General vertex-distinguishing total coloring of complete bipartite graphs

Hong Yang

School of Information Science & Technology,
Chengdu University, Chengdu, 610106, China

Abstract. The general vertex-distinguishing total chromatic number of a graph G is the minimum integer k, for which the vertices and edges of G are colored using k colors such that there are no two vertices possessing the same color-set, where a color-set of a vertex is a set of colors of the vertex and its incident edges. In this paper, we discuss the general vertex-distinguishing total chromatic number of complete bipartite graphs $K_{m,n}$, and obtain the exact value of this number for some cases in terms of m and n. Particularly, we give the bounds of this number for $K_{n,n}$.

Keywords. Complete bipartite graph, general vertex-distinguishing total coloring, general vertex-distinguishing total chromatic number, color-set

1 Introduction

Graphs considered in this paper are simple (without loops or multiple edges), finite, and undirected. Given a graph G, we denote by $V(G)$, $E(G)$, $\Delta(G)$ and $\delta(G)$ the set of vertices, edges, maximum degree and minimum degree of G, respectively. For a vertex v of G, $d_G(v)$ is the degree of v in G. For any undefined terms, the reader is referred to the book [20].

Graph coloring is an important research problem. It can be widely applied in practice [1, 2, 3]. Given that many practical problems can be abstracted into coloring problems, many new colorings have been introduced [4]. In 1985, Harary [5] proposed general vertex-distinguishing edge coloring of graphs.

Definition 1.1 [5] *Let G be a graph and k be a non-negative integer. A general vertex-distinguishing k-edge coloring of G, abbreviated as k-$GVDEC$, is a mapping f from E to $\{1,2,\ldots,k\}$ such that for $\forall u,v \in V(G)$, $C(u) \neq C(v)$, where $C(u) = \{f(uv)|uv \in E(G)\}$. The general vertex-distinguishing edge chromatic number of G, denoted by $\chi'_{gvd}(G)$, is the minimum k such that G has a k-$GVDEC$.*

In [6] and [7], Horňák, Soták and Sakvi studied general vertex-distinguishing edge chromatic number of complete bipartite graphs.

In 2008, Győri, Horňák, Palmer and Woźniak [14] introduced the *general adjacent vertex-distinguishing k-edge coloring* of a graph.

Definition 1.2 [14] *The general adjacent vertex-distinguishing k-edge coloring of a graph G is a mapping f from $E(G)$ to $\{1,2,\ldots,k\}$ such that $C(u) \neq C(v)$ for any $uv \in E(G)$, where $C(u) = \{f(uv)|uv \in E(G)\}$. The general adjacent vertex-distinguishing index of G, denoted by $\chi_{gav}(G)$, is the minimum k for which there exists a general adjacent vertex-distinguishing k-edge coloring of G.*

A total k-coloring of a graph G is a mapping from $V(G) \cup E(G)$ to $\{1,2,\ldots,k\}$. A total coloring is called as *proper* if any two adjacent or incident elements receive distinct colors. Given a total k-coloring f of G, we denote by $C_t^f(v)$ the set of colors of v and its incident edges under f. We also call $C_t^f(v)$ the color set of v (under f). In 2005, Zhang, Chen, Li, Yao, Lu and Wang [15] introduced a variant of proper total coloring.

Definition 1.3 [15] *Let f be a proper total k-coloring of a graph G. If $\forall uv \in E(G)$, $C_t^f(u) \neq C_t^f(v)$, then f is called an adjacent vertex-distinguishing total k-coloring of G, or a k-$AVDTC$ of G for short. The minimum number k for which G has a k-$AVDTC$ is the adjacent vertex-distinguishing total chromatic number of G, denoted by $\chi_{at}(G)$.*

Zhang et al. [15] conjectured that

Conjecture 1.1 *For any graph G, it follows that*

$$\chi_{at}(G) \leq \Delta(G) + 3$$

In [16], [17], and [18], authors independently proved that there exist a 6-$AVDTC$ of graphs with $\Delta = 3$, which indicates Conjecture 1.1 holds for such graphs.

Let f be a proper total k-coloring of a graph G. If for any two distinct vertices u, v, it has $C_t^f(u) \neq C_t^f(v)$, then f is referred to as a *vertex-distinguishing total k-coloring* of G, abbreviated as k-$VDTC$. The minimum number k such that G has a k-$VDTC$ is called the vertex-distinguishing total chromatic number, denoted by $\chi_{vt}(G)$ [19]. Zhang et al. [19] conjectured that

Conjecture 1.2 *For any graph G, it has that*

$$\mu_t(G) \le \chi_{vt}(G) \le \mu_t(G) + 1$$

where $\mu_t(G) = min\{k|\binom{k}{i+1} \ge n_i, \delta \le i \le \Delta\}$, n_i is the number of vertices with degree i in G.

In [21], Liu and Zhu proposed the general vertex-distinguishing total coloring of graphs.

Definition 1.4 *Let G be a graph and k be a positive integer. A total coloring f of G using k colors is called a general vertex-distinguishing total k-coloring of G (or k-GVDTC of G briefly) if $\forall u, v \in V(G)$, $C_t^f(u) \ne C_t^f(v)$. The minimum number k for which G has a k-GVDTC is the general vertex-distinguishing total chromatic number*, denoted by $\chi_{gvt}(G)$.

Obviously, $\chi_{gvt}(G)$ does exist for every graph G. In this paper, we study the general vertex-distinguishing total coloring of complete bipartite graphs $K_{m,n}$.

2 Main results

Recall that a *bipartite graph* is a graph whose vertices can be divided into two disjoint sets X and Y (that is, X and Y are each independent sets) such that every edge connects a vertex in X to one in Y. A *complete bipartite graph* is a special bipartite graph such that every vertex of the X is connected to every vertex of Y. In what follows, we denote by $K_{m,n}$ a complete bipartite graph with partitions of size $|X| = m$ and $|Y| = n$, and let $V(K_{m,n}) = X \cup Y$, where $X = \{x_i | i = 1, 2, \ldots, m\}$ and $Y = \{y_i | i = 1, 2, \ldots, n\}$.

We first give some simple but useful results as follows.

Lemma 2.1 *Let f be a k-GVDTC of $K_{m,n}$. Then for $\forall x \in X, y \in Y$, it follows that*

$$C_t^f(x) \cap C_t^f(y) \ne \emptyset.$$

Proof. The proof of Lemma 2.1 is straightforward, so we omit it. □

Let f be a k-GVDTC of $K_{m,n}$, and v be a vertex of $K_{m,n}$. We will denote by $\overline{C}_t^f(v)$ the set of colors not appearing at v and its incident edges, i.e. $\overline{C}_t^f(v) = \{1, 2, \ldots, k\} \setminus C_t^f(v)$, where $\{1, 2, \ldots, k\}$ is the set of k colors.

Lemma 2.2 *Let f be a k-GVDTC of $K_{m,n}$. Denote by q the number of vertices $v \in Y$ such that $|C_t^f(v)| = 1$. Then $m \le 2^{k-q}$.*

Proof. Let $\{1\},\{2\},\ldots,\{q\}$ be the q color sets appearing at vertices in Y. Then by Lemma 2.1 each vertex $x \in X$, $\{1,2,\ldots,q\} \subseteq C_t^f(x)$. In addition, because $C_t^f(x_i) \neq C_t^f(x_j)$ for any two distinct vertices $x_i, x_j \in X$, it follows that $m \leq \sum_{\ell=0}^{k-q} \binom{k-q}{\ell} = 2^{k-q}$. □

We call the $K_{1,n}$ a *Star*. In [21], Liu and Zhu have obtained the general vertex distinguishing total chromatic number of $K_{1,n}$. In this paper, we mainly discuss this parameter of $K_{m,n}$ for $m \geq 2$ and $n \geq 2$.

For any set S, we denote

$$\binom{S}{r} = \text{the set of all } r\text{-subsets of } S.$$

Theorem 2.1 *Let $n \geq 2$ be an integer. Then*

$$\chi_{gvt}(K_{2,n}) = \begin{cases} 3, & n = 2,3,4; \\ 4, & n = 5,6,7,8,9,10,11,12; \\ k^*, & n \geq 13. \end{cases}$$

Where k^* is the minimum value satisfying $n \leq \binom{k^*}{1} + \binom{k^*}{2} + \binom{k^*}{3} - 1$.

Proof. When $n \leq 12$, the proof of the conclusion is straightforward. We now consider the situation for $n \geq 13$. Suppose $\chi_{gvt}(K_{2,n}) = k$. Let $N_k = \binom{k}{1} + \binom{k}{2} + \binom{k}{3} - 1$. Let f be a k-*GVDTC* of $K_{2,n}$. It is obvious that $|C_t^f(x_i)| \leq n+1$, and $|C(y_j)| \leq 3$ for $i = 1,2$ and $j = 1,2,\ldots,n$. By Lemma 2.2 there must exist some $\{\ell\} \notin \cup_{j=1}^{n} C_t^f(y_j)$, $\ell \in \{1,2,\ldots,k\}$, so $n \leq N_k$ which implies that $k \geq k^*$. In particular, when $k = 4$, because there exists some $i \in \{1,2\}$ such that $|C_t^f(x_i)| \leq 3$, we have $n \leq N_4 - 1 = 12$. When $k \geq 5$, in order to show $k = k^*$, it suffices to prove $K_{2,n}$ has a k^*-*GVDTC*.

We can always assume $n \geq k^* - 1$ because the case $n < k^* - 1$ is trivial. Let $K^* = \{1,2,\ldots,k^*\}$. First, we (arbitrarily) assign a color set C_j to each y_j, $j = 1,2,\ldots,n$, such that (1) $C_j \in ((\binom{K^*}{1} \setminus \{k^*\}) \cup \binom{K^*}{2} \cup \binom{K^*}{3}$, (2) $C_{j_1} \neq C_{j_2}$ for any two distinct $j_1, j_2 \in \{1,2,\ldots,n\}$, (3) $\{\{1\},\{2\},\ldots,\{k^*-1\}\} \subseteq \{C_j : j = 1,2,\ldots,n\}$. Because $N_k = |((\binom{K^*}{1} \setminus \{k^*\}) \cup \binom{K^*}{2} \cup \binom{K^*}{3}|$, $n \leq N_k$ and $n \geq k^* - 1$, it follows that such C_j does exist for $j = 1,2,\ldots,n$. Now, we define a k^*-*GVDTC* of K_{2,N_k} according to C_j.

Color x_1 by k^* and x_2 by $k^* - 1$; For each y_j and its incident edges, if $|C_j| = |\{c_1\}| = 1$, then color y_j, $y_j x_1$, $y_j x_2$ by c_1; If $|C_j| = |\{c_1, c_2\}| = 2$, assume $c_1 < c_2$, then color y_j by c_2, color $y_j x_1$ and $y_j x_2$ by c_1; If $|C_j| = |\{c_1, c_2, c_3\}| = 3$, assume $c_1 < c_2 < c_3$, then color y_j by c_3, $y_j x_1$ by c_1, and $y_j x_2$ by c_2. Where $c_1, c_2, c_3 \in K^*$.

By the above coloring, because color sets $\{1\}$, $\{2\}$,..., $\{k^* - 1\}$ appear at vertices of Y, it has that the color set of x_1 is $\{1,2,\ldots,k^*\}$, and the color set of x_2 is $\{1,2,\ldots,k^*-1\}$ by Lemma 2.1. Given that for any two $j_1 \neq j_2 \in \{1,2,\ldots,n\}$ $C_{j_1} \neq C_{j_2}$ and $|C_j| \leq 3$, we can see that x_i has different color set with y_j, $i=1,2$, $j=1,2,\ldots,n$. So the coloring defined above is a k^*-$GVDTC$ of $K_{2,n}$. □

As an illustration of the above Theorem, we consider the complete bipartite graph $K_{2,20}$. We now define a 5-$GVDTC$ of $K_{2,20}$ as follows.

First, we assign a color set C_j to y_j for $j=1,2,\ldots,20$; See the following table.

C_1	C_2	C_3	C_4	C_5	C_6	C_7	C_8
{1}	{2}	{3}	{4}	{1,2}	{1,3}	{1,4}	{1,5}
C_9	C_{10}	C_{11}	C_{12}	C_{13}	C_{14}	C_{15}	C_{16}
{1,2,3}	{1,2,4}	{1,2,5}	{1,3,4}	{1,3,5}	{1,4,5}	{2,3}	{2,4}
C_{17}	C_{18}	C_{19}	C_{20}				
{2,5}	{2,3,4}	{2,3,5}	{3,5}				

Then, according to the C_j defined above, we color vertices and edges of K_{2,N_5} as follows:

Color x_1 by 5 and x_2 by 4;

Color $y_1, y_2, \ldots, y_{20}$ by 1, 2, 3, 4, 2, 3, 4, 5, 3, 4, 5, 4, 5, 5, 3, 4, 5, 4, 5, 5, respectively;

Color $x_1y_1, x_1y_2, \ldots, x_1y_{20}$ by 1, 2, 3, 4, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 2, 2, 2, 2, 2, 3, respectively;

Color $x_2y_1, x_2y_2, \ldots, x_2y_{20}$ by 1, 2, 3, 4, 1, 1, 1, 1, 2, 2, 2, 3, 3, 4, 2, 2, 2, 3, 3, 3, respectively.

Under the above coloring, it follows that the color sets of x_1 and x_2 are {1,2,3,4,5} and {1,2,3,4}, respectively. So, this coloring is a 5-$GVDTC$ of $K_{2,20}$.

Theorem 2.2 *Let $n \geq 3$ be any positive integer. Then*

$$\chi_{gvt}(K_{3,n}) = \begin{cases} 4, & n = 3,4,5,6,7,8,9,10; \\ 5, & n = 11,12,\ldots,26; \\ 6, & n = 27,28,29; \\ k^*, & n \geq 30. \end{cases}$$

Where $k^*(\geq 6)$ is the minimum value satisfying $\sum_{i=1}^{4}\binom{k^*-1}{i} - 1 \leq n \leq \sum_{i=1}^{4}\binom{k^*}{i} - 2$.

Proof. Suppose that $\chi_{gvt}(K_{3,n}) = k$ and let f be a k-$GVDTC$ of $K_{3,n}$. By Lemma 2.2 there must exist some $\{\ell_1\} \notin \cup_{j=1}^{n} C_t^f(y_j)$ and $\{\ell_2\} \notin \cup_{j=1}^{n} C_t^f(y_j)$, $\ell_1, \ell_2 \in \{1,2,\ldots,k\}$. Thus, by Lemma 2.1, one can readily check that $n \leq 10$ and $n \leq 26$ when $k=4$ and $k=5$, respectively. When $k \geq 6$, because $|C_t^f(y)| \leq 4$ for each $y \in Y$, it follows that $n \leq \sum_{i=1}^{4} \binom{k}{i}$-2 which implies that $k \geq k^*$. Additionally, we can easily give a 6-*GVDTC* of $K_{3,n}$ when $n = 27, 28, 29$. So, it suffices to consider the case of $k \geq 6$ and $n \geq 30$. In order to show $k = k^*$ in this case, we are sufficient to show that $K_{3,n}$ has a k^*-*GVDTC*.

Since the case $n < k^* - 1$ is trivial, we assume $n \geq k^* - 1$. Analogously to the proof of Theorem 2.1, we first assign a color set C_j to each y_j, $j = 1, 2, \ldots, n$, such that (1) $C_j \in ((\binom{K^*}{1} \setminus \{\{k^*\}, \{k^* - 1\}\}) \cup \binom{K^*}{2} \cup \binom{K^*}{3} \cup \binom{K^*}{4}$ (2) $C_{j_1} \neq C_{j_2}$ for any two distinct $j_1, j_2 \in \{1, 2, \ldots, n\}$, (3) $\{\{1\}, \{2\}, \ldots, \{k^* - 2\}, \{k^* - 1, k^*\}\} \subseteq \{C_j : j = 1, 2, \ldots, n\}$, where $K^* = \{1, 2, \ldots, k^*\}$. Given that $N_k = |((\binom{K^*}{1}) \setminus \{\{k^*\}, \{k^* - 1\}\}) \cup \binom{K^*}{2} \cup \binom{K^*}{3} \cup \binom{K^*}{4}|$ and $n \leq N_k$, we can see that such C_j does exist, $j = 1, 2, \ldots, n$. Now, we define a k^*-*GVDTC* of K_{2,N_k} according to C_j.

Color x_1, x_2 by k^* and x_3 by $k^* - 1$; For each y_j and its incident edges, if $|C_j| = |\{c_1\}| = 1$, then color y_j, $y_j x_1$, $y_j x_2$ and $y_j x_3$ by c_1; If $|C_j| = |\{c_1, c_2\}| = 2$, assume $c_1 < c_2$, then color y_j by c_2, color $y_j x_1$ and $y_j x_3$ by c_1, and color $y_j x_2$ by c_2; If $|C_j| = |\{c_1, c_2, c_3\}| = 3$, assume $c_1 < c_2 < c_3$, then color y_j by c_3, color $y_j x_1$ and $y_j x_3$ by c_2, and color $y_j x_2$ by c_1; If $|C_j| = |\{c_1, c_2, c_3, c_4\}| = 4$, assume $c_1 < c_2 < c_3 < c_4$, then color y_j by c_4, color $y_j x_1$ by c_1, $y_j x_2$ by c_2, and color $y_j x_3$ by c_3; Where $c_1, c_2, c_3 \in K^*$.

By the above coloring, because $\{k^* - 1, k^*\}$ is the color set of some vertex of Y, it follows that $k^* - 1$ appears at vertices x_1 and also x_3. In addition, because $\{1\}$, $\{2\}$,.., $\{k^* - 2\}$ are color sets of vertices of Y and $k^* - 1$ does not appear at vertex x_2, it follows that the color set of x_1 is $\{1, 2, \ldots, k^*\}$, the color set of x_2 is $\{1, 2, \ldots, k^* - 2, k^*\}$ and the color set of x_2 is $\{1, 2, \ldots, k^* - 2, k^* - 1\}$, by Lemma 2.1. Given that for any two $j_1 \neq j_2 \in \{1, 2, \ldots, n\}$ $C_{j_1} \neq C_{j_2}$ and $|C_j| \leq 4$, we can see that x_i has different color set with y_j for $i = 1, 2$, $j = 1, 2, \ldots, n$. So the coloring defined above is a k^*-*GVDTC* of $K_{3,n}$. □

We now consider the case of $K_{n,n}$.

Theorem 2.3 *For positive integer n, we have*

$$1 + \lceil \log_2 (n + \frac{1}{2}) \rceil \leq \chi_{gvt}(K_{n,n}) \leq 2 + \lceil \log_2 n \rceil.$$

Proof. Let $\chi_{gvt}(K_{n,n}) = k$, and suppose that f is a k-*GVDTC* of $K_{n,n}$. Denote the set of colors by $C = \{1, 2, \ldots, k\}$. In order to ensure that for any two vertices u, v, $C_t^f(u) \neq C_t^f(v)$, it demands that

$$2n \leq 2^k - 1.$$

So

$$k \geq 1 + \lceil \log_2(n + \frac{1}{2}) \rceil.$$

As for the upper bound, it suffices to prove that $K_{n,n}$ has a $(2+\lceil \log_2 n \rceil)$-*GVDTC*. Note that when $k = 2 + \lceil \log_2 n \rceil$, $n \leq 2^{k-2}$. In addition, we shall assume $n > 2^{k-3}$.

When $k = 2, 3, 4$, one can readily check that $\chi_{gvt}(K_{1,1}) = 2$; $\chi_{gvt}(K_{2,2}) = 3$; $\chi_{gvt}(K_{3,3}) = 4$; $\chi_{gvt}(K_{4,4}) = 4$.

When $k \geq 5$, it follows $n \geq k$. We now present a method to prove that $K_{n,n}$ has a *k-GVDTC*. Denote by S the family of all subsets of C that contain element 1, i.e. $S=\{\{1\},\{1,2\},\{1,3\}, \ldots,\{1,k\},\{1,2,3\},\{1,2,4\},\ldots,\{1,2,k\},\ldots\ldots, \{1,2,\ldots,k-1\},\{1,2,\ldots,k-2,k-1\}, \{1,2,\ldots,k-3,k-2,k-1\},\ldots,\{1,3,4,\ldots,k\},\{1,2,\ldots,k\}\}$. We intend to construct a *k-GVDTC* f with the following properties, denoted by (*)-rule:

$C_t^f(x_1) = \{1\}$; $C_t^f(x_i) = \{1, i\}$ for $i = 1, 2, \ldots, k$; $C_t^f(y_j) = C\backslash\{k+1-j\}$ for $j = 1, 2, \ldots, k-1$; $C_t^f(y_k) = \{1, 2, \ldots, k\}$; $C_t^f(x_i)$, $C_t^f(y_j)$, for $i, j = k+1, k+2, \ldots, n$ are any $2(n-k)$ different sets in $S\backslash(\cup_{i=1}^{k}\{C_t^f(x_i), C_t^f(y_i)\})$ (Since $|S|=2^{k-1}$, such $2(n-k)$ sets do exist).

If f meets the above demand, then f is a *k-GVDTC* of $K_{n,n}$. We now show that such f does exist.

Let f be:

(1) $f(x_i) = i, i = 1, 2, \ldots, k$; $f(x_i) = 1, i = k+1, k+2, \ldots, 2^{k-2}$; $f(y_i) = 1$.

(2) For $i = 1, 2, \ldots, k, j = 1, 2, \ldots, n$, $f(x_i y_j) = i$ when $i \in C_t^f(y_j)$, otherwise, $f(x_i y_j) = 1$.

Obviously, after the above two steps, all of $C_t^f(y_j), j = 1, 2, \ldots, n$ satisfy (*)-rule. In the following, we show that edges $x_i y_j$ for $i = k+1, k+2, \ldots, n, j = 1, 2, \ldots, n$ can be colored properly to make sure that each $C_t^f(x_i)$ for $i = k+1, k+2, \ldots, n$ has the properties of (*)-rule.

(3) For an arbitrary x_i, $i \in \{k+1, k+2, \ldots, n\}$, denote by $X_{ij} = C_t^f(x_i) \cap C_t^f(y_j)$ for $j = 1, 2, \ldots, k$. Let $f(x_i y_1) = 1$, and for $j = 2, \ldots, k$, let $f(x_i y_j)$ be the smallest number of $X = X_{ij} \setminus \{f(x_i y_1), f(x_i y_2), \ldots, f(x_i y_{j-1})\}$ when $X \neq \emptyset$; Otherwise let $f(x_i y_j) = 1$ when $X = \emptyset$.

By (3), it can be seen that each x_i, $i \in \{k+1, k+2, \ldots, n\}$, has the properties of (*)-rule. So, we need only color all the remaining edges by color 1.

To sum up, f is a *k-GVDTC* satisfying (*)-rule, and the conclusion holds. □

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