions (3) and (4), we can get the following polynomials Q_1 and Q_0:

when $l > 1$,

$$Q_1(x_1, x_2, \ldots, x_l) = \prod_{1 \leq i < j \leq l} (x_i - x_j) \prod_{i=1}^{m} \Big(\sum_{j=1}^{l} x_j + \sum_{j=1}^{m} \phi(vv_j) + \phi(v) - f(v_i)\Big);$$

when $l = 1$,

$$Q_0(x_1) = \prod_{i=1}^{m} \Big(x_1 + \sum_{j=1}^{m} \phi(vv_j) + \phi(v) - f(v_i)\Big).$$

We can get $\widetilde{Q_1}$ and $\widetilde{Q_0}$ as following:

when $l > 1$,

$$\widetilde{Q_1}(x_1, x_2, \ldots, x_l) = \prod_{1 \leq i < j \leq l} (x_i - x_j) \Big(\sum_{j=1}^{l} x_j\Big)^m;$$

when $l = 1$,

$$\widetilde{Q_0}(x_1) = x_1^m.$$

We notice that the coefficient of the monomial which with the highest degree in Q_1 and Q_0 are equal to that in $\widetilde{Q_1}$ and $\widetilde{Q_0}$, respectively.

If $m=0$, then $l \geq 3$, $c_{Q_1}(x_1^{l-1}x_2^{l-2}x_3^{l-3}\cdots x_{l-1}) = c_{\widetilde{Q_1}}(x_1^{l-1}x_2^{l-2}x_3^{l-3}\cdots x_{l-1}) \neq 0$ by Lemma 2. Similarly, if $m=2$, then $l \geq 1$, when $l=1$, $c_{Q_0}(x_1) = c_{\widetilde{Q_0}}(x_1) \neq 0$, when $l>1$, $c_{Q_1}(x_1^{l-1}x_2^{l}x_3^{l-3}\cdots x_{l-1}) = c_{\widetilde{Q_1}}(x_1^{l-1}x_2^{l} x_3^{l-3}\cdots x_{l-1}) \neq 0$. If $m=1$, then $l \geq 2$, $c_{Q_1}(x_1^{l}x_2^{l-2}x_3^{l-3}\cdots x_{l-1}) = c_{\widetilde{Q_1}}(x_1^{l}x_2^{l-2}x_3^{l-3}\cdots x_{l-1}) \neq 0$ by Lemma 3. At last, we can recolor $u_1, \ldots, u_l$ easily. By Lemma 1, those imply that we can get a desired coloring ϕ' of G, which is a contradiction.

Claim 2 For each $v \in V(H)$, if $d_H(v) = k$, then $d_G(v) = k$ $(k = 3, 4, 5)$.

Proof. If not, G contains a subgraph isomorphic to configuration F_1 in Figure 1, where $m = k$ and $l \geq 1$.

Case 2.1: $l = 1$. Let $G' = G - \{u_1v\}$, thus G' has an α-tnsd-coloring ϕ. According to conditions (2) to (4) we have $|F(u_1v)| \leq 2m+2$, then $|A(u_1v)| \geq \alpha - (2m+2) \geq 11+3-(2\times 5+2) = 2$. We can recolor u_1 easily. This implies that we get a desired coloring ϕ' of G, which is a contradiction.

Case 2.2: $l = 2$. Let $G' = G - \{u_1v, u_2v\}$, thus G' has an α-tnsd-coloring ϕ. According to conditions (2) and (3), we have $|F(u_iv)| \leq m+2$, then $|A(u_iv)| \geq \alpha - (m+2) \geq 11+3-(5+2) = 7$, where $i = 1$ or 2. Associate with u_1v, u_2v a variable x_1, x_2, respectively. Based on the coloring conditions (3) and (4), we get the following polynomial Q_2:

$$Q_2(x_1,x_2) = (x_1 - x_2)\prod_{i=1}^{m}(\sum_{j=1}^{m}\phi(vv_j) + \sum_{j=1}^{2}x_j + \phi(v) - f(v_i)).$$

Similar to Claim 1's calculation method, we can get $c_{Q_2}(x_1^{m+1}) \neq 0$, where $m+1 = k+1 \leq 6$. At last we can recolor u_1, u_2 easily. By Lemma 1, we get a desired coloring ϕ' of G, which is a contradiction.

Case 2.3: $l \geq k$. Let $G' = G - \{u_1v, u_2v, \ldots, u_kv\}$, thus we have an α-tnsd-coloring ϕ of G'. According to conditions (2) and (3), we have $|F(u_iv)| \leq m + (l-k) + 2$, then $|A(u_iv)| \geq \alpha - (m+l-k+2) \geq k+1$, where $1 \leq i \leq k$. Associate with $u_1v, u_2v, \ldots, u_kv$ a variable $x_1, x_2, \ldots, x_k$, respectively. Based on the conditions (3) and (4), we get Q_3:

$$Q_3(x_1,x_2,\ldots,x_k) = \prod_{1\leq i<j\leq k}(x_i - x_j)\prod_{i=1}^{m}(\sum_{j=1}^{m}\phi(vv_j) + \sum_{j=k+1}^{l}\phi(u_jv) + \sum_{j=1}^{k}x_j + \phi(v) - f(v_i)),$$

where if $k+1 > l$, set $\sum_{j=k+1}^{l}\phi(u_jv) = 0$. Similar to Claim 1's calculation method, we can calculate by Matlab and get $c_{Q_3}(x_1^{k}x_2^{k-1}\cdots x_k) \neq 0$. At last, we recolor $u_1, u_2, \ldots, u_k$ easily. By Lemma 1, we get a desired coloring ϕ' of G, which is a contradiction.

Case 2.4: $k = 4$ or 5, $l = 3$ or $k = 5$, $l = 4$. Let $G' = G - \{u_1v, u_2v, \ldots, u_lv\}$. Similar to Case 2.1, we have $|F(u_iv)| \leq m+2$, then $|A(u_iv)| \geq \alpha - (m+2) = \Delta(G) - m + 1 \geq 11 - 5 + 1 = 7$. Associate with $u_1v, u_2v, \ldots, u_lv$ a variable $x_1, x_2, \ldots, x_l$, respectively. Based on the coloring conditions (3) and (4), we get the following polynomial Q_4:

$$Q_4(x_1, x_2, \ldots, x_l) = \prod_{1\leq i<j\leq l}(x_i - x_j)\prod_{i=1}^{m}(\sum_{j=1}^{m}\phi(vv_j) + \sum_{j=1}^{l}x_j + \phi(v) - f(v_i)).$$

Similar to Claim 1's calculation method, and calculated by Matlab we can get if $l = 3$, $k = 4$, $c_{Q_4}(x_1^6x_2) \neq 0$; if $l = 3$, $k = 5$, $c_{Q_4}(x_1^5x_2^3) \neq 0$; if $l = 4$, $k = 5$, $c_{Q_4}(x_1^5x_2^4x_3^2) \neq 0$. At last, we can recolor $u_1, u_2, \ldots, u_l$ easily. By Lemma 1, we get a desired coloring ϕ' of G, which is a contradiction.

Claim 3 In H, any 4^--vertex is not adjacent to another 4^--vertex.

Proof. If otherwise, by Claim 1 and Claim 2, we have G contains a subgraph isomorphic to configuration F_2 in Figure 1, where $2 \leq l_1, l_2 \leq 3$. Let $G' = G - \{uv\}$, thus we have an α-tnsd-coloring ϕ of G'. To get a desired coloring ϕ' of G, based on conditions (2) to (4), $|F(uv)| = |\{\phi(uu_1), \phi(uu_2), \ldots, \phi(uu_{l_2}), \phi(vv_1), \phi(vv_2), \ldots, \phi(vv_{l_1})\}| \leq 6$, then $|A(uv)| \geq \alpha - 6 \geq 8$. At last recolor $u, u_1, u_2, \ldots, u_{l_2}, v, v_1, \ldots, v_{l_1}$. This implies that we can get a desired coloring ϕ' of G, which is a contradiction.

Claim 4 In H, any k-vertex is not adjacent to 4^--vertex, where $k = 5, 6$.

Proof. If not, then G contains a subgraph isomorphic to configuration F_3 in Figure 1, where $l_1 = k-1, l_2 = 1, l_3 \geq 0$ and u_1 is a 4^--vertex.

Case 4.1: $l_3 = 0$. Let $G' = G - \{u_1v\}$, thus we have an α-tnsd-coloring ϕ of G'. To get a desired coloring ϕ' of G, we first erase the color on v. Then according to conditions (1) to (3), we have $|F(u_1v)| \leq k+2$, $|F(v)| \leq 2(k-1)$, then $|A(u_1v)| \geq \alpha - (k+2) \geq 6$, $|A(v)| \geq \alpha - 2(k-1) \geq 4$. Associate with u_1v, v a variable x_1, x_2, respectively. Based on the coloring conditions (2) and (4), we get the following polynomial Q_5:

$$Q_5(x_1, x_2) = (x_1 - x_2)\prod_{i=1}^{l_1}(\sum_{j=1}^{l_1}\phi(vv_j) + x_1 + x_2 - f(v_i)).$$

It can be easily calculated that $c_{Q_5}(x_1^5) \neq 0$, if $k = 5$; $c_{Q_5}(x_1^4x_2^2) \neq 0$, if $k = 6$. At last we can recolor u_1 easily. This implies that we can get a desired coloring ϕ' of G, which is a contradiction.

Case 4.2: $l_3 \geq 1$. By Claim 2, $k = 6$. Let $G' = G - \{u_1v\}$, thus we have an α-tnsd-coloring ϕ of G'. To prove there is a desired coloring ϕ' of G, erase the colors of $v, vw_1, vw_2, \ldots, vw_t$, where $t = l_3$, if $l_3 = 1$ or 2; otherwise $t = 3$. Then we have $|F(u_1v)| \leq l_3 - t + 8$, $|F(vw_i)| \leq l_3 - t + 6$, $|F(v)| \leq l_3 - t + 10$ according to conditions (1) to (3). Associate with

$u_1v, vw_1, \ldots, vw_t, v$ a variable $x_1, x_2, \ldots, x_{t+1}, x_{t+2}$, respectively. Based on conditions (2) to (4), we get Q_6:

$$Q_6(x_1, x_2, \ldots, x_{t+2}) = \prod_{1\le i<j\le t+2}(x_i - x_j)\prod_{i=1}^{l_1}(\sum_{j=1}^{l_1}\phi(vv_j) + \sum_{j=t+1}^{l_3}\phi(vw_j) + \sum_{j=1}^{t+2}x_j - f(v_i)),$$

where if $k+1 > l_3$, set $\sum_{j=k+1}^{l_3}\phi(vw_j) = 0$. If $l_3 = 1$ or 2, $|A(u_1v)| \ge \alpha - (l_3 - t + 8) = \Delta(G) - 5 \ge 6$, $|A(vw_i)| \ge \alpha - (l_3 - t + 6) \ge 8$, $|A(v)| \ge \alpha - (l_3 - t + 10) \ge 4$, and $c_{Q_6}(x_1^4x_2^3x_3) \neq 0$, or $c_{Q_6}(x_1^4x_2^5x_3^2) \neq 0$, respectively. If $l_3 \ge 3$, $|A(u_1v)| \ge \alpha - (l_3 - t + 8) \ge k + t - 5 \ge 4$, $|A(vw_i)| \ge \alpha - (l_3 - t + 6) \ge 6$, $|A(v)| \ge \alpha - (l_3 - t + 10) \ge 2$, and $c_{Q_6}(x_1^2x_2^5x_3^4x_4^3x_5) \neq 0$. At last, we can recolor $u_1, w_1, \ldots, w_{l_3}$ easily. This implies that we can get a desired coloring ϕ' of G, which is a contradiction.

Claim 5 In H, each 7-vertex is adjacent to at most one 4^--vertex.

Proof. If otherwise, then G contains a subgraph isomorphic to configuration F_3 in Figure 1, where $l_1 = 5, l_2 = 2, l_3 \ge 0$ and u_1, u_2 are 4^--vertices.

Case 5.1: $l_3 = 0$. Let $G' = G - \{u_1v, u_2v\}$, thus we have an α-tnsd-coloring ϕ of G'. To get a desired coloring ϕ' of G, we first erase the color of v. $|F(u_iv)| \le 8$, $|F(v)| \le 10$ according to conditions (1) to (3), then $|A(u_iv)| \ge \alpha - 8 \ge 6$, $|A(v)| \ge \alpha - 10 \ge 4$, where $i = 1$ or 2. Associate with u_1v, u_2v, v a variable x_1, x_2, x_3, respectively. Based on the coloring conditions (2) to (4), we get the following polynomial Q_7:

$$Q_7(x_1, x_2, x_3) = \prod_{1\le i<j\le 3}(x_i - x_j)\prod_{i=1}^{l_1}(\sum_{j=1}^{l_1}\phi(vv_j) + \sum_{j=1}^{3}x_j - f(v_i)).$$

Similar to Claim 1's calculation method, we can calculate that $c_{Q_7}(x_1^4x_2^3x_3) \neq 0$. At last, we can recolor u_1, u_2 easily. According to Lemma 1, this implies that we can get a desired coloring ϕ' of G, which is a contradiction.

Case 5.2: $l_3 \ge 1$. Let $G' = G - \{u_1v, u_2v\}$ and erase the colors of $v, vw_1, vw_2, \ldots, vw_k$, where if $l_3 = 1$, set $k = 1$, otherwise set $k = 2$. Then $|F(u_iv)| \le l_3 - k + 8$, $|F(vw_j)| \le l_3 - k + 6$, $|F(v)| \le l_3 - k + 10$ according to conditions (1) to (3), where $1 \le i \le 2$, $1 \le j \le k$. Then if $l_3 = 1$, $|A(u_iv)| \ge \alpha - (l_3 - k + 8) = \Delta(G) - 5 \ge 6$, $|A(vw_1)| \ge \alpha - (l_3 - k + 6) \ge 8$, $|A(v)| \ge \alpha - (l_3 - k + 10) \ge 4$; if $l_3 \ge 2$, $|A(u_iv)| \ge \alpha - (l_3 - k + 8) \ge l_1 + l_2 + k - 5 = 4$, $|A(vw_j)| \ge \alpha - (l_3 - k + 6) \ge 6$, $|A(v)| \ge \alpha - (l_3 - k + 10) \ge 2$, where $1 \le i \le 2$, $1 \le j \le k$. Associate with $u_1v, u_2v, vw_1, \ldots, vw_k, v$ a variable $x_1, x_2, \ldots, x_{k+3}$, respectively. Based on the coloring conditions (2) to (4), we get the following polynomial Q_8:

$$Q_8(x_1,x_2,\ldots,x_{k+3}) = \prod_{1\le i<j\le k+3}(x_i-x_j)\prod_{i=1}^{l_1}(\sum_{j=1}^{l_1}\phi(vv_j)+\sum_{j=k+1}^{l_3}\phi(vw_j)+\sum_{j=1}^{k+3}x_j - f(v_i)),$$

where if $k+1 > l_3$, set $\sum_{j=k+1}^{l_3}\phi(vw_j) = 0$. Similar to Claim 1, we have $c_{Q_8}(x_1^4x_2^2x_3^5) \neq 0$, if $l_3 = 1$; $c_{Q_8}(x_1^3x_2^2x_3^5x_4^4x_5) \neq 0$, if $l_3 \geq 2$. At last, we can recolor u_1, u_2 easily. This implies that we can get a desired coloring ϕ' of G according to Lemma 1, which is a contradiction.

Claim 6 In H, for each 8-vertex $v \in V(H)$, the followings hold.

(a) v is adjacent to at most three 3-vertices.

(b) If v adjacent to three 3-vertices, then v is adjacent to at most one 4-vertex.

Proof. (a) If otherwise, G contains a subgraph isomorphic to configuration F_3 in Figure 1, where $l_1 = 4, l_2 = 4, l_3 \geq 0$ and $u_1, u_2, \ldots, u_{l_2}$ are 3-vertices.

Case 6.1: $l_3 = 0$. Let $G' = G - \{u_1v, u_2v, \ldots, u_{l_2}v\}$, then we have an α-tnsd-coloring ϕ of G'. According to conditions (2) and (3), $|F(u_iv)| \leq 7$, then $|A(u_iv)| \geq \alpha - 7 \geq 7$, where $1 \leq i \leq l_2$. Associate with $u_1v, u_2v, \ldots, u_{l_2}v$ a variable $x_1, x_2, \ldots, x_{l_2}$, respectively. Based on the coloring conditions (3) and (4), we get the following polynomial Q_9:

$$Q_9(x_1,x_2,\ldots,x_{l_2}) = \prod_{1\le i<j\le l_2}(x_i-x_j)\prod_{i=1}^{l_1}(\sum_{j=1}^{l_1}\phi(vv_j)+\sum_{j=1}^{l_2}x_j+\phi(v)-f(v_i)).$$

We can calculate that $c_{Q_9}(x_1^5x_2^4x_3) = 2$. At last, recolor $u_1, u_2, \ldots, u_{l_2}$. By Lemma 1, we get a contradiction.

Case 6.2: $l_3 \geq 1$. Let $G' = G - \{u_1v, u_2v, \ldots, u_{l_2}v\}$, thus we have an α-tnsd-coloring ϕ of G'. To get a desired coloring ϕ' of G, we first erase the colors of $vw_1, vw_2, \ldots, vw_k$, where if $l_3 = 1$, set $k = 1$; otherwise set $k = 2$. We have $|F(vw_j)| \leq l_3 - k + 6$, $|F(u_iv)| \leq l_3 - k + 7$ according to conditions (2) and (3), where $1 \leq i \leq l_2, 1 \leq j \leq k$. Then if $l_3 = 1$, $|A(vw_j)| \geq \alpha - (l_3 - k + 6) = \Delta(G) - 3 \geq 8$, $|A(u_iv)| \geq \alpha - (l_3 - k + 7) \geq 7$; if $l_3 \geq 2$, $|A(vw_j)| \geq \alpha - (l_3 - k + 6) \geq l_1 + l_2 + k - 3 = 7$, $|A(u_iv)| \geq \alpha - (l_3 - k + 7) \geq 6$, where $1 \leq i \leq l_2, 1 \leq j \leq k$. Associate with $vw_1, vw_2, \ldots, vw_k, u_1v, u_2v, \ldots, u_{l_2}v$ a variable $x_1, x_2, \ldots, x_{l_2+k}$, respectively. Based on the coloring conditions (3) and (4), we get the following polynomial Q_{10}:

$$Q_{10}(x_1,x_2,\ldots,x_{l_2+k}) = \prod_{1\le i<j\le l_2+k}(x_i-x_j)\prod_{i=1}^{l_1}(\sum_{j=1}^{l_1}\phi(vv_j)+\sum_{j=1}^{l_2+k}x_j+\sum_{j=k+1}^{l_3}\phi(vw_j)+\phi(v)-f(v_i)),$$

where if $k+1 > l_3$, set $\sum_{j=k+1}^{l_3} \phi(vw_j) = 0$. Similar to Claim 1, we have $c_{Q_{10}}(x_1^5 x_2^4 x_3^3 x_4^2) \neq 0$, if $l_3 = 1$; $c_{Q_{10}}(x_1^5 x_2^4 x_3^3 x_4 x_5^6) \neq 0$, if $l_3 \geq 2$. At last, recolor $u_1, u_2, \dots, u_{l_2}, w_1, w_2, \dots, w_k$. By Lemma 1, we get a contradiction.

(b) If otherwise, G contains a subgraph isomorphic to configuration F_3 in Figure 1, where $l_1 = 3, l_2 = 5, l_3 \geq 0$ and $u_1, u_2, \dots, u_{l_2-2}$ are 3-vertices, u_{l_2-1}, u_{l_2} are 4-vertices.

Case 6.3: $l_3 = 0$. Let $G' = G - \{u_1v, u_2v, \dots, u_{l_2}v\}$, thus we have an α-tnsd-coloring ϕ of G'. Now we prove that we can get a desired coloring ϕ' of G. We have $|F(u_iv)| \leq 6$, $|F(u_jv)| \leq 7$ according to conditions (2) and (3), then $|A(u_iv)| \geq \alpha - 6 \geq 8$, $|A(u_jv)| \geq \alpha - 7 \geq 7$, where $1 \leq i \leq l_2 - 2, l_2 - 1 \leq j \leq l_2$. Associate with $u_1v, u_2v, \dots, u_{l_2}v$ a variable $x_1, x_2, \dots, x_{l_2}$ respectively. Based on the coloring conditions (3) and (4), we get the following polynomial Q_{11}:

$$Q_{11}(x_1, x_2, \dots, x_{l_2}) = \prod_{1 \leq i < j \leq l_2} (x_i - x_j) \prod_{i=1}^{l_1} \Big(\sum_{j=1}^{l_1} \phi(vv_j) + \sum_{j=1}^{l_2} x_j + \phi(v) - f(v_i)\Big).$$

Similar to the calculation method in Claim 1 and by Lemma 2, we get $c_{Q_{11}}(x_1^4 x_2^6 x_3^2 x_4) \neq 0$. At last, we can recolor $u_1, u_2, \dots, u_{l_2}$ easily. By Lemma 1, we get a contradiction.

Case 6.4: $l_3 \geq 1$. Let $G' = G - \{u_1v, u_2v, \dots, u_{l_2}v\}$, thus we have an α-tnsd-coloring ϕ of G'. To get a desired coloring ϕ' of G, first we erase the colors of $vw_1, vw_2, \dots, vw_k$, where if $l_3 = 1$, set $k = 1$; otherwise set $k = 2$. We have $|F(u_iv)| \leq l_3 - k + 6$, $|F(u_jv)| \leq l_3 - k + 7$, $|F(vw_t)| \leq l_3 - k + 5$ according to conditions (2) and (3), where $1 \leq i \leq l_2 - 2, l_2 - 1 \leq j \leq l_2, 1 \leq t \leq k$. Then if $l_3 = 1$, $|A(vw_t)| \geq \alpha - (l_3 - k + 5) = \Delta(G) - 2 \geq 9$, $|A(u_iv)| \geq \alpha - (l_3 - k + 6) \geq 8$, $|A(u_jv)| \geq \alpha - (l_3 - k + 7) \geq 7$; if $l_3 \geq 2$, $|A(vw_t)| \geq \alpha - (l_3 - k + 5) \geq l_1 + l_2 + k - 2 = 8$, $|A(u_iv)| \geq \alpha - (l_3 - k + 6) \geq 7$, $|A(u_jv)| \geq \alpha - (l_3 - k + 7) \geq 6$, where $1 \leq i \leq l_2 - 2, l_2 - 1 \leq j \leq l_2, 1 \leq t \leq k$. Associate with $vw_1, vw_2, \dots, vw_k, u_1v, u_2v, \dots, u_{l_2}v$ a variable $x_1, x_2, \dots, x_{l_2+k}$ respectively. Based on the coloring conditions (3) and (4), we get the following polynomial Q_{12}:

$$Q_{12}(x_1, x_2, \dots, x_{l_2+k}) = \prod_{1 \leq i < j \leq l_2+k} (x_i - x_j) \prod_{i=1}^{l_1} \Big(\sum_{j=1}^{l_1} \phi(vv_j) + \sum_{j=k+1}^{l_3} \phi(vw_j) + \sum_{j=1}^{l_2+k} x_j + \phi(v) - f(v_i)\Big),$$

where if $k+1 > l_3$, set $\sum_{j=k+1}^{l_3} \phi(vw_j) = 0$. By Lemma 2, we get $c_{Q_{12}}(x_1^5 x_2^7 x_3^3 x_4^2 x_5) \neq 0$, if $l_3 = 1$; if $l_3 \geq 2$, similar to the calculation method in Claim 1, we can easily calculate that $c_{Q_{12}}(x_1^7 x_2^6 x_4 x_5^5 x_6^2 x_7^3) \neq 0$. At last, we can recolor $u_1, u_2, \dots, u_{l_2}, w_1, w_2, \dots, w_k$ easily. By Lemma 1, we get a desired coloring ϕ' of G, which is a contradiction.

Claim 7 In H, each 9-vertex is adjacent to at most four 3-vertices.

Proof. If otherwise, that is G contains a subgraph isomorphic to configuration F_3 in Figure 1, where $l_1 = 4, l_2 = 5, l_3 \geq 0$ and $u_1, u_2, \ldots, u_{l_2}$ are 3-vertices.

The proof of this situation is the same as Claim 6(a). Similar to Claim 1, we can calculate that in Case 6.1, $c_{Q_9}(x_1^5 x_2^4 x_3^3 x_4^2) \neq 0$. In Case 6.2, if $l_3 = 1$, $c_{Q_{10}}(x_1^5 x_2^4 x_3 x_5^3 x_6^6) \neq 0$; if $l_3 \geq 2$, we have $|A(u_i v)| \geq \alpha - (l_3 - k + 7) \geq 7$, $|A(vw_j)| \geq \alpha - (l_3 - k + 6) \geq 8$, $c_{Q_{10}}(x_1^6 x_2^5 x_4^4 x_5 x_6^7 x_7^2) \neq 0$.

In conclusion, those all imply that we can get a desired coloring ϕ' of G, which is a contradiction.

Claim 8 In H, for each 10-vertex $v \in V(H)$, the followings hold.

(a) v is adjacent to at most seven 3-vertices.

(b) If v adjacent to seven 3-vertices, then v is not adjacent to 4-vertex.

Proof. (a) If otherwise, G contains a subgraph isomorphic to configuration F_3 in Figure 1, where $l_1 = 2, l_2 = 8, l_3 \geq 0$ and $u_1, u_2, \ldots, u_{l_2}$ are 3-vertices.

Case 8.1: $l_3 = 0$. Let $G' = G - \{u_1 v, u_2 v, \ldots, u_{l_2} v\}$, thus we have an α-tnsd-coloring ϕ of G'. To get a desired coloring ϕ' of G, we first erase the color of v. We have $|F(u_i v)| \leq 4$, $|F(v)| \leq 4$ according to conditions (1) to (3), then $|A(u_i v)| \geq 10$, $|A(v)| \geq 10$, where $1 \leq i \leq l_2$. Associate with $v, u_1 v, u_2 v, \ldots, u_{l_2} v$ a variable $x_1, x_2, \ldots, x_{l_2+1}$, respectively. Based on the conditions (2) to (4), we get the following polynomial Q_{13}:

$$Q_{13}(x_1, x_2, \ldots, x_{l_2+1}) = \prod_{1 \leq i < j \leq l_2+1} (x_i - x_j) \prod_{i=1}^{l_1} \Big(\sum_{j=1}^{l_1} \phi(vv_j) + \sum_{j=1}^{l_2+1} x_j - f(v_i)\Big).$$

Similar to Claim 1 and by Lemma 2, we have $c_{Q_{13}}(x_1^8 x_2^9 x_3^6 x_4^5 x_5^4 x_6^3 x_7^2 x_8) \neq 0$.

Case 8.2: $l_3 \geq 1$. The proof of this situation is the same as Case 6.2, where $|F(u_i v)| \leq l_3 - k + 5$, $|F(vw_j)| \leq l_3 - k + 4$, then if $l_3 = 1$, $|A(u_i v)| \geq \alpha - (l_3 - k + 5) = \Delta(G) - 2 \geq 9$, $|A(vw_j)| \geq \alpha - (l_3 - k + 4) \geq 10$; if $l_3 \geq 2$, $|A(u_i v)| \geq \alpha - (l_3 - k + 5) \geq l_1 + l_2 + k - 2 = 10$, $|A(vw_j)| \geq \alpha - (l_3 - k + 4) \geq 11$, where $1 \leq i \leq l_2$, $1 \leq j \leq k$. According to Lemma 2, if $l_3 = 1$, $c_{Q_{10}}(x_1^8 x_2^9 x_3^6 x_4^5 x_5^4 x_6^3 x_7^2 x_8) \neq 0$; if $l_3 \geq 2$, $c_{Q_{10}}(x_1^9 x_2^{10} x_3^7 x_4^6 x_5^5 x_6^4 x_7^3 x_8^2 x_9) \neq 0$.

In conclusion, those all imply that we can get a desired coloring ϕ' of G, which is a contradiction.

(b) If otherwise, G contains a subgraph isomorphic to configuration F_3 in Figure 1, where $l_1 = 2, l_2 = 8, l_3 \geq 0$ and $u_1, u_2, \ldots, u_{l_2-1}$ are 3-vertices, u_{l_2} is 4-vertex.

Case 8.3: $l_3 = 0$. The proof of this situation is the same as Case 8.1, where $|F(u_i v)| \leq 4$, $|F(u_{l_2} v)| \leq 5$, $|F(v)| \leq 4$, then $|A(u_i v)| \geq 10$, $|A(u_{l_2} v)| \geq 9$, $|A(v)| \geq 10$, where $1 \leq i \leq l_2 - 1$. According to Lemma 2, we have $c_{Q_{13}}(x_1^8 x_2^9 x_3^6 x_4^5 x_5^4 x_6^3 x_7^2 x_8) \neq 0$.

Case 8.4: $l_3 \geq 1$. The proof of this situation is the same as Case 6.2, where $|F(u_i v)| \leq l_3 - k + 5$, $|F(u_{l_2} v)| \leq l_3 - k + 6$, $|F(vw_j)| \leq l_3 - k + 4$, then if $l_3 = 1$, $|A(u_iv)| \geq 9$, $|A(u_{l_2}v)| \geq 8$, $|A(vw_j)| \geq 10$; if $l_3 \geq 2$, $|A(u_iv)| \geq 10$, $|A(u_{l_2}v)| \geq 9$, $|A(vw_j)| \geq 11$, where $1 \leq i \leq l_2-1, 1 \leq j \leq k$. According to Lemma 2, if $l_3 = 1$, $c_{Q_{10}}(x_1^8x_2^9x_3^6x_4^5x_5^4x_6^3x_7^2x_8) \neq 0$; if $l_3 \geq 2$, $c_{Q_{10}}(x_1^9x_2^{10}x_3^7x_4^6x_5^5x_6^4x_7^3x_8^2x_9) \neq 0$.

In conclusion, those all imply that we can get a desired coloring ϕ' of G, which is a contradiction.

Claim 9 In H, for each 11^+-vertex $v \in V(H)$, the followings hold.

(a) Vertex v is adjacent to at most $d_H(v) - 2$ 3-vertices.

(b) If v adjacent to $d_H(v) - 2$ 3-vertices, then v doesn't adjacent to 4-vertex.

(c) If v adjacent to $d_H(v) - 3$ 3-vertices, then v is adjacent to at most two 4-vertices.

Proof. (a) If not, G contains a subgraph isomorphic to configuration F_3, where $l_1 = 1, l_2 = d_H(v) - 1, l_3 \geq 0$, and $u_1, u_2, \ldots, u_{l_2}$ are 3-vertices.

Case 9.1: $l_3 = 0$.The proof of this situation is the same as Case 8.1, where $|F(v)| \leq 2$, $|F(u_iv)| \leq 3$, then $|A(v)| \geq l_2 + 2$, $|A(u_iv)| \geq l_2 + 1$, where $1 \leq i \leq l_2$. Similar to the calculation method of Claim 1 and by Lemma 3, we have $c_{Q_{13}}(x_1^{l_2+1}x_2^{l_2-1}x_3^{l_2-2}\cdots x_{l_2-1}^2x_{l_2}) \neq 0$.

Case 9.2: $l_3 \geq 1$. The proof of this situation is the same as Case 6.2, where $|F(vw_j)| \leq l_3 - k + 3$, $|F(u_iv)| \leq l_3 - k + 4$, then if $l_3 = 1$, $|A(vw_j)| \geq l_2 + 2$, $|A(u_iv)| \geq l_2 + 1$; if $l_3 \geq 2$, $|A(vw_j)| \geq l_2 + 3$, $|A(u_iv)| \geq l_2 + 2$, where $1 \leq i \leq l_2, 1 \leq j \leq k$. According to Lemma 3, we have if $l_3 = 1$, $c_{Q_{10}}(x_1^{l_2+1}x_2^{l_2-1}x_3^{l_2-2}\cdots x_{l_2-1}^2x_{l_2}) \neq 0$; if $l_3 \geq 2$, $c_{Q_{10}}(x_1^{l_2+2}x_2^{l_2}x_3^{l_2-1}\cdots x_{l_2}^2x_{l_2+1}) \neq 0$.

In conclusion, those all imply that we can get a desired coloring ϕ' of G, which is a contradiction.

(b): If otherwise, G contains a subgraph isomorphic to configuration F_3 in Figure 1, where $l_1 = 1, l_2 = d_H(v) - 1, l_3 \geq 0$ and $u_1, u_2, \ldots, u_{l_2-1}$ are 3-vertices, u_{l_2} is 4-vertex.

Case 9.3: $l_3 = 0$. Let $G' = G - \{u_1v, u_2v, \ldots, u_{l_2}v\}$, thus we have a desired coloring ϕ of G'. To get a desired coloring ϕ' of G, we first erase the color of v. We have $|F(v)| \leq 2$, $|F(u_iv)| \leq 3$, $|F(u_{l_2}v)| \leq 4$ according to conditions (1) to (3), then $|A(v)| \geq \alpha - 2 \geq l_2 + 2$, $|A(u_iv)| \geq l_2 + 1$, $|A(u_{l_2}v)| \geq l_2$, where $1 \leq i \leq l_2 - 1$. Associate with $v, u_1v, u_2v, \ldots, u_{l_2}v$ a variable $x_1, \ldots, x_{l_2+1}$, respectively. Based on the coloring conditions (2) to (4), we get the following polynomial Q_{14}:

$$Q_{14}(x_1, x_2, \ldots, x_{l_2+1}) = \prod_{1 \leq i < j \leq l_2+1} (x_i - x_j)(\sum_{j=1}^{l_2+1} x_j + \phi(vv_1) - f(v_1)).$$

By Lemma 3, we get $c_{Q_{14}}(x_1^{l_2+1}x_2^{l_2-1}x_3^{l_2-2}\cdots x_{l_2-1}^2x_{l_2}) \neq 0$. At last, we can recolor $u_1, u_2, \ldots, u_{l_2}$ easily. This implies that we can get a desired coloring ϕ' of G, which is a contradiction.

Case 9.4: $l_3 \geq 1$. The proof of this situation is the same as Case 6.4, where $|F(vw_j)| \leq l_3 - k + 3$, $|F(u_iv)| \leq l_3 - k + 4$, $|F(u_{l_2}v)| \leq l_3 - k + 5$, then if $l_3 = 1$, $|A(vw_j)| \geq \alpha - (l_3 - k + 3) \geq l_2 + 2$, $|A(u_iv)| \geq \alpha - (l_3 - k + 4) \geq l_2 + 1$, $|A(u_{l_2}v)| \geq \alpha - (l_3 - k + 5) \geq l_2$, if $l_3 \geq 2$, $|A(vw_j)| \geq \alpha - (l_3 - k + 3) \geq l_2 + 3$, $|A(u_iv)| \geq \alpha - (l_3 - k + 4) \geq l_2 + 2$, $|A(u_{l_2}v)| \geq \alpha - (l_3 - k + 5) \geq l_2 + 1$, where $1 \leq i \leq l_2 - 1, 1 \leq j \leq k$. According to Lemma 3, we have if $l_3 = 1$, $c_{Q_{12}}(x_1^{l_2+1}x_2^{l_2-1}x_3^{l_2} \cdots x_{l_2-1}^2 x_{l_2}) \neq 0$; if $l_3 \geq 2$, $c_{Q_{12}}(x_1^{l_2+2}x_2^{l_2}x_3^{l_2-1} \cdots x_{l_2}^2 x_{l_2+1}) \neq 0$. At last, we can recolor $u_1, u_2, \ldots, u_{l_2}, w_1, w_2, \ldots, w_k$ easily. This implies that we can get a desired coloring ϕ' of G, which is a contradiction.

(c): Suppose this claim is false, that is G contains a subgraph isomorphic to configuration F_3, where $v_1, v_2, \ldots, v_{l_1}$ don't exist, $l_2 = d_H(v), l_3 \geq 0$ and $u_1, u_2, \ldots, u_{l_2-3}$ are 3-vertices, $u_{l_2-2}, u_{l_2-1}, u_{l_2}$ are 4-vertices.

Case 9.5: $l_3 = 0$. Let $G' = G - \{u_1v, u_2v, \ldots, u_{l_2}v\}$, thus we have an α-tnsd-coloring ϕ of G'. We have $|F(u_iv)| \leq 3$, $|F(u_jv)| \leq 4$, according to conditions (2) and (3), then $|A(u_iv)| \geq \alpha - 3 \geq l_2$, $|A(u_jv)| \geq l_2 - 1$, where $1 \leq i \leq l_2 - 3, l_2 - 2 \leq j \leq l_2$. Associate with $u_1v, u_2v, \ldots, u_{l_2}v$ a variable $x_1, x_2, \ldots, x_{l_2}$, respectively. Based on the coloring condition (3), we get the following polynomial Q_{15}:

$$Q_{15}(x_1, x_2, \ldots, x_{l_2}) = \sum_{1 \leq i < j \leq l_2} (x_i - x_j).$$

By Lemma 2, we have $c_{Q_{15}}(x_1^{l_2-1}x_2^{l_2-2}x_3^{l_2-3} \cdots x_{l_2-1}) \neq 0$. We can recolor $u_1, u_2, \ldots, u_{l_2}$ easily. This implies that we can get a desired coloring ϕ' of G, which is a contradiction.

Case 9.6: $l_3 \geq 1$. Let $G' = G - \{u_1v, u_2v, \ldots, u_{l_2}v\}$, thus we have an α-tnsd-coloring ϕ of G'. To get a desired coloring ϕ' of G, we erase the color of vw_1. We have $|F(vw_1)| \leq l_3 + 1$, $|F(u_iv)| \leq l_3 + 2$, $|F(u_jv)| \leq l_3 + 3$ according to conditions (2) and (3), then $|A(vw_1)| \geq \alpha - (l_3 + 1) \geq l_2 + 2$, $|A(u_iv)| \geq l_2 + 1$, $|A(u_jv)| \geq l_2$, where $1 \leq i \leq l_2 - 3, l_2 - 2 \leq j \leq l_2$. Associate with $vw_1, u_1v, u_2v, \ldots, u_{l_2}v$ a variable $x_1, x_2, \ldots, x_{l_2+1}$ respectively. Based on the coloring condition (3), we get the following polynomial Q_{16}:

$$Q_{16}(x_1, x_2, \ldots, x_{l_2+1}) = \prod_{1 \leq i < j \leq l_2+1} (x_i - x_j).$$

By Lemma 2, we have $c_{Q_{16}}(x_1^{l_2}x_2^{l_2-1}x_3^{l_2-2} \cdots x_{l_2}) \neq 0$. At last, recolor $u_1, u_2, \ldots, u_{l_2}, w_1$. By Lemma 1, we get a contradiction.

By Claim 1 and Claim 3, we have $\Delta(H) \geq 5$. If $\Delta(H) = 5$ or 6, vertices in H are all 5^+-vertices according to Claim 3 and Claim 4, then we have $5 \leq \text{mad}(H) \leq \text{mad}(G) < 5$, which is a contradiction. Therefore we have $\Delta(H) \geq 7$. In the following, in order to complete the proof, we use the discharging method. For every $v \in V(H)$, we define the original charge of

v to be $w(v) = d_H(v) - 5$. The total charge of the vertices of H is equal to $\sum_{v \in V(H)} (d_H(v) - 5) = |V(H)| \times (\text{ad}(H) - 5) \le |V(H)| \times (\text{mad}(H) - 5) < 0$.
Now we give the following discharging rule:

(R) In graph H, Every 7^+-vertex gives $\frac{2}{3}$ to each of its adjacent 3-vertex and gives $\frac{1}{4}$ to each of its adjacent 4-vertex.

Let $w'(v)$ denote the new charge of a vertex $v \in V(H)$ after the discharging is finished. If $\sum_{v \in V(G)} w'(v) \ge 0$ can be deduced, then we can show that the assumption is wrong.

Now let us check the new charge of each vertices in H.

For each $v \in V(H)$, if $d_H(v) = 3$, by Claim 3 and Claim 4, v is adjacent to three 7^+-vertices, by (R), $w'(v) = w(v) + 3 \times \frac{2}{3} = 0$.

If $d_H(v) = 4$, by Claim 3 and Claim 4, v is adjacent to four 7^+-vertices, by (R), $w'(v) = w(v) + 4 \times \frac{1}{4} = 0$.

If $d_H(v) = 5$ or 6, $w'(v) = w(v) \ge 0$.

If $d_H(v) = 7$, by Claim 5 and (R), $w'(v) \ge w(v) - \frac{2}{3} = \frac{4}{3} > 0$.

If $d_H(v) = 8$, by Claim 6 and (R), if v adjacent to three 3-vertices, then $w'(v) \ge w(v) - 3 \times \frac{2}{3} = 1 > 0$; if v adjacent to k 3-vertices, where $0 \le k \le 2$, then $w'(v) \ge w(v) - k \times \frac{2}{3} - (8-k) \times \frac{1}{4} = 1 - \frac{5}{12}k \ge \frac{1}{6} > 0$.

If $d_H(v) = 9$, by Claim 7 and (R), if v adjacent to four 3-vertices, then $w'(v) \ge w(v) - 4 \times \frac{2}{3} - 5 \times \frac{1}{4} = 1 - \frac{5}{12}k > 0$; if v adjacent to k 3-vertices, where $0 \le k \le 3$, then $w'(v) \ge w(v) - k \times \frac{2}{3} - (9-k) \times \frac{1}{4} = \frac{7}{4} - \frac{5}{12}k \ge \frac{1}{2} > 0$.

If $d_H(v) = 10$, if v adjacent to seven 3-vertices, by Claim 8(b) and (R), then $w'(v) \ge w(v) - 7 \times \frac{2}{3} = \frac{1}{3} > 0$; if v adjacent to k 3-vertices, where $0 \le k \le 6$, then $w'(v) \ge w(v) - k \times \frac{2}{3} - (10-k) \times \frac{1}{4} = \frac{5}{2} - \frac{5}{12}k \ge 0$.

If $d_H(v) \ge 11$, if v adjacent to $d_H(v) - 2$ 3-vertices, by Claim 9(b) and (R), $w'(v) = w(v) - (d_H(v) - 2) \times \frac{2}{3} = \frac{1}{3}((d_H(v) - 11)) \ge 0$; if v adjacent to $d_H(v) - 3$ 3-vertices, by Claim 9(c), $w'(v) \ge w(v) - (d_H(v) - 3) \times \frac{2}{3} - 2 \times \frac{1}{4} = \frac{1}{3}d_H(v) - \frac{7}{2} \ge \frac{1}{6} > 0$; if v adjacent to k 3-vertices, where $0 \le k \le d_H(v) - 4$, then $w'(v) \ge w(v) - k \times \frac{2}{3} - (d_H(v) - k) \times \frac{1}{4} = \frac{3}{4}d_H(v) - \frac{5}{12}k - 5 \ge \frac{1}{3}(d_H(v) - 10) \ge \frac{1}{3} > 0$.

So $\sum_{v \in V(H)} w'(v) \ge 0$, which contradicts to $\sum_{v \in V(H)} w(v) < 0$. This completes the proof.

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