

# Lower bounds for algebraic connectivity of graphs in terms of matching number or edge covering number\*

Jing Xu<sup>1</sup>, Yi-Zheng Fan<sup>1,†</sup>, Ying-Ying Tan<sup>1,2</sup>

1. School of Mathematical Sciences,

Anhui University, Hefei 230601, P. R. China

2. School of Mathematics and Physics,

Anhui Jianzhu University, Hefei 230601, P. R. China

**Abstract:** In this paper we characterize the unique graph whose algebraic connectivity is minimum among all connected graphs with given order and fixed matching number or edge covering number, and present two lower bounds for the algebraic connectivity in terms of the matching number or edge covering number.

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## 1 Introduction

Let  $G$  be a connected simple graph of order  $n$  with vertex set  $V = V(G) = \{v_1, v_2, \dots, v_n\}$  and edge set  $E = E(G)$ . The *adjacency matrix* of the graph  $G$  is defined to be the matrix  $A = A(G) = [a_{ij}]$ , where  $a_{ij} = 1$  if  $v_i$  is adjacent to  $v_j$ , and  $a_{ij} = 0$  otherwise. The matrix  $L(G) = D(G) - A(G)$  is called the *Laplacian matrix* of the graph  $G$ , where  $D(G) = \text{diag}\{d_G(v_1), d_G(v_2), \dots, d_G(v_n)\}$  is a diagonal matrix, and  $d_G(v)$  denotes the degree of a vertex  $v$  in the graph  $G$ . It is easy to see that  $L(G)$  is a real and positive semidefinite, so that its eigenvalues can be arranged as

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†Corresponding author. E-mail addresses: fanyz@ahu.edu.cn(Y.-Z. Fan), xujing@ahu.edu.cn (J. Xu), tansusan1@ahjzu.edu.cn (Y.-Y. Tan).

follows:

$$0 = \mu_n(G) \leq \mu_{n-1}(G) \leq \dots \leq \mu_1(G),$$

where  $\mu_n(G) = 0$  as each row sum of  $L$  is zero, with the all-one vector  $\mathbf{1}$  as an corresponding eigenvector. It is well known that the multiplicity of eigenvalue 0 is equal to the number of components of  $G$ . The eigenvalue  $\mu_{n-1}(G)$ , also denoted by  $\alpha(G)$ , is called the *algebraic connectivity* of  $G$  by Fiedler [10]; and the eigenvectors corresponding to  $\alpha(G)$  are usually called the *Fiedler vectors* of  $G$ .

The algebraic connectivity has received much attention; see [1, 2, 5, 13, 14, 16, 19, 20, 21, 22, 23]. For example, upper bounding or maximizing the algebraic connectivity has been discussed by Lu et. al [18] in terms of the domination number, Lal et. al [17] subject to the number of pendant vertices, Zhu [24] by means of matching number. Lower bounding or minimizing the algebraic connectivity has also been discussed by Fallat et. al [5, 6] subject to diameter or girth, Biyikoğlu and Leydold [3, 4] subject to degree sequence or size, and Fan and Tan [8] subject to domination number.

Recently Fan and Tan [8] obtain a perturbation result for the algebraic connectivity of a graph when a branch of the graph is relocated from one vertex to another vertex. The result motivates us to do a lot of work on minimizing the algebraic connectivity subject to graph parameters, which provides some lower bounds for the algebraic connectivity. In this paper, we characterize the unique graph whose algebraic connectivity is minimum among all connected graphs with given order and fixed matching number or edge covering number, and present two lower bounds for the algebraic connectivity in terms of the matching number or edge covering number.

At the end of this section, we introduce some notions. Recall that a *matching* of a graph  $G$  is an set of independent edges of  $G$ ; and the *matching number* of  $G$  is the maximal cardinalities of all the matchings of  $G$ , denoted by  $\beta(G)$ . Clearly,  $n \geq 2\beta(G)$ . In particular,  $G$  has *perfect matchings* if  $n = 2\beta(G)$ . An *edge cover* of a graph  $G$  without isolated vertices is a set of edges of  $G$  that covers all vertices of  $G$ . The *edge covering number* of a graph  $G$  is the minimum cardinality of all edge covers of  $G$ , denoted by  $\gamma(G)$ . It is known that  $\beta(G) + \gamma(G) = |V(G)|$  if  $G$  contains no isolated vertices [12].

Denote by  $\mathcal{M}_{n,\beta}$  (respectively,  $\mathcal{C}_{n,\gamma}$ ) the set of connected graphs of order  $n$  with matching number  $\beta$  (respectively, edge covering number  $\gamma$ ). Let  $P_d$  denote a path of order  $d$ , and  $S_{1,m}$  a star on  $m + 1$  vertices. Denote by  $T(k, l, d)$  a tree obtained from a path  $P_d$  by attaching two stars  $S_{1,k}, S_{1,l}$  at its two end points respectively; see Fig. 1.1. In particular, if  $d = 1$ , then  $T(k, l, d) = S_{1,k+l}$ ; if  $k = 1$  and  $l = 0$ , then  $T(k, l, d) := P_{d+1}$ . For convenience, a graph is called *minimizing* in a certain class if  $\alpha(G)$  is

minimum among all graphs in the class.

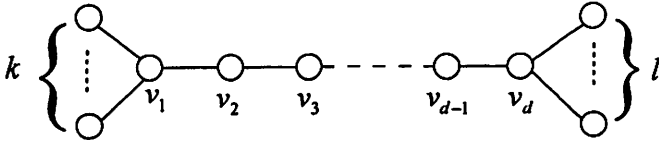


Fig. 1.1 The tree  $T(k, l, d)$

Let  $x = (x_1, x_2, \dots, x_n) \in \mathbb{R}^n$  and  $G$  be a graph on vertices  $v_1, v_2, \dots, v_n$ . The vector  $x$  can be considered as a function defined on  $V(G)$ , which maps each vertex  $v_i$  of  $G$  to the value  $x_i$ , i.e.  $x(v_i) = x_i$ . If  $x$  is an eigenvector of  $L(G)$ , then it defines on  $G$  naturally, i.e.  $x(v)$  is the entry of  $x$  corresponding to  $v$ . One can find that the quadratic form  $x^T L(G)x$  can be written as

$$x^T L(G)x = \sum_{uv \in E(G)} [x(u) - x(v)]^2. \quad (1.1)$$

The eigenvector equation  $L(G)x = \lambda x$  can be interpreted as

$$[d_G(v) - \lambda]x(v) = \sum_{u \in N_G(v)} x(u), \quad \text{for each } v \in V(G), \quad (1.2)$$

where  $N_G(v)$  denotes the neighborhood of  $v$  in  $G$ . In addition, for an arbitrary unit vector  $x \in \mathbb{R}^n$  orthogonal to  $\mathbf{1}$ ,

$$\alpha(G) \leq x^T L(G)x, \quad (1.3)$$

with equality if and only if  $x$  is a Fiedler vector of  $G$ .

## 2 Preliminaries

First we introduce the property of Fiedler vectors of a tree.

**LEMMA 2.1** [11] *Let  $T$  be a tree with a Fiedler vector  $x$ . Then exactly one of the two cases occurs:*

*Case A. All values of  $x$  are nonzero. Then  $T$  contains exactly one edge  $pq$  such that  $x(p) > 0$  and  $x(q) < 0$ . The values in vertices along any path in  $T$  which starts in  $p$  and does not contain  $q$  strictly increase, the values in vertices along any path starting in  $q$  and not containing  $p$  strictly decrease.*

*Case B. The set  $N_0 = \{v : x(v) = 0\}$  is non-empty. Then the graph induced by  $N_0$  is connected and there is exactly one vertex  $z \in N_0$  having at least one neighbor not belonging to  $N_0$ . The values along any path in  $T$  starting in  $z$  are strictly increasing, or strictly decreasing, or zero.*

If the Case B in Lemma 2.1 occurs, the vertex  $z$  is called the *characteristic vertex*, and  $T$  is called a *Type I* tree; otherwise,  $T$  is called a *Type II* tree in which case the edge  $pq$  is called the *characteristic edge*. The characteristic vertex or characteristic edge of a tree is independent of the choice of Fiedler vectors; see [19].

Next we introduce the perturbation result of the algebraic connectivity of a graph. Let  $G_1, G_2$  be two vertex-disjoint graphs, and let  $v \in V(G_1), u \in V(G_2)$ . The *coalescence* of  $G_1$  and  $G_2$  with respect to  $v$  and  $u$ , denoted by  $G_1(v) \diamond G_2(u)$ , is obtained from  $G_1$  and  $G_2$  by identifying  $v$  with  $u$  and forming a new vertex  $p$ , which is also denoted as  $G_1(p) \diamond G_2(p)$ . If a connected graph  $G$  can be expressed as  $G = G_1(p) \diamond G_2(p)$ , where  $G_1$  and  $G_2$  are nontrivial subgraphs of  $G$  both containing  $p$ , then  $G_1$  or  $G_2$  is called a *branch* of  $G$  rooted at  $p$ . Let  $G = G_1(v_2) \diamond G_2(u)$  and  $G^* = G_1(v_1) \diamond G_2(u)$ , where  $v_1$  and  $v_2$  are two distinct vertices of  $G_1$  and  $u$  is a vertex of  $G_2$ . We say that  $G^*$  is obtained from  $G$  by *relocating*  $G_2$  from  $v_2$  to  $v_1$ ; see Fig. 2.1.

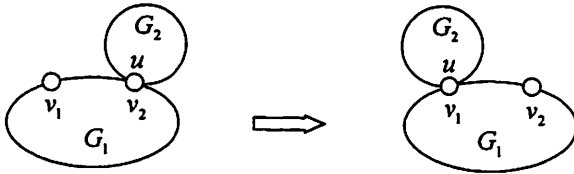


Fig. 2.1. Relocating  $G_2$  from  $v_2$  to  $v_1$ .

LEMMA 2.2 [8] Let  $G_1$  be a connected graph containing at least two vertices  $v_1, v_2$ , and let  $G_2$  be a nontrivial connected graph containing a vertex  $u$ . Let  $G = G_1(v_2) \diamond G_2(u)$  and  $G^* = G_1(v_1) \diamond G_2(u)$ . If there exist a Fiedler vector  $x$  of  $G$  such that  $x(v_1) \geq x(v_2) \geq 0$  and all vertices in  $G_2$  are nonnegatively valuated by  $x$ , then

$$\alpha(G^*) \leq \alpha(G),$$

with equality if and only if  $x(v_1) = x(v_2) = 0, \sum_{w \in N_{G_2}(u)} x(w) = 0$ , and  $x$  is also a Fiedler vector of  $G^*$ .

Now we investigate the property of the algebraic connectivity of the graph  $T(k, l, d)$  listed in Section 1. Denote  $T_d := T(\lceil \frac{n-d}{2} \rceil, \lfloor \frac{n-d}{2} \rfloor, d)$ .

LEMMA 2.3 [5] Among all trees of order  $n$  and diameter  $d + 1$ , the tree  $T_d$  is the unique graph with minimum algebraic connectivity.

LEMMA 2.4 [8] (1) If  $k \geq 2, \alpha(T(k, l, d)) > \alpha(T(k - 1, l, d + 1))$ ;  
 (2) if  $l \geq 2, \alpha(T(k, l, d)) > \alpha(T(k, l + 1, d + 1))$ .

LEMMA 2.5 Let  $\beta(T_d) = \beta \geq 2$ , where  $n \geq d + 2$  and  $n \geq 2\beta + 1$ . Then  $2(\beta - 1) \leq d \leq 2\beta - 1$ . Furthermore,

$$\alpha(T_d) \geq \alpha(T_{2\beta-1}),$$

with equality if and only if  $d = 2\beta - 1$ .

*Proof:* Since  $\beta(T_d) = \lfloor \frac{d+2}{2} \rfloor$ , then  $\beta \leq \frac{d+2}{2} < \beta + 1$ , that is  $2(\beta - 1) \leq d < 2\beta$ . Hence  $2\beta - 2 \leq d \leq 2\beta - 1$ . It suffices to show that  $\alpha(T_{2\beta-2}) \geq \alpha(T_{2\beta-1})$ . By Lemma 2.4 and Lemma 2.3, we have

$$\begin{aligned} \alpha(T_{2\beta-2}) &= \alpha\left(T\left(\left\lfloor \frac{n-(2\beta-2)}{2} \right\rfloor, \left\lfloor \frac{n-(2\beta-2)}{2} \right\rfloor, 2\beta-2\right)\right) \\ &> \alpha\left(T\left(\left\lfloor \frac{n-(2\beta-2)}{2} \right\rfloor - 1, \left\lfloor \frac{n-(2\beta-2)}{2} \right\rfloor, 2\beta-1\right)\right) \\ &\geq \alpha(T_{2\beta-1}). \end{aligned}$$

■

LEMMA 2.6 Let  $T_{2\beta_1-1}, T_{2\beta_2-1}$  be two trees of order  $n$  with matching number  $\beta_1, \beta_2$ . If  $\beta_1 < \beta_2$  and  $n \geq 2\beta_2 + 1$ , then

$$\alpha(T_{2\beta_1-1}) > \alpha(T_{2\beta_2-1}).$$

*Proof:* Let  $T_{2\beta_1-1} := T(k, l, 2\beta_1 - 1)$ . By Lemma 2.4, we have

$$\begin{aligned} \alpha[T(k, l, 2\beta_1 - 1)] &> \alpha[T(k - 1, l, 2\beta_1)] \\ &> \alpha[T(k - 1, l - 1, 2\beta_1 + 1)] \\ &= \alpha[T(k - 1, l - 1, 2(\beta_1 + 1) - 1)]. \end{aligned}$$

That is  $\alpha(T_{2\beta_1-1}) > \alpha(T_{2(\beta_1+1)-1})$ . The result follows by induction on the matching number. ■

LEMMA 2.7 Let  $G \in \mathcal{M}_{n,\beta}$ . Then  $G$  contains a spanning tree also with matching number  $\beta$ .

*Proof:* Let  $G \in \mathcal{M}_{n,\beta}$  and let  $M$  be a maximum matching of  $G$ . Denote  $e(G) = |E(G)|$ . Clearly,  $e(G) \geq n - 1$  as  $G$  is connected. The result is certainly true if  $e(G) = n - 1$ , in which case  $G$  is a tree. So we assume that  $e(G) > n - 1$ .

Delete an edge  $e_1$  of some cycle of  $G$ , where  $e_1 \notin M$ , producing a graph  $G_1$  such that  $\beta(G_1) = \beta(G)$ . If  $e(G_1) = n - 1$ , then  $G_1$  is a spanning tree of  $G$ . If  $e(G_1) > n - 1$ , delete an edge  $e_2$  of some cycle of  $G_1$ , where  $e_2 \notin M$ , producing a graph  $G_2$  such that  $\beta(G_2) = \beta(G_1) = \beta(G)$ . We continue the above process until we arrive at an spanning tree  $G_k$  of  $G$  such that  $\beta(G_k) = \beta(G_{k-1}) = \dots = \beta(G_1) = \beta(G)$ , where  $k = e(G) - n + 1$ . ■

**COROLLARY 2.8** *Let  $G \in \mathcal{M}_{n,\beta}$ . If  $G$  contains cycles, then  $G$  contains a spanning unicyclic graphs with matching number  $\beta$ .*

*Proof:* By Lemma 2.7,  $G$  contains a spanning tree  $T$  with matching number  $\beta$ . The result follows by adding an edge  $e \in E(G) \setminus E(T)$  to the tree  $T$ . ■

### 3 Main results

We first restrict our discussion to trees with minimum algebraic connectivity.

**THEOREM 3.1** *Among all trees of order  $n$  with matching number  $\beta$ , where  $n \geq 2\beta + 1$ , the tree  $T_{2\beta-1}$  is the unique graph with minimum algebraic connectivity.*

*Proof:* Here we adopt a similar technique used in the paper [8]. If  $n = 2\beta + 1$ , the result follows obviously since  $T_{2\beta-1} = P_n$  is the unique minimizing graph among all connected graphs of order  $n$ .

Now assume that  $n \geq 2\beta + 2$ . Let  $T$  be a minimizing tree of order  $n$  with matching number  $\beta$ . If  $T$  has exactly two pendant stars (i.e. the star with maximum possible size centered at a quasi-pendant vertex), then  $T = T(k, l, d)$  for some  $k, l, d$ , where  $d \geq 2$ . By Lemma 2.3,  $k = \lceil \frac{n-d}{2} \rceil$ ,  $l = \lfloor \frac{n-d}{2} \rfloor$  and  $T = T_d$ ; by Lemma 2.5,  $d = 2\beta - 1$ . The result follows.

Next suppose that  $T := T_0$  has more than two pendant stars, which has  $p_0$  pendent vertices and  $q_0$  quasi-pendent vertices. Let  $x$  be a Fiedler vector of  $T_0$ . First assume  $T_0$  is of Type I. Let  $N_0 = \{v \in V(T_0) : x(v) = 0\}$ . If  $|N_0| \geq 2$ , then there exists at least one zero pendant star  $S$  attached at some vertex say  $u$ , and at least one positive quasi-pendant vertex  $w$ . Relocating the zero star  $S$  at  $u$  to  $w$ , we will arrive at a new tree  $T_1$  such that  $\alpha(T_1) < \alpha(T_0)$  by Lemma 2.2. Note that  $\beta(T_1) \leq \beta(T_0)$  (in fact,  $\beta(T_1) < \beta(T_0)$ ); otherwise we will get a contradiction to the fact that  $T_0$  is minimizing. If  $|N_0| = 1$ , there exist at least two pendant stars  $S_1, S_2$  both being positive or negative valued by  $x$ , attached at  $u_1, u_2$  respectively. Without loss of generality, assume  $S_1, S_2$  are both positive and  $x(u_1) \geq x(u_2) > 0$ . Relocating  $S_2$  from  $u_2$  to  $u_1$ , we arrive at a new tree  $T_1$  such that  $\alpha(T_1) < \alpha(T_0)$  by Lemma 2.2 and  $\beta(T_1) < \beta(T_0)$ .

If  $T_0$  is of Type II, then there exist at least two pendant stars  $S_1, S_2$  both being positive or negative valued by  $x$ , attached at  $u_1, u_2$  respectively. By the similar way with the case  $|N_0| = 1$  above, we also arrive at a new tree  $T_1$  such that  $\alpha(T_1) < \alpha(T_0)$  and  $\beta(T_1) < \beta(T_0)$ .

Repeat the above procession on  $T_1$  if  $T_1$  has more than two pendant stars and continue a similar discussion to the resulting tree. Note that from the

$k$ -th step to the  $(k + 1)$ -th step, either  $p_{k+1} = p_k$  and  $q_{k+1} = q_k - 1$ , or  $p_{k+1} = p_k + 1$  and  $q_{k+1} = q_k$ . So the above procession will be terminated at the  $n$ -th step in which the tree  $T_n$  has exactly two pendant stars, i.e.  $T_n = T(k, l, d)$  for some  $k, l, d$ , where  $d \geq 2$ . Hence

$$\alpha(T) = \alpha(T_0) > \alpha(T_1) > \dots > \alpha(T_n), \beta(T) = \beta(T_0) > \beta(T_1) > \dots > \beta(T_n).$$

Therefore, noting that  $T_{2\beta-1}$  has matching number  $\beta$ , by Lemma 2.3 and Lemma 2.5,

$$\alpha(T_{2\beta-1}) \geq \alpha(T) > \alpha(T_n) \geq \alpha(T_d) \geq \alpha(T_{2\beta(T_n)-1}).$$

However, since  $\beta(T_n) < \beta(T) = \beta$ , by Lemma 2.6, we have  $\alpha(T_{2\beta-1}) < \alpha(T_{2\beta(T_n)-1})$ , a contradiction. So this case cannot happen and the result follows. ■

**THEOREM 3.2** *Let  $G \in \mathcal{M}_{n,\beta}$ . Then  $G$  is minimizing in  $\mathcal{M}_{n,\beta}$  if and only if  $G = T_{2\beta-1}$ .*

*Proof:* If  $\beta = 1$ , the result holds as  $T_1 = S_{1,n-1}$  is the unique graph of matching number 1 for  $n \geq 2$  and  $n \neq 3$ . When  $n = 3$ , there are exactly two graphs:  $S_{1,2}$  and the triangle  $C_3$ , both having matching number 1. Since  $\alpha(S_{1,2}) < \alpha(C_3)$ , the result also holds in this case.

Assume that  $\beta \geq 2$ . If  $n = 2\beta$ , the result surely holds as  $P_{2\beta}$  is the unique minimizing graph. So suppose that  $n \geq 2\beta + 1$  in the following. Let  $G$  be a minimizing graph in  $\mathcal{M}_{n,\beta}$ . Then  $G$  contains a spanning tree  $T$  with matching number  $\beta$  by Lemma 2.7. Furthermore, by Theorem 3.1,

$$\alpha(G) \geq \alpha(T) \geq \alpha(T_{2\beta-1}). \quad (3.1)$$

Hence  $\alpha(G) = \alpha(T) = \alpha(T_{2\beta-1})$ , which implies that  $T = T_{2\beta-1}$  also by Theorem 3.1.

We claim that  $G = T_{2\beta-1}$ ; otherwise  $E(G) \setminus E(T_{2\beta-1}) \neq \emptyset$ . Let  $x$  be a unit Fiedler vector of  $G$ . Then

$$\begin{aligned} \alpha(G) &= \sum_{uv \in E(G)} [x(u) - x(v)]^2 \\ &= \sum_{uv \in E(T_{2\beta-1})} [x(u) - x(v)]^2 + \sum_{uv \in E(G) \setminus E(T_{2\beta-1})} [x(u) - x(v)]^2 \\ &\geq \sum_{uv \in E(T_{2\beta-1})} [x(u) - x(v)]^2 \\ &\geq \alpha(T_{2\beta-1}). \end{aligned}$$

Since  $\alpha(G) = \alpha(T_{2\beta-1})$ , then  $x$  is also a Fiedler vector of  $T_{2\beta-1}$ , and  $x(u) = x(v)$  for each edge  $uv \in E(G) \setminus E(T_{2\beta-1})$ . By Lemma 2.1, whenever

$T_{2\beta-1}$  is of Type I or Type II,  $u, v$  should be both the pendent vertices lying in a same pendent star. However, in this case  $\beta(T_{2\beta-1} + uv) > \beta = \beta(G)$  for any  $uv \in E(G) \setminus E(T_{2\beta-1})$ ; a contradiction.

The sufficiency result follows from the discussion of (3.1). ■

As a byproduct, we get the following result on edge covering number.

**COROLLARY 3.3** *Let  $G \in C_{n,\gamma}$ . Then  $G$  is minimizing in  $C_{n,\gamma}$  if and only if  $G = T_{2(n-\gamma)-1}$ .*

*Proof:* The result follows by Theorem 3.2 and the fact that  $\beta(G) + \gamma(G) = n$ . ■

**LEMMA 3.4** [15] *Suppose that  $d \geq 3, k \geq 1, l \geq 1$  and  $n := k + l + d - 1$ . Then*

$$\alpha(T(k, l, d - 1)) \geq \left( \frac{nd}{4} - \frac{2n + d^2 - 4d - 5}{8} \right)^{-1}.$$

**COROLLARY 3.5** *Let  $G \in \mathcal{M}_{n,\beta}$ . Then*

$$\alpha(G) \geq \frac{8}{-4\beta^2 + 4\beta(n + 2) - 2n + 5}.$$

*Proof:* By Theorem 3.2,  $\alpha(G) \geq \alpha(T_{2\beta-1})$ . If  $\beta = 1$ , surely

$$\alpha(T_{2\beta-1}) = \alpha(T_1) = 1 > \frac{8}{-4\beta^2 + 4\beta(n + 2) - 2n + 5} = \frac{8}{2n + 9}.$$

If  $\beta \geq 2$  and  $n = 2\beta$ , noting that in this case  $T_{2\beta-1} = P_n$ ,

$$\alpha(P_n) = 2 \left( 1 - \cos \frac{\pi}{n} \right) > \frac{8}{-4\beta^2 + 4\beta(n + 2) - 2n + 5} = \frac{8}{n^2 + 2n + 5}.$$

If  $\beta \geq 2$  and  $n \geq 2\beta + 1$ , the result follows by taking  $d = 2\beta$  in Lemma 3.4. ■

Similarly we have the following corollary.

**COROLLARY 3.6** *Let  $G \in C_{n,\gamma}$ . Then*

$$\alpha(G) \geq \frac{8}{-4\gamma^2 + 4\gamma(n - 2) + 6n + 5}.$$

## References

- [1] N. M. M. de Abreu, Old and new results on algebraic connectivity of graphs, *Linear Algebra Appl.*, 423(2007), 53-73.



- [2] R. B. Bapat, S. Pati, R. Merris, Algebraic connectivity and the characteristic set of a graph, *Linear Multilinear Algebra*, 45(1998), 247-273.
- [3] T. Biyikoğlu, J. Leydold, Algebraic connectivity and degree sequences of trees, *Linear Algebra Appl.*, 430(2009), 811-817,
- [4] T. Biyikoğlu, J. Leydold, Graphs of given order and size and minimum algebraic connectivity, *Linear Algebra Appl.*, 436(2012), 2067-2077.
- [5] S. Fallat, S. Kirkland, Extremizing algebraic connectivity subject to graph theoretic constraints, *Electron. J. Linear Algebra*, 3(1998), 48-74.
- [6] S. Fallat, S. Kirkland, S. Pati, Minimizing algebraic connectivity over connected graphs with fixed girth, *Discrete Math.*, 254(2002) 115-142.
- [7] S. Fallat, S. Kirkland, S. Pati, Maximizing algebraic connectivity over unicyclic graphs, *Linear Multilinear Algebra*, 51 (2003), 221-241.
- [8] Y.-Z. Fan, Y.-Y. Tan, A lower bound for the algebraic connectivity of a graph in terms of the domination number, *Acta Math. Appl. Sinica English Ser.*, in press.
- [9] L.-H. Feng, G.-H. Yu, Q. Li, Minimizing the Laplacian eigenvalues for trees with given domination number, *Linear Algebra Appl.*, 419(2006), 648-655.
- [10] M. Fiedler, Algebraic connectivity of graphs, *Czechoslovak Math. J.*, 23 (1973), 298-305.
- [11] M. Fiedler, A property of eigenvectors of nonnegative symmetric matrices and its application to graph theory, *Czechoslovak Math. J.*, 25 (1975), 619-633.
- [12] T. Gallai, Über extreme Punkt- und Kantenmengen. *Ann. Univ. Sci. Budapest. Eötvös Sect. Math.*, 2(1959), 133-138.
- [13] R. Grone, S. Diego, R. Merris, Hayward, Algebraic connectivity of trees, *Czechoslovak Math. J.*, 37(1987), 660-670.
- [14] R. Grone, R. Merris, V.S. Sunder, The Laplacian Spectrum of a graph, *SIAM J. Matrix Anal. Appl.*, 11(1990), 218-238.
- [15] S. Kirkland, M. Neumann, Algebraic connectivity of weighted trees under perturbation, *Linear Multilinear Algebra*, 42(1997), 187-203.
- [16] S. Kirkland, A bound on the algebraic connectivity of a graph in terms of the number of cut points, *Linear Multilinear Algebra*, 47(2000), 93-103.
- [17] A. K. Lal , K. L. Patra, B. K. Sahoo, Algebraic connectivity of connected graphs with fixed number of pendant vertices, *Graphs Combin.*, 27(2011), 215-229.
- [18] M. Lu, H.-Q. Liu, F. Tian, Bounds of Laplacian spectrum of graphs based on the domination number, *Linear Algebra Appl.*, 402(2005), 390-396.
- [19] R. Merris, Characteristic vertices of trees, *Linear Multilinear Algebra*, 22(1987), 115-131.
- [20] R. Merris, Degree maximal graphs are Laplacian integral, *Linear Algebra Appl.*, 199(1994), 381-389.

- [21] R. Merris, Laplacian matrices of graphs: a survey, *Linear Algebra Appl.*, 197/198(1994), 143-176.
- [22] R. Merris, A survey of graph Laplacian, *Linear Multilinear Algebra*, 392(1995), 19-31.
- [23] R. Merris, Laplacian graph eigenvectors, *Linear Algebra Appl.*, 278(1998), 221-236.
- [24] B.-X. Zhu, The algebraic connectivity of graphs with given matching number, *Graphs Combin.*, 29(2013), 1989-1995.