Implicit degree condition for hamiltonicity of graphs

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Abstract: In order to find more sufficient conditions for the existence of hamiltonian cycles of graphs, Zhu, Li and Deng proposed the definition of implicit degree of a vertex. In this paper, we consider the relationship between implicit degrees of vertices and the hamiltonicity of graphs, and obtain that: If the implicit degree sum for each pair of nonadjacent vertices of an induced claw or an induced modified claw in a 2-connected graph G is more than or equal to $|V(G)| - 1$, then G is hamiltonian with some exceptions. This extends a previous result of Cai et al. [J. Cai, H. Li and W. Ning, An implicit degree condition for hamiltonian cycles, Ars Combin. 108 (2013) 365-378.] on the existence of hamiltonian cycles.

Keywords: Implicit degree; Hamiltonian cycles; Induced claw; Induced modified claw

1 Introduction

Throughout this paper, we consider only finite, undirected and simple graphs. Let G be a graph and H be a subgraph of G, $G[H]$ denotes the subgraph of G induced by $V(H)$. For a vertex $u \in V(G)$, $N_H(u)$ and $d_H(u)$ denote the neighborhood and the degree of u in H, respectively. If $H = G$, we can use $N(u)$ and $d(u)$ in place of $N_G(u)$ and $d_G(u)$, respectively. Let $N_2(u) = \{v \in V(G) : d(u, v) = 2\}$, where $d(u, v)$ indicates the distance from u to v in G. Let A and B be the subsets of $V(G)$, $e(A, B)$ denotes the number of edges xy of G with $x \in A$ and $y \in B$. We write $e(A, y)$ instead of $e(A, \{y\})$.

A cycle (or path) containing all the vertices of G is called a *hamiltonian cycle* (or *hamiltonian path*) of G, G is called *hamiltonian* if it contains a hamiltonian cycle. We call a cycle C an *l-cycle* if $|V(C)| = l$. Other notation and terminology not defined here can be found in [2].

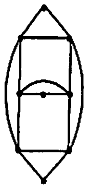

Fig. 1. H

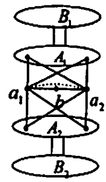

Fig. 2. $\mathscr{G}_n$

Hamiltonian problem is an important problem in graph theory. Various sufficient conditions for a graph to be hamiltonian have been given in terms of degree conditions. We have the following classic result due to Fan.

Theorem 1. *([7]) Let G be a 2-connected graph of order $n \geq 3$. If $\max\{d(u), d(v)\} \geq n/2$ for every pair of vertices u and v at distance 2, then G is hamiltonian.*

In 1987, Benhocine and Wojda [1] extended the result of Fan as follows. Where H is the graph of order 9 depicted in Fig.1 and $\mathscr{G}_n$ denotes the family of graphs such that $G \in \mathscr{G}_n$ if and only if $|V(G)| = n$ and the vertex-set of G is the disjoint union of the sets A_1, A_2, B_1, B_2 and $\{a_1, a_2, b\}$ so that (i) $|A_i \cup B_i| = \frac{n-3}{2}, i = 1, 2$; (ii) $|A_i| \geq 2, i = 1, 2$; (iii) $G[A_i \cup B_i]$ and $G[A_i \cup \{a_j\}]$ are both complete subgraphs of G for $i = 1, 2$ and $j = 1, 2$; (iv) $e(a_1, a_2) \leq 1$; (v) $|A_1 \cup A_2| \geq \frac{n-3}{2} - e(a_1, a_2)$; and and (vi) $d(b) = 2$ and the neighbors of b are a_1 and a_2. (See Fig.2)

Theorem 2. *([1]) Let G be a 2-connected graph of order $n \geq 3$ with independent number $\alpha(G) \leq \frac{n}{2}$ such that $\max\{d(u), d(v)\} \geq \frac{n-1}{2}$ for each pair of vertices u and v at distance 2, then either G is hamiltonian or $G \in \mathscr{G}_n \cup H$.*

In the case that some vertices may have small degrees, we hope to use some large degree vertices to replace some small degree vertices in the right position considered in the proofs, so that we may construct a longer cycle. This idea leads to the definition of *implicit degree* given by Zhu, Li and Deng [9] in 1989.

Definition 1. *([9]) Let v be a vertex of a graph G. If $N_2(v) \neq \emptyset$ and $d(v) \geq 2$, then set $l = d(v) - 1$, $m_2^v = \min\{d(u) : u \in N_2(v)\}$ and $M_2^v = \max\{d(u) : u \in N_2(v)\}$. Suppose that $d_1^v \leq d_2^v \leq \ldots \leq d_{l+1}^v \leq \ldots$ is the degree sequence of vertices of $N(v) \cup N_2(v)$. Let*

$$d^*(v) = \begin{cases} m_2^v, & \text{if } m_2^v > d_l^v; \\ d_{l+1}^v, & \text{if } d_{l+1}^v > M_2^v; \\ d_l^v, & \text{otherwise.} \end{cases}$$

Then the implicit degree of v is defined as $id(v) = \max\{d(v), d^*(v)\}$. If $N_2(v) = \emptyset$ or $d(v) \leq 1$, then $id(v) = d(v)$.

Clearly, $id(v) \geq d(v)$ for every vertex v from the definition of implicit degree. The authors in [9] used implicit degree in place of degree in Ore's theorem [8] and gave a sufficient condition for a 2-connected graph to be hamiltonian.

Theorem 3. *([9]) Let G be a 2-connected graph of order $n \geq 3$. If $id(u) + id(v) \geq n$ for each pair of nonadjacent vertices u and v in G, then G is hamiltonian.*

In 2006, Chen and Zhang extended Theorem 3 as follows.

Theorem 4. *([3]) Let G be a 2-connected graph such that $\max\{id(u), id(v)\} \geq c/2$ for each pair of nonadjacent vertices u and v that are vertices of an induced claw ($K_{1,3}$) or an induced modified claw ($K_{1,3} + e$). Then G contains either a hamiltonian cycle or a cycle of length at least c.*

The join of two disjoint graphs G and H, denoted by $G \vee H$, is defined as: $V(G \vee H) = V(G) \cup V(H)$ and $E(G \vee H) = E(G) \cup E(H) \cup \{uv : u \in V(G)$ and $v \in V(H)\}$. Recently, Cai, Li and Ning [4] extended Theorem 2 as follows. Where $\mathscr{H}_n = (kK_1 \cup 2K_{\frac{n-1}{2}-k}) \vee K_{k+1}$, $\mathscr{B}_n$ denotes the family of graphs such that $G \in \mathscr{B}_n$ if and only if $|V(G)| = n$ and $V(G)$ is the disjoint union of the sets A_1, A_2, B_1, B_2 and $\{a_1, a_2, b\}$ so that they satisfy the above (i),(iv),(v),(vi) and (vii) $G[A_i \cup \{a_j\}]$ is complete subgraph of G and $uv \in E(G)$ for any vertex $u \in A_i$ and any vertex $v \in B_i$ for $i = 1, 2$ and $j = 1, 2$; (viii) $|A_i| \geq \max\{2, |\{b : d(b) < \frac{n-5}{2}$ and $b \in B_i\}| + 1\}, i = 1, 2$.

Theorem 5. *([4]) Let G be a 2-connected graph of order $n \geq 3$ such that $id(u) + id(v) \geq n - 1$ for each pair of vertices u and v at distance 2, then either G is hamiltonian or $G \in \mathscr{B}_n \cup H$ or G is a subgraph of $\mathscr{H}_n \cup (\frac{n+1}{2}K_1 \vee K_{\frac{n-1}{2}})$.*

Motivated by the results of Theorem 2 and Theorem 5, we study implicit degrees and the hamiltonicity of graphs and extend Theorem 5 as follows.

Theorem 6. *Let G be a 2-connected graph of order $n \geq 3$. If $id(u) + id(v) \geq n - 1$ for each pair of vertices u and v that are vertices of an induced claw or an induced modified claw, then either G is hamiltonian or $G \in \mathscr{B}_n \cup H$ or G is a subgraph of $\mathscr{H}_n \cup (\frac{n+1}{2}K_1 \vee K_{\frac{n-1}{2}})$.*

2 Lemmas

For a cycle C in G with a given orientation and a vertex x in C, x^+ and x^- denote the successor and the predecessor of x in C, respectively. Define

$x^{(h+1)+} = (x^{h+})^{+}$ for every integer $h \geq 0$, with $x^{0+} = x$. And for any $I \subseteq V(C)$, let $I^{-} = \{x : x^{+} \in I\}$ and $I^{+} = \{x : x^{-} \in I\}$. For two vertices $x, y \in C$, xCy denotes the subpath of C from x to y. We use $y\bar{C}x$ for the path from y to x in the reversed direction of C.

Lemma 1. *([6]) Let G be a 2-connected graph of order $n \geq 3$.If xPy is a longest path of G such that $d(x) + d(y) \geq n$, then G is a hamiltonian.*

Lemma 2. *([1]) If a graph G of order $n \geq 3$ has a cycle C of length $n-1$, such that the vertex not in C has degree at least $\frac{n}{2}$, then G is hamiltonian.*

Lemma 3. *([9]) Let G be a 2-connected graph and $P = x_1x_2 \ldots x_p$ be a longest path of G. If $d(x_1) < id(x_1)$ and $x_1x_p \notin E(G)$, then either*
(1) there is some vertex $x_j \in (N(x_1))^{-}$ such that $d(x_j) \geq id(x_1)$; or
(2) $N(x_1) = \{x_2, x_3, \ldots, x_{d(x_1)+1}\}$ and $id(x_1) = m_2^{x_1}$.

Lemma 4. *([4]) Let $P = x_1x_2 \ldots x_p$ be a path and y_1, y_2 be two vertices not in $V(P)$. If $(N_P(y_1))^{-} \cap N_P(y_2) = \emptyset$ and $x_1y_1 \notin E(G)$, then $d_P(y_1) + d_P(y_2) \leq |V(P)|$.*

3 Proof of Theorem 6

Let G be a graph satisfying the condition in Theorem 6 and suppose G is not hamiltonian. By Lemma 4, G contains an $(n-1)$-cycle. We choose an $(n-1)$-cycle C such that the degree of the vertex not in C is as large as possible. Let x be the vertex not in C of G. Without loss of generality, we give C a clockwise orientation, and define $y_1, y_2, \ldots, y_{k+1} (k \geq 1)$ to be the neighbors of x. Since G is not hamiltonian, $\{x, y_1^{+}, y_2^{+}, \ldots, y_{k+1}^{+}\}$ is an independent set and $d(x, y_i^{+}) = 2$ for every $i = 1, 2, \ldots, k+1$. Similarly, $\{x, y_1^{-}, y_2^{-}, \ldots, y_{k+1}^{-}\}$ is an independent set and $d(x, y_i^{-}) = 2$ for every $i = 1, 2, \ldots, k+1$. Moreover, $d(x) \leq \frac{n-1}{2}$ by Lemma 2.

If $d(x) = \frac{n-1}{2}$, then $\{x, y_1^{+}, y_2^{+}, \ldots, y_{\frac{n-1}{2}}^{+}\}$ is an independent set of G with $\frac{n+1}{2}$ elements. It is easy to check that G is the subgraph of $\frac{n+1}{2}K_1 \vee K_{\frac{n-1}{2}}$. So next we can assume $d(x) < \frac{n-1}{2}$.

Let $P = y_1^{+}y_1^{2+} \ldots y_1^{h+}y_2xy_1y_2^{l+}y_2^{(l-1)+} \ldots y_2^{+}$, where h and l are the minimum integers such that $y_1^{h+} = y_2^{-}$ and $y_2^{l+} = y_1^{-}$, respectively. For convenience, let $P = x_1x_2 \ldots x_n$, where $x_1 = y_1^{+}, x_2 = y_1^{2+}$, and so on. Without loss of generality, suppose $x_m = x$.

Claim 1. $id(x) \geq \frac{n-1}{2}$.

Proof. Suppose to the contrary that $id(x) < \frac{n-1}{2}$. For every $i = 1, 2, \ldots, k+1$, since $\{y_i, x, y_i^{-}, y_i^{+}\}$ induces a claw or a modified claw, $id(y_i^{+}) > \frac{n-1}{2}$.

Since P is a hamiltonian path of G, $x_1x_n \notin E(G)$. By Lemma 1, we can assume, without loss of generality, that $id(x_1) > d(x_1)$. Since $y_1^+y_1 \in E(G)$ and $y_1^+x \notin E(G)$, $N_P(x_1) \neq \{x_2, x_3, \ldots, x_{d(x_1)+1}\}$. Therefore, there exists a vertex $x_i \in (N_P(x_1))^-$ such that $d(x_i) \geq id(x_1)$ by Lemma 3. Then $P' = x_nx_{n-1}\ldots x_{i+1}x_1x_2\ldots x_i$ is hamiltonian path of G. Since $d(x) < \frac{n-1}{2}$, $i \neq m$. If $id(x_n) = d(x_n)$, then $d(x_n) + d(x_i) \geq id(x_n) + id(x_1) > n-1$, and hence by Lemma 1, G is hamiltonian, a contradiction.

Suppose $d(x_n) < id(x_n)$. For convenience, let $P' = z_1z_2\ldots z_n$. Since $y_2^+y_2 \in E(G), y_2^+x \notin E(G)$ if $i < m$ and $y_2^+y_2 \in E(G), y_2^+y_1^+ \notin E(G)$ if $i > m$, $N_{P'}(x_n) \neq \{z_2, z_3, \ldots, z_{d(x_n)+1}\}$. Therefore, there is a vertex $z_j \in (N_{P'}(x_n))^-$ such that $d(z_j) \geq id(x_n)$. Then $P'' = z_jz_{j-1}\ldots z_1z_{j+1}z_{j+2}\ldots z_n$ is a hamiltonian path of G with $d(z_j) + d(z_n) \geq id(x_n) + id(x_1) > n-1$. Thus G is hamiltonian by Lemma 1, a contradiction. □

By Claim 1, we know $d(x) < id(x)$. Moreover, by the proof of Claim 1, we have $id(y_i^+) \leq \frac{n-1}{2}$ for each $i = 1, 2, \ldots, k+1$. Since $d(x, y_i^+) = 2$, $|N_2(x)| \geq k+1$. By the definition of implicit degree, we can easily get that $id(x) \neq d_{k+1}^x$. We consider the following two cases.

Case 1. $id(x) = m_2^x$.

For each $i = 1, 2, \ldots, k+1$, since $d(x, y_i^+) = 2$, $d(y_i^+) \geq m_2^x = id(x) \geq \frac{n-1}{2}$. Since G is not hamiltonian, it is easy to check that (1) $e(y_1^+, z^+) + e(y_2^+, z) \leq 1$ for every $z \in A = \{y_1^+, y_1^{2+}, \ldots, y_1^{h+}\}$; and (2) $e(y_1^+, z) + e(y_2^+, z^+) \leq 1$ for every $z \in B = \{y_2^+, y_2^{2+}, \ldots, y_2^{l+}\}$. As $y_1^+x \notin E(G)$ and $y_2^+x \notin E(G)$, (1) and (2) imply

$$\begin{aligned} n-1 &\leq d(y_1^+) + d(y_2^+) \\ &= \sum_{z\in A}[e(y_1^+, z^+) + e(y_2^+, z)] + \sum_{z\in B}[e(y_1^+, z) + e(y_2^+, z^+)] \\ &\quad + e(y_1^+, y_1) + e(y_2^+, y_2) \\ &\leq h + l + 2 = n-1, \end{aligned}$$

which implies that all the inequalities above are equalities. In particular, $d(y_1^+) = d(y_2^+) = \frac{n-1}{2}$, n is odd and $id(x) = \frac{n-1}{2}$.

Claim 2. $d(x) = 2$.

Proof. Suppose $d(x) \geq 3$. Then $e(y_1^+, y_3^+) + e(y_2^+, y_3^{2+}) = 1$. Since $y_1^+y_3^+ \notin E(G)$, $y_2^+y_3^{2+} \in E(G)$. So $C' = y_2^+Cy_3xy_2\bar{C}y_3^{2+}y_2^+$ is an $(n-1)$-cycle avoiding y_3^+ whose degree is at least $\frac{n-1}{2}$, contrary to the choice of C. □

By Claim 2 and by the choice of C, we can assume that whenever we have an $(n-1)$-cycle, then the vertex not in the cycle has degree precisely 2. By analogous argument as in the proof of Claim 2, we can get that $y_1^+y_1^{3+} \in E(G)$, $y_2^+y_2^{3+} \in E(G)$, $y_2^+y_1^{2+} \notin E(G)$ and $y_1^+y_2^{2+} \notin E(G)$.

Observe that y_1^+ and y_2^+ have degree precisely $\frac{n-1}{2}$ and are joined by the hamiltonian path P. We can easily deduce the following useful properties:
Property 1. $e(x_1, x_{i+1}) + e(x_n, x_i) = 1$ for every $i = 1, 2, \dots, n-1$.
Property 2. If $e(x_1, x_{i+1}) + e(x_n, x_{i-1}) = 2$ for some $i = 2, 3, \dots, n-1$, then $d(x_i) = 2$. Moreover, by the definition of implicit degree, we have $d(x_{i-2}) \geq id(x_i) \geq \frac{n-1}{2}$ and $d(x_{i+2}) \geq id(x_i) \geq \frac{n-1}{2}$.
Property 3. $x_1x_{n-1} \notin E(G)$ and $x_nx_2 \notin E(G)$.

Since $x_1x_3 = y_1^+y_1^{3+} \in E(G), y_1^+y_1 \in E(G)$ and $y_1^+x \notin E(G)$, only two cases can arise.

Case 1.1. There are i and j with $j \geq i+1$, such that $x_1x_{i-1}, x_1x_{j+1} \in E(G)$ and $x_1x_s \notin E(G)$ for each $s = i, i+1, \dots, j$.

Choose such i such that i is as small as possible. By Property 1 and Property 3, we have $i \geq 4$, $j \leq n-3$ and $x_nx_s \in E(G)$ for all $s = i-1, i, \dots, j-1$.

Claim 3. If $z_1z_2\dots z_n$ is a hamiltonian path of G such that there are i and j with $i+1 \leq j$, $z_1z_{i-1} \in E(G), z_1z_{j+1} \in E(G), z_1z_s \notin E(G)$ for each $s = i, i+1, \dots, j$, then $d(x_{j-2}) \geq \frac{n-1}{2}$ and $d(x_{j+2}) \geq \frac{n-1}{2}$. Moreover, $j = i+1$.

Proof. Suppose $j \geq i+2$. By Property 2, $d(z_j) = 2$. By similar proof as in Claim 1, $id(z_j) \geq \frac{n-1}{2}$. Moreover, by the definition of implicit degree, $d(x_{j-2}) \geq id(x_j) \geq \frac{n-1}{2}$ and $d(x_{j+2}) \geq id(x_j) \geq \frac{n-1}{2}$.

Since $z_{j-2}z_{j-3}\dots z_1z_{j+1}z_jz_{j-1}z_nz_{n-1}\dots z_{j+2}$ is a hamiltonian path, $z_1z_{j-2} \notin E(G)$ and $z_{j+2}z_2 \in E(G)$. Then $z_{j+1}z_jz_{j-1}\dots z_iz_nz_{n-1}\dots z_{j+2}z_2z_3 \dots z_{i-1}z_1z_{j+1}$ is a hamiltonian cycle, a contradiction. So $j = i+1$. □

Claim 4. $x_1x_s \in E(G)$ for each $s \leq i-2$.

Proof. By the choice of i, we suppose to the contrary that there exists some $s, (4 \leq s \leq i-2)$ such that $x_1x_{s-1}, x_1x_{s+1} \in E(G)$ and $x_1x_s \notin E(G)$. Since $x_1x_2, x_1x_3 \in E(G)$, $s \geq 4$. By Property 1, $x_nx_{s-1} \in E(G)$ and $x_nx_{s-2} \notin E(G)$; by Property 2, $d(x_s) = 2$, thus $d(x_{s+2}) \geq id(x_s) \geq \frac{n-1}{2}$ and $d(x_{s-2}) \geq id(x_s) \geq \frac{n-1}{2}$. So $x_1x_{s-2} \in E(G)$.

Next, we will distinguish the following two cases to discuss.
(1) $x_1x_{s+2} \notin E(G)$.

By Property 1, $x_nx_{s+1} \in E(G)$. Since $d(x_{s+2}) \geq \frac{n-1}{2}$, $x_1x_{s+3} \notin E(G)$ by Property 2. Thus, $x_nx_{s+2} \in E(G)$. By the choice of i, we have $i = s+2$. Then $i \geq 6$ and $d(x_i) \geq \frac{n-1}{2}$.

By Claim 3, $d(x_{i-1}), d(x_{i+3}) \geq \frac{n-1}{2}$. Let $P' = x_1x_2 \ldots x_{i-1}x_nx_{n-1} \ldots x_i$. Then P' is a hamiltonian path with $x_1x_{i-1}, x_1x_{i+2} \in E(G)$ and $x_1x_n, x_1x_{n-1} \notin E(G)$. By Claim 3 again, $x_1x_{n-2} \in E(G)$. Moreover, $d(x_{n-1}) = 2$ and $d(x_{i+2}) \geq \frac{n-1}{2}$. Then use P we can obtain $x_nx_{n-3} \notin E(G)$.

If $i+3 < n-2$, then since $x_{i-1}x_{i-2} \ldots x_1x_{i+2}x_{i+1}x_ix_nx_{n-1} \ldots x_{i+3}$ is a hamiltonian path of G and $x_{n-1}x_{i+3} \notin E(G)$, $x_{i-1}x_{n-2} \in E(G)$ by Property 1. Moreover, considering the hamiltonian path $x_1x_2 \ldots x_{i-1}x_{n-2}x_{n-1}x_n$ $x_ix_{i+1} \ldots x_{n-3}$ and observing that $x_{n-3}x_n \notin E(G)$ implies $x_1x_i \in E(G)$ by Property 1, but this contradicts the hypothesis in Case 1.1.

Suppose $i+3 = n-2$. Since $x_2x_3 \ldots x_{i-1}x_nx_{n-1}x_{n-2}x_1x_{i+2}x_{i+1} \cdots x_i$ is a hamiltonian path, $x_ix_2 \notin E(G)$. Considering the hamiltonian path $x_ix_{i+1}x_{i+2}x_1x_2 \ldots x_{i-1}x_{i+5}x_{i+4}x_{i+3}$ and $d(x_{i+1}) = 2$, we have $x_ix_{i+2} \in E(G)$ by Property 1. Since $x_ix_1 \notin E(G)$ and $x_ix_{i-1} \in E(G)$, $x_ix_3 \in E(G)$by Claim 3. This implies that $d(x_2) = 2$. Then $d(x_4) \geq \frac{n-1}{2}$. Since $x_{i+2}x_{i+1}x_ix_3x_2x_1x_{i+3}x_{i+4}x_{i+5}x_{i-1}x_{i-2} \ldots x_4$ is a hamiltonian path, $x_{i+2}x_{i+5} \in E(G)$ by Property 1 and the fact $d(x_{i+4}) = 2$. Then $x_{i+2}x_{i+1} \cdots$ $x_1x_{i+3}x_{i+4}x_{i+5}x_{i+2}$ is a hamiltonian cycle, a contradiction.

(2) $x_1x_{s+2} \in E(G)$.

By Property 1, $x_nx_{s+1} \notin E(G)$. Since $x_{s-2}x_{s-3} \ldots x_1x_{s-1}x_sx_{s+1} \cdots x_n$ is a hamiltonian path and $x_nx_{s+1} \notin E(G)$, $x_{s-2}x_{s+2} \in E(G)$. Then $x_{s-2}x_{s-3} \ldots x_1x_{s+1}x_sx_{s-1}x_nx_{n-1} \ldots x_{s+2}x_{s-2}$ is a hamiltonian cycle of G, a contradiction. □

Claim 5.([4]) $x_1x_{i+3} \in E(G)$.

Claim 6.([4]) $x_1x_s \notin E(G)$ for each $s = i+4, i+5, \ldots, n$.

By Claim 6, $e(x_1, \{x_{i+4}, x_{i+5}, \ldots, x_n\}) = 0$. So $e(x_n, \{x_{i-1}, x_i, x_{i+3}, x_{i+4}, \ldots, x_{n-1}\}) = n-i-1$ by Property 1. Thus, $i = \frac{n-1}{2}$. For every $s \leq i-2$ and $t \geq i+4$, we have $x_sx_t, x_sx_i, x_tx_{i+2} \notin E(G)$, for $x_sx_{s-1} \ldots x_1x_{s+1}x_{s+2} \cdots$ $x_{t-1}x_nx_{n-1} \ldots x_t$, $x_sx_{s-1} \ldots x_1x_{s+1}x_{s+2} \cdots x_{i-1}x_nx_{n-1} \ldots x_i$, $x_tx_{t+1} \cdots x_n$ $x_{t-1}x_{t-2} \ldots x_{i+3}x_1x_2 \ldots x_{i+2}$ are hamiltonian paths of G, respectively. Then $\{x_{i-1}, x_i, x_{i+1}x_{i+2}, x_{i+3}\}$ is a cut-set of G. Let $U_1 = \{x_1, x_2, \ldots, x_{i-2}\}$ and $U_2 = \{x_{i+4}, x_{i+5}, \ldots, x_n\}$, we see that $|U_1| = |U_2| = \frac{n-5}{2}$. Moreover, $d(x_s) \leq \frac{n-1}{2}$ for any $x_s \in U_1 \cup U_2$, and if $d(x_s) = \frac{n-1}{2}$ for some $x_s \in U_1 \cup U_2$, then $N(x_s) = (U_1 \setminus \{x_s\}) \cup \{x_{i-1}, x_{i+2}, x_{i+3}\}$ when $x_s \in U_1$ and $N(x_s) = (U_2 \setminus \{x_s\}) \cup \{x_{i-1}, x_i, x_{i+3}\}$ when $x_s \in U_2$.

Case 1.1.1 $d(x_i) \geq \frac{n-1}{2}$.

Then $P' = x_1x_2\ldots x_{i-1}x_nx_{n-1}\ldots x_i$ is a hamiltonian path with $d(x_1) \geq \frac{n-1}{2}$ and $d(x_i) \geq \frac{n-1}{2}$. Since $x_1x_{i-1} \in E(G), x_1x_n \notin E(G), x_1x_{n-1} \notin E(G), x_1x_{i+2} \in E(G)$, we have $x_1x_{n-2} \in E(G)$ by Claim 3. Therefore, $n-2 \leq i+3$ by Claim 6. So $n \leq 9$. Since $x_1x_i \notin E(G)$, $i \geq 4$. Then $n = 9$ and G is isomorphic to H.

Case 1.1.2 $d(x_i) < \frac{n-1}{2}$.

Claim 7. $id(x_i) \geq \frac{n-1}{2}$.

Proof. Suppose $id(x_i) < \frac{n-1}{2}$. Since $x_{i-2}x_{i-3}\ldots x_1x_{i-1}x_nx_{n-1}\ldots x_i$ is a hamiltonian path, $x_{i-2}x_i \notin E(G)$. Then $\{x_{i-1}, x_1, x_{i-2}, x_i\}$ induced a modified claw and $id(x_{i-2}) > \frac{n-1}{2}$. Considering the hamiltonian path $P' = x_{i-2}x_{i-3}\ldots x_1x_{i-1}x_i\ldots x_n = z_1z_2\ldots z_n$ and $d(x_i) < \frac{n-1}{2}$ with $x_i \in N_2(x_{i-2})$, by Lemma 3, there must exist a vertex $z_s \in (N_{P'}(z_1))^-$ such that $d(z_s) \geq id(z_1) > \frac{n-1}{2}$. Then $P'' = z_sz_{s-1}\ldots z_1z_{s+1}z_{s+2}\ldots z_n$ is a hamiltonian path with $d(z_s) + d(z_n) > \frac{n-1}{2} + \frac{n-1}{2} = n-1$. Then by Lemma 1, G is hamiltonian, a contradiction. □

Claim 8.([4]) If $x_ix_t \in E(G)$ for some $x_t \in U_2$, then $x_ix_{t+1}, x_ix_{t+2} \notin E(G)$.

Let $d(x_i) = s+1$. By the above, we can get that $((N(x_i))^- \cup (N(x_i))^+) \cap U_2 \subseteq N_2(x_i)$ and $(N(x_i))^- \cap (N(x_i))^+ = \emptyset$. Thus, $|(N(x_i))^- \cup (N(x_i))^+| \geq 2s-3 \geq s$ and $d(x_t) < \frac{n-1}{2}$ for any $x_t \in (N(x_i))^- \cup (N(x_i))^+$. It is contrary to the definition of implicit degree.

Case 1.2. $x_1x_{i-1} \in E(G), x_1x_{i+1} \in E(G)$ and $x_1x_i \notin E(G)$ for some $i = 4, 5, \ldots, n-3$.

Choose such i such that i is as small as possible, then $e(x_1, \{x_2, x_3, \ldots, x_{i-1}\}) = i-2$ and $e(x_n, \{x_1, x_2, \ldots, x_{i-2}\}) = 0$. By Property 1, $x_nx_{i-1} \in E(G)$ and $x_nx_{i-2} \notin E(G)$; by Property 2, $d(x_i) = 2$, thus $d(x_{i+2}) \geq id(x_i) \geq \frac{n-1}{2}$ and $d(x_{i-2}) \geq id(x_i) \geq \frac{n-1}{2}$. So $x_nx_{i-3} \notin E(G)$.

Since $x_{i-2}x_{i-3}\ldots x_1x_{i-1}x_i\ldots x_n$ is a hamiltonian path and $x_nx_i \notin E(G)$, $x_{i-2}x_{i+1} \in E(G)$ by Property 1. But since $x_1x_2\ldots x_{i-2}x_{i+1}x_ix_{i-1}x_n x_{n-1}\ldots x_{i+2}$ is a hamiltonian path of G, we have $x_1x_{i+2} \notin E(G)$. Which implies by Property 1, $x_nx_{i+1} \in E(G)$ and by Property 2, $x_1x_{i+3} \notin E(G)$. Now, we can suppose that $e(x_1, \{x_{i+2}, x_{i+3}, \ldots, x_n\}) = 0$, otherwise Case 1.1 holds. Thus $e(x_n, \{x_{i+1}, x_{i+2}, \ldots, x_{n-1}\}) = n-i-1$. The degree of x_1 and x_n impose $i = \frac{n+1}{2}$.

For every $s \leq i-2$ and $t \geq i+2$, we have $x_sx_t \notin E(G)$ for $x_sx_{s-1}\ldots x_1 x_{s+1}x_{s+2}\ldots x_{t-1}x_nx_{n-1}\ldots x_t$ is a hamiltonian path of G. We deduce that

$\{x_{i-1}, x_i, x_{i+1}\}$ is a cut-set of G, and $d(u) \le \frac{n-1}{2}$ for any $u \in V_1 \cup V_2$, where $V_1 = \{x_1, x_2, \ldots, x_{i-2}\}$ and $V_2 = \{x_{i+2}, x_{i+3}, \ldots, x_n\}$. We see that $|V_1| = |V_2| = \frac{n-3}{2}$ and $x_i = x$. Moreover, if $x_s \in V_1 \cup V_2$ such that $x_s x_{i-1} \in E(G)$ or $x_s x_{i+1} \in E(G)$, then $d(x_s) = \frac{n-1}{2}$ for $id(x) = m_2^x$ and $x_s \in N_2(x)$. And $e(x_s, \{x_{i-1}, x_{i+1}\} = 0, 2$ for each $x_s \in V_1 \cup V_2$.

Claim 9. *$id(x_{i-1}) \ge \frac{n-1}{2}$ and $id(x_{i+1}) \ge \frac{n-1}{2}$. Moreover, $d(x_{i-1}) = id(x_{i-1}) \ge \frac{n-1}{2}$ and $d(x_{i+1}) = id(x_{i+1}) \ge \frac{n-1}{2}$.*

Proof. Firstly, suppose, without loss of generality, that $id(x_{i-1}) < \frac{n-1}{2}$. Then $d(x_{i-1}) \le id(x_{i-1}) < \frac{n-1}{2}$. Since $i = \frac{n+1}{2}$, there exists some vertex, say x_j, in $\{x_2, x_3, \ldots, x_{i-2}\}$ such that $x_{j-1}x_{i-1} \in E(G)$ and $x_j x_{i-1} \notin E(G)$. Since $d(x_i) = 2$, $d(x_{j-1}) \ge \frac{n-1}{2}$ and thus $x_{j-1}x_{i+1} \in E(G)$. Then $\{x_{j-1}, x_j, x_{i-1}, x_{i+1}\}$ induces a claw or a modified claw. Thus $id(x_j) > \frac{n-1}{2}$. Considering the hamiltonian path $P' = x_j x_{j-1} \ldots x_1 x_{j+1} x_{j+2} \ldots x_n = z_1 z_2 \ldots z_n$ and using the fact that $x_{i-1} \in N_2(x_j)$ and $d(x_{i-1}) < \frac{n-1}{2}$, we can get that there exists some vertex $z_s \in (N(z_1))^-$ such that $d(z_s) \ge id(z_1) > \frac{n-1}{2}$ by Lemma 3. Then $P'' = z_s z_{s-1} \ldots z_1 z_{s+1} z_{s+2} \ldots z_n$ is a hamiltonian path with $d(z_s) + d(z_n) > n - 1$. Then, by Lemma 1, G is hamiltonian, a contradiction. So $id(x_{i-1}) \ge \frac{n-1}{2}$.

Secondly, suppose $d(x_{i-1}) < id(x_{i-1})$ and let $d(x_{i-1}) = l+1$. Since x_1 is adjacent to each vertex of $\{x_2, x_3, \ldots, x_{i-1}, x_{i+1}\}$ and x_n is adjacent to each vertex of $\{x_{i-1}, x_{i+1}, x_{i+2}, \ldots, x_{n-1}\}$, we get that $|N(x_{i-1}) \cup N_2(x_{i-1})| = n - 1$. Since each vertex with degree at least $\frac{n-1}{2}$ must be adjacent to x_{i-1} and x_{i+1}, we get that $d(u) < \frac{n-1}{2}$ for each $u \in N_2(x_{i-1})$. By the definition of implicit degree, we can easily check that $id(x_{i-1}) \ne m_2^{x_{i-1}}, d_l^{x_{i-1}}$. Therefore, $id(x_{i-1}) = d_{l+1}^{x_{i-1}}$, then $d_{l+1}^{x_{i-1}} > M_2^{x_{i-1}}$, but $|N_2(x_{i-1})| > l$, a contradiction. So $d(x_{i-1}) = id(x_{i-1})$. Similarly, $d(x_{i+1}) = id(x_{i+1})$. □

For $j = 1, 2, V_j$ can be partitioned into $A_j \cup B_j$ such that $d(a) \ge \frac{n-1}{2}$ for each $a \in A_1 \cup A_2$ and $d(b) < \frac{n-1}{2}$ for each $b \in B_1 \cup B_2$. Since $x_1, x_{i-2}, x_{i+2}, x_n$ have degree at least $\frac{n-1}{2}$, we have $|A_j| \ge 2, j = 1, 2$. Moreover, taking $a \in A_1$, we have

$$\begin{aligned} \frac{n-1}{2} &\le d(a) \\ &\le |A_1| - 1 + |B_1| + e(a, \{x_{i-1}, x_{i+1}\}) \\ &\le |V_1| + 1. \end{aligned}$$

And similarly, $\frac{n-1}{2} \le |V_2| + 1$. Then $n - 1 \le |V_1| + |V_2| + 2 = n - 1$, that implies $e(A_1 \cup A_2, \{x_{i-1}, x_{i+1}\}) = 2|A_1 \cup A_2|$.

If $B_1 \cup B_2 = \emptyset$, then $d(u, x_i) = 2$ for any $u \in V_1 \cup V_2$. Therefore, by the definition of implicit degree, we have $d(u) = \frac{n-1}{2}$ for any $u \in V_1 \cup V_2$. Then $G \in \mathscr{B}_n$.

So suppose $B_1 \cup B_2 \neq \emptyset$. Since each vertex of $B_j, j = 1, 2$, is not adjacent to $\{x_{i-1}, x_{i+1}\}$, $d(x_{i-1}) = d(x_{i+1}) = |A_1| + |A_2| + 1 + e$ where $e = e(x_{i-1}, x_{i+1})$. Since $d(x_{i-1}) = d(x_{i+1}) \geq \frac{n-1}{2}$, we get $|A_1| + |A_2| + 1 + e \geq \frac{n-1}{2}$, so $|A_1| + |A_2| \geq \frac{n-3}{2} - e$.

Claim 10. ([4]) *For any two vertices $a, b \in B_1$, if $ab \notin E(G)$, then $id(a) \geq \frac{n-1}{2}$ and $id(b) \geq \frac{n-1}{2}$. Similar for B_2.*

If $G[B_1]$ is not a complete graph, then choose two vertices $a, b \in B_1$ such that $ab \notin E(G)$. By Claim 10, $d(b) < id(b)$. Let $d(b) = \alpha + 1$, $|A_1| = m$, $|N(b) \cap B_1| = k_1$ and $|N_2(b) \cap B_1| = k_2$. Then $k_1 + k_2 + m = \frac{n-5}{2}$ and $\alpha + 1 = k_1 + m$. Since $d(x_{i-1}, b) = 2$ and $d(x_{i-1}) \geq \frac{n-1}{2}$, $id(b) \neq d^b_{\alpha+1}, m^b_2$. So $id(b) = d^b_\alpha$. Therefore, $k_1 + k_2 \leq \alpha - 1 = k_1 + m - 2$. Then $k_2 \leq m - 2$. By the arbitrary of b, we have $|A_1| \geq \max\{|N_2(b) \cap B_1| + 2 : b \in B_1\}$. If $G[B_1]$ is a complete graph, then $N_2(u) = \emptyset$ for each vertex $u \in B_1$. Since $|A_1| \geq 2 = \{|N_2(b) \cap B_1| + 2 : b \in B_1\}$. Therefore, $|A_1| \geq \max\{|N_2(b) \cap B_1| + 2 : b \in B_1\}$. Similarly, $|A_2| \geq \max\{|N_2(b) \cap B_2| + 2 : b \in B_2\}$. Consequently, $G \in \mathscr{B}_n$.

Case 2. $id(x) = d^x_k$.

Then $d^x_k > m^x_2$ and $k \geq 2$. Let $W_1 = \{y_i : |V(C(y_i, y_{i+1}))| = 1\}$ and $W_2 = \{y_i : |V(C(y_i, y_{i+1}))| \geq 2\}$. Set $|W_1| = w_1$ and $|W_2| = w_2$. Then $w_1 + w_2 = k + 1$. Moreover, $\{y_i^+, y_{i+1}^- : y_i \in W_2\} \subseteq N_2(x)$ and $\{y_i^+ : y_i \in W_1\} \subseteq N_2(x)$. So $|N_2(x)| \geq w_1 + 2w_2$. By the choice of C, we can get that $d(y_i^+) \leq d(x) < \frac{n-1}{2}$ for any $y_i \in W_1$. Since $id(x) = d^x_k$, there are at least $w_2 + 2$ vertices in $N_2(x)$ with degree at least $id(x) \geq \frac{n-1}{2}$.

Claim 11. $w_2 = 2$.

Proof. If $w_2 \leq 1$, then since there are at least $w_2 + 2$ vertices in $N_2(x)$ with degree at least $\frac{n-1}{2}$, we can easily check that there exists at least one vertex, without loss of generality, say y_1, in W_1 such that $d(y_1^+) \geq \frac{n-1}{2}$, contrary to the choice of C.

If $w_2 \geq 3$, then there are at least three vertices in $\{y_i^+ : y_i \in W_2\}$ with degrees at least $id(x)$ or at least three vertices in $\{y_{i+1}^- : y_i \in W_2\}$ with degrees at least $id(x)$. Without loss of generality, suppose there are at least three vertices in $\{y_i^+ : y_i \in W_2\}$ with degrees at least $id(x) \geq \frac{n-1}{2}$. Let $y_r, y_s, y_t \in W_2$ such that $d(y_r^+), d(y_s^+), d(y_t^+) \geq \frac{n-1}{2}$. Set $P' = y_r^+ C y_s x y_r \bar{C} y_s^+$. By similar argument as in Case 1 to the path P', we can get an $(n-1)$-cycle avoiding y_t^+, contrary to the choice of C. □

By Claim 11, we can assume $W_2 = \{y_1, y_s\}$ with $2 \leq s \leq k + 1$. Then

$d(y_1^+), d(y_2^-), d(y_s^+), d(y_{s+1}^-) \geq id(x)$. Since $y_1^+Cy_sxy_1\bar{C}y_s^+$ is a hamiltonian path of G, $d(y_1^+) + d(y_s^+) \leq n-1$ by Lemma 1. Hence, $id(x) = \frac{n-1}{2}$. For each vertex $y_j \in W_1$, since $\{y_{j+1}, y_j^+, y_{j+1}^+, x\}$ induces a claw, $id(y_j^+) \geq \frac{n-1}{2}$.

Claim 12. *$N(y_j^+) = N(x)$ for any $y_j \in W_1$.*

Proof. Let $d(y_j^+) = l+1$. Since $x \in N_2(y_j^+)$ and $d(x) < \frac{n-1}{2}$, we can get that $id(y_j^+) \neq m_2^{y_j^+}$. Since $C' = y_jxy_{j+1}Cy_j$ is an $(n-1)$-cycle of G avoiding y_j^+, by the choice of C, we have $d(y_j^+) < \frac{n-1}{2}$. Since G is not hamiltonian, $id(y_j^+) \neq d_{l+1}^{y_j^+}$.

Therefore, $id(y_j^+) = d_l^{y_j^+}$. If there exists some vertex $y_t \in W_1$ such that $y_ty_j^+ \in E(G)$ and $y_{t+1}y_j^+ \notin E(G)$, then by similar argument as in Claim 11 to the cycle C', we can get that $d(y_t^+) \geq id(y_j^+) \geq \frac{n-1}{2}$, a contradiction. Therefore, if there exists some vertex $y_t \in W_1$ such that $y_ty_j^+ \in E(G)$, then $y_{t+1}y_j^+ \in E(G)$. Similarly, if there exists some vertex $y_t \in W_1$ such that $y_ty_j^+ \in E(G)$, then $y_{t-1}y_j^+ \in E(G)$.

Since $y_j^+y_j \in E(G)$ and $y_j^+y_{j+1} \in E(G)$, we have $y_ry_j^+ \in E(G)$ for each $r = 2, 3, 4, \ldots, s$. By similar argument as in Claim 11 to the cycle C', there must exist some $y_t \in W_1$ for $t = s+1, s+2, \ldots, k+1$. Therefore, $y_ry_j^+ \in E(G)$ for each $r = 1, s+1, s+2, \ldots, k+1$. So $N(y_j^+) = N(x)$. □

Claim 13. *$N(x) \subseteq N(u)$ for any $u \in \{y_1^+, y_2^-, y_s^+, y_{s+1}^-\}$.*

Proof. Considering the hamiltonian path $P' = y_1^+Cy_sxy_1\bar{C}y_s^+$ and using the fact $d(y_1^+) \geq \frac{n-1}{2}$ and $d(y_s^+) \geq \frac{n-1}{2}$, we deduce $d(y_1^+) = d(y_s^+) = \frac{n-1}{2}$. Since $y_s^+y_r^+ \notin E(G)$ for any $y_r \in W_1$ and $xy_s^+ \notin E(G)$, we have $N(x)\backslash\{y_2\} \subseteq N(y_1^+)$. Since $y_1^+y_r^+ \in E(G)$ for any $y_r \in W_1$ and $xy_1^+ \notin E(G)$, $N(x)\backslash\{y_{s+1}\} \subseteq N(y_s^+)$.

By Claim 12, $y_s^+Cy_{s+1}xy_s\bar{C}y_2y_{s+1}^+Cy_2^-$ and $y_1^+Cy_2xy_1\bar{C}y_{s+1}y_2^+Cy_{s+1}^-$ are hamiltonian paths of G. Then $y_s^+y_2^- \notin E(G)$ and $y_1^+y_{s+1}^- \notin E(G)$. By using P', we get that $y_1^+y_2 \in E(G)$ and $y_s^+y_{s+1} \in E(G)$. Therefore, $N(x) \subseteq N(y_1^+)$ and $N(x) \subseteq N(y_s^+)$. Similarly, $N(x) \subseteq N(y_1^-)$ and $N(x) \subseteq N(y_s^-)$. □

Let $C_1 = C[y_1^+, y_2^-], C_2 = C[y_s^+, y_{s+1}^-]$ and $C_3 = C[y_2, y_s] \cup C[y_{s+1}, y_1]$. By the proof of Claim 13, $y_s^+y_2^- \notin E(G)$ and $y_1^+y_{s+1}^- \notin E(G)$. Since G is not hamiltonian, we have $(N_{C_1}(y_s^+))^+ \cap N_{C_1}(y_1^+) = \emptyset$ and $(N_{C_2}(y_1^+))^+ \cap N_{C_2}(y_s^+) = \emptyset$. By Lemma 4, we can get that $d_{C_1}(y_1^+) + d_{C_1}(y_s^+) \leq |V(C_1)| - 1$ and $d_{C_2}(y_1^+) + d_{C_2}(y_s^+) \leq |V(C_2)| - 1$. Similarly, $d_{C_2}(y_2^-) + d_{C_2}(y_{s+1}^-) \leq$

$|V(C_2)|-1$ and $d_{C_1}(y_2^-)+d_{C_1}(y_{s+1}^-) \leq |V(C_1)|-1$. By the above inequalities

$$\begin{aligned} 2(n-1) &\leq d_C(y_1^+)+d_C(y_s^+)+d_C(y_2^-)+d_C(y_{s+1}^-) \\ &\leq 4(k+1)+2(|V(C_1)|-1)+2(|V(C_2)|-1) \\ &\leq 2(n-1), \end{aligned}$$

which implies that all the inequalities are equalities. If there exists some vertex $y \in V(C_1)$ such that $y_s^+y \in E(G)$, then $y_2^-y^-, y_2^-y^+, y_2^-y_s^+, y_2^-y_s^{2+} \notin E(G)$ and $y_{s+1}^-y^- \notin E(G)$. By Lemma 4, we can get that $d_{C_1}(y_{s+1}^-) + d_{C_1}(y_2^-) < |C_1| - 1$, a contradiction. Hence, $N_{C_1}(y_s^+) = \emptyset$. Similarly, we can get that $N_{C_1}(y_{s+1}^-) = \emptyset, N_{C_2}(y_1^+) = \emptyset$ and $N_{C_2}(y_2^-) = \emptyset$. Hence, $d_{C_1}(y_1^+) = |V(C_1)|-1$ and $d_{C_2}(y_s^+) = |V(C_2)|-1$. Since $d(y_1^+) = \frac{n-1}{2}$ and $d(y_s^+) = \frac{n-1}{2}$, we can get that $|V(C_1)| = |V(C_2)| = \frac{n-1}{2} - k$. Therefore, we can get that G is the subgraph of $\mathscr{H}_n$. Then Theorem 6 holds. □

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