Rainbow vertex connection number of dense and sparse graphs *

Mengmeng Liu
Department of Mathematics
Lanzhou jiaotong University, Lanzhou 730070, China
liumm05@163.com

Abstract

A vertex-colored graph G is rainbow connected, if any two vertices are connected by a path whose internal vertices have distinct colors. The rainbow vertex connection number of a connected graph G, denoted $rvc(G)$, is the smallest number of colors that are needed in order to make G rainbow vertex connected. In this paper, we show that $rvc(G) \leq k$, if $|E(G)| \geq \binom{n-k}{2} + k$, for $k = 2, 3, n-4, n-5, n-6$. These bounds are sharp.

Keywords: vertex-colored graph, rainbow vertex coloring, rainbow vertex connection number.

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1 Introduction

All graphs considered in this paper are simple, finite and undirected. We follow the terminology and notation of [1].

An edge-colored graph G is called rainbow connected, if any two vertices are connected by a path whose edges have distinct colors. The concept of rainbow connection in graphs was introduced by Chartrand et al. in [2]. The *rainbow connection number* of a connected graph G, denoted $rc(G)$, is the smallest number of colors that are needed in order to make G rainbow connected. Obviously, we always have $diam(G) \leq rc(G) \leq n-1$, where $diam(G)$ denotes the diameter of G. Notice that $rc(G) = 1$ if and only if G is a complete graph, and that $rc(G) = n-1$ if and only if G is a tree.

In [4], Krivelevich and Yuster proposed the concept of rainbow vertex connection. A vertex-colored graph G is called rainbow vertex connected, if any two vertices are connected by a path whose internal vertices have

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distinct colors. The *rainbow vertex connection number* of a connected graph G, denoted $rvc(G)$, is the smallest number of colors that are needed in order to make G rainbow vertex connected. Observe that $rvc(G) \leq n-2$ and $rvc(G) = 0$ if and only if G is a complete graph. It is easy to verify that $rvc(G) \geq diam(G) - 1$ with equality if the diameter is 1 or 2. The rainbow connection has been studied for several graph classes. These results are presented in a recent survey [5].

In [3] the following problem was suggested:

Problem 1.1 *For every $k, 1 \leq k \leq n-1$, compute and minimize the function $f(n,k)$ with the following property: If $|E(G)| \geq f(n,k)$, then $rc(G) \leq k$.*

They give the lower bound of $f(n,k)$ and compute $f(n,2) = \binom{n-1}{2} + 2$. In [6], they give a simple method to prove $f(n,2) = \binom{n-1}{2} + 2$, and also showed that $f(n,3) = \binom{n-2}{2} + 2$ and $f(n,4) = \binom{n-3}{2} + 3$. In this paper, we consider an analogous problem of rainbow vertex connection:

Problem 1.2 *For every $k, 0 \leq k \leq n-2$, compute and minimize the function $g(n,k)$ with the following property: If $|E(G)| \geq g(n,k)$, then $rvc(G) \leq k$.*

We compute $g(n,k)$ for $k \in \{0, 1, 2, 3, n-3, n-2\}$ in section 2 and for $k \in \{n-6, n-5, n-4\}$ in section 3.

2 Main results for dense graphs

At first, we give some notation which will be used in the sequel.

Definition 2.1 *Let G be a connected graph. The distance between two vertices u and v in G, denoted by $d(u,v)$, is the length of a shortest path between them in G. Distance between a vertex v and a set $S \in V(G)$ is defined as $d(v,S) = min_{x \in S} d(v,x)$. The k-step open neighborhood of a set $S \in V(G)$ is defined as $N^k(S) = \{x \in V(G) | d(x,S) = k\}, k \in \{0,1,2\cdots\}$. When $k=1$, we may omit the qualifier "1-step" in the above name and the superscript 1 in the notation. The neighborhood of a vertex v in $\overline{G}$, denoted by $\overline{N}(v)$, is defined as $\overline{N}(v) = \{x | xv \notin E(G)\}$.*

Now, we give a lower bound for $g(n,k)$.

Proposition 2.2 $g(n,k) \geq \binom{n-k}{2} + k$.

Proof. We can construct a graph G with $|E(G)| = \binom{n-k}{2} + k - 1$ but $rvc(G) > k$. Let G be the graph constructed from $K_{n-k} - e$ with $e = v_1 v_2$

and the path $P_{k+1} : u_1, \cdots, u_{k+1}$ by identifying v_1 and u_{k+1}. We know $|E(G)| = \binom{n-k}{2} + k - 1$ and $diam(G) = k + 2$. So $rvc(G) \geq k + 1$. ∎

In [7], Li et al. investigated the vertex rainbow connection number for 2-connected graphs.

Theorem 2.3 *Let G be a 2-connected graph of order $n \geq 3$, then*

$$rvc(G) \leq \begin{cases} \lceil \frac{n}{2} \rceil - 2, & \text{if } n = 3, 5, 9; \\ \lceil \frac{n}{2} \rceil - 1, & \text{if } n = 4, 6, 7, 8, 10, 11, 12, 13, 15; \\ \lceil \frac{n}{2} \rceil, & \text{if } n \geq 16 \text{ or } n = 14. \end{cases}$$

Lemma 2.4 *Let G be a connected graph of order $n \geq 4$, then $rvc(G) = n - 2$ if and only if G is a path.*

Proof. If G is a path, then $rvc(G) = n-2$ is obvious. On the other hand, if $rvc(G) = n-2$, then G has at least one cut vertex by Theorem 2.3. If there is an end block with k vertices which is 2-connected, then by Theorem 2.3, $rvc(G) \leq \lceil \frac{k}{2} \rceil$ for $k \geq 8$. We can assign B_1 except the cut vertex with $\lceil \frac{k}{2} \rceil$ colors and assign $n-k+1$ distinct colors to the remaining $n-k+1$ vertices of G, thus $rvc(G) \leq \lceil \frac{k}{2} \rceil + n - k + 1 \leq n - 3$, for $k \geq 8$. For $k = 3, 4, 5, 6, 7$, we can also get $rvc(G) \leq n - 3$ by Theorem 2.3. Now we can assume that all end blocks are K_2, then G has exactly two pendent vertices. Otherwise, we can assign $n - 3$ distinct colors to the $n - 3$ vertices except the three pendent vertices, thus G is rainbow vertex connected. Now let v_1, v_2 be two pendent vertices. If G is not a path, then there exists a block $B_2 \neq K_2$, which is not an end block. Let $w \in B_2$ be one cut vertex and $ww' \in E(B_2)$. We assign color 1 to w, w', v_1, v_2 and $n - 4$ distinct colors different from color 1 to the remaining $n - 4$ vertices of G, then $rvc(G) \leq n - 3$, which is a contradiction. Hence G is a path. ∎

We can easily compute $g(n, k)$ for $k \in \{0, 1, n-3, n-2\}$ from Lemma 2.4.

Theorem 2.5 $g(n, 0) = \binom{n}{2}, g(n, 1) = \binom{n-1}{2} + 1, g(n, n-3) = n, g(n, n-2) = n - 1$.

Theorem 2.6 *Let G be a connected graph of order $n \geq 4$. If $|E(G)| \geq \binom{n-2}{2} + 2$, then $rvc(G) \leq 2$.*

Proof. Our proof will be by induction on n. For $n = 4$, we have $g(n, n-2) = n - 1 = 3 = \binom{4-2}{2} + 2$, for $n = 5$, we have $g(n, n-3) = n = 5 = \binom{5-2}{2} + 2$. So we may assume $n \geq 6$.

Claim 1: $diam(G) \leq 3$.

Proof. Suppose $diam(G) \geq 4$ and consider a diameter path $v_1, v_2, \cdots, v_{D+1}$ with $D \geq 4$. Then $d(v_1) + d(v_4) \leq n-2$ and $d(v_2) + d(v_5) \leq n-2$ implying

$|E(G)| \leq \binom{n}{2} - 2(2n-3-(n-2)) = \binom{n}{2} - 2(n-1) = \binom{n-2}{2} - 1 < \binom{n-2}{2} + 2$, a contradiction.

Claim 2: If $\delta(G) = 1$, then $rvc(G) \leq 2$.

Proof. Let w be a vertex with $d(w) = \delta(G) = 1$, let $H = G - w$. Then $|E(H)| \geq \binom{n-2}{2} + 2 - 1 = \binom{n-2}{2} + 1 = \binom{(n-1)-1}{2} + 1$, hence $rvc(H) \leq 1$ by Theorem 2.5. Let v is the neighbor of w, and set c(u)=1 for $u \in V(G)\backslash\{v\}$ and c(v)=2. Then G is 2-rainbow vertex connected.

Hence we may assume $\delta(G) \geq 2$. Let $w_1, w_2 \in V(G)$ with $w_1w_2 \notin E(G)$. Suppose $N(w_1) \cap N(w_2) = \emptyset$. Let $H = G - \{w_1, w_2\}$. Then $|E(H)| \geq \binom{n-2}{2} + 2 - (n-2) = \binom{n-3}{2} + 1 = \binom{(n-2)-1}{2} + 1$. Thus H is connected. Hence $rvc(H) \leq 1$ by Theorem 2.5. We may assume that $c(u) = 1$ for all $u \in V(H)$. Since $diam(G) \leq 3$, there is a w_1w_2-path $w_1u_1u_2w_2$, then change $c(u_1) = 2$. Since $\delta(G) \geq 2$, there is a vertex u with $uw_2 \in N(G)$, now change $c(u) = 2$. For any two vertices $x, y \in V(H)$, the rainbow path from x to y doesn't contain both u_1 and u, so G is 2-rainbow vertex connected.

Hence we may assume if $w_1, w_2 \in V(G), w_1w_2 \notin E(G)$, then $N(w_1) \cap N(w_2) \neq \emptyset$. Thus $diam(G) \leq 2$, so $rvc(G) \leq 1$. ∎

Theorem 2.7 *Let G be a connected graph of order $n \geq 5$. If $|E(G)| \geq \binom{n-3}{2} + 3$, then $rvc(G) \leq 3$.*

Proof. We apply the proof idea from the proof of Theorem 2.6.

Our proof will be by induction on n. For $n = 5$, we have $g(n, n-2) = n-1 = 4 = \binom{5-3}{2}+3$ and for $n = 6$, we have $g(n, n-3) = n = 5 = \binom{6-3}{2}+3$. So we may assume $n \geq 7$.

By Theorem 2.6, we have $rvc(G) \leq 2$ for $|E(G)| \geq \binom{n-2}{2} + 2$. Hence we may assume $|E(G)| \leq \binom{n-2}{2} + 1$. This implies $\delta(G) \leq \frac{(n-2)(n-3)+2}{n} = n - 5 + \frac{8}{n} < n - 3$.

Claim 4: $diam(G) \leq 4$.

Proof. Suppose $diam(G) \geq 5$ and consider a diameter path $v_1, v_2, \cdots, v_{D+1}$ with $D \geq 5$. Then $d(v_i) + d(v_{i+3}) \leq n - 2$ for $i = 1, 2, 3$ implying $|E(G)| \leq \binom{n}{2} - 3(2n-3-(n-2)) = \binom{n}{2} - 3(n-1) = \binom{n-3}{2} - 3 < \binom{n-3}{2} + 3$, a contradiction.

Claim 5: If $\delta(G) = 1$, then $rvc(G) \leq 3$.

Proof. Let w be a vertex with $d(w) = \delta(G) = 1$, let $H = G - w$. Then $|E(H)| \geq \binom{n-3}{2} + 3 - 1 = \binom{n-3}{2} + 2 = \binom{(n-1)-2}{2} + 2$, hence $rvc(H) \leq 2$ by Theorem 2.6. Take a 2-rainbow vertex coloring for H and change $c(v) = 3$ for the vertex incident with w. Then $rvc(G) \leq 3$.

Hence we may assume $\delta(G) \geq 2$.

Case 1 : If there are $w_1, w_2 \in V(G), w_1w_2 \notin E(G)$, with $N(w_1) \cap N(w_2) = \emptyset$ such that $d(w_1) + d(w_2) \leq n - 3$.

Let $H = G - \{w_1, w_2\}$. Then $|E(H)| \geq \binom{n-3}{2} + 3 - (n-3) = \binom{n-4}{2} + 2 = \binom{(n-2)-2}{2} + 2$. We claim that H is connected. If not, by computing edges, we know H has at most 2 components and one of them is a single vertex. Thus $\delta(G) = 1$, a contradiction. Hence $rvc(H) \leq 2$ by Theorem 2.6. Consider a 2-rainbow coloring of H with colors $1, 2$. If $d(w_1, w_2) = 3$, there is a path $w_1u_1u_2w_2$. If there is a vertex $v \in N(w_2)$ such that $u_1v \notin E(G)$, then change $c(u_1) = c(v) = 3$. For any two vertices $x, y \in V(H)\backslash\{u_1, v\}$, the rainbow path from x to y doesn't contain both u_1 and v, otherwise contradict with $rvc(H) \leq 2$. So G is 3-rainbow vertex connected. If for every $v \in N(w_2), u_1v \in E(G)$, now just change $c(u_2) = 3$. Then G is 3-rainbow vertex connected. If $d(w_1, w_2) = 4$, then we can claim that there is a path $w_1u_1u_2u_3w_2$ such that at most two internal vertices with the same color. Otherwise, all paths w_1xyzw_2 from w_1 to w_2 satisfy $c(x) = c(y) = c(z)$, then there is no rainbow path from the vertex $u \in N(w_1)$ with $c(u) = 1$ to the vertex $v \in N(w_2)$ with $c(v) = 2$. Hence we can assume that $c(u_1) = c(u_2) = 1, c(u_3) = 2$, since $\delta(G) \geq 2$, there is a vertex $v \in N(w_2)\backslash\{u_3\}$, we change $c(u_1) = 3, c(v) = 3$. Obviously $u_1v \notin E(G)$, then G is 3-rainbow vertex connected.

Case 2 : For all $w_1, w_2 \in V(G), w_1w_2 \notin E(G)$, such that $N(w_1) \cap N(w_2) \neq \emptyset$ or $d(w_1) + d(w_2) \geq n - 2$.

We know that in this case $diam(G) \leq 3$. Choose a vertex w with $d(w) = \delta(G)$ and set $d(w) = n - 2 - t$ with $2 \leq t \leq n - 4$.

Subcase 2.1 : $N^3(w) = \emptyset$.

Let $H = G - w$, then $|E(H)| \geq \binom{n-3}{2} + 3 - (n - 2 - t) = \binom{n-3}{2} - n + 7 + t - 2 \geq \binom{(n-1)-3}{2} + 3$.

If H is connected, by induction, H is 3-rainbow vertex connected. Now take a 3-rainbow coloring of H. Set $c(w) = 1$, for every $u \in N^2(w)$, there is a vertex $u' \in N(w)$ such that $uu' \in E(G)$, so there is a rainbow path from w to u. Then G is 3-rainbow vertex connected.

If H is disconnected, since H has no isolated vertices then H has at most 2 components. Otherwise $|E(H)| < \binom{n-4}{2} + 3$. Hence H has exactly 2 components H_1, H_2. we may assume that $|H_2| \geq |H_1| \geq 2$. Since $n \geq 7$, then $|H_2| \geq 3$. Thus we have

$$
\begin{aligned}
|E(H_1)| &\geq \binom{n-4}{2} + 3 - \binom{|H_2|}{2} \\
&= \frac{1}{2}\left[|H_1|^2 - 3|H_1| + 4\right] + |H_1||H_2| - 3|H_2| - 2|H_1| + 7 \\
&\geq \binom{|H_1| - 1}{2} + 1 + 3(n-4) - 3(n-1) + |H_1| + 7 \\
&\geq \binom{|H_1| - 1}{2} + 1
\end{aligned}
$$

Similar $|E(H_2)| \geq \binom{|H_2|-1}{2} + 1$. Hence both H_1, H_2 are 1-rainbow vertex connected. Set $c(u) = 1$, for all $u \in V(H_1), c(u) = 2$, for all $u \in V(H_2), c(w) = 3$, since $d(w,x) \leq 2$ for $x \in V(G)$, then G is 3-rainbow vertex connected.

Subcase 2.2 : $N^3(w) \neq \emptyset$.

For every $u \in N^3(w), wu \notin E(G)$ and $N(w) \cap N(u) = \emptyset$, then $d(w) + d(u) = n-2$, that is $N(u) = N^2(w) \cup N^3(w) \backslash \{u\}$. Set $c(w) = 3, c(u) = 2$ for $u \in N(w), c(u) = 1$ for $u \in N^2(w) \cup N^3(w)$. It's easy to check that G is 3-rianbow vertex connected. ∎

3 Main result for sparse graphs

The following result of this paper is the solution of Problem 1.2 for $n-6 \leq k \leq n-4$. The proof of this result consists of two parts. Firstly, we prove for 2-connected graph of order n and size at least $\binom{n-k}{2}+k$ that $rvc(G) \leq k$, if $n-6 \leq k \leq n-4$. The same statement for G is not 2-connected is proved in second step.

Theorem 3.1 *Let G be a 2-connected graph of order n, if $|E(G)| \geq \binom{n-k}{2}+k$, then $rvc(G) \leq k$ for $n-6 \leq k \leq n-4$.*

Proof. We may assume that $k \geq 4$.
If $k = n-4$, then $n \geq 8$, thus $rvc(G) \leq \lceil \frac{n}{2} \rceil \leq n-4$.
If $k = n-5$, then $n \geq 9$, thus $rvc(G) \leq \lceil \frac{n}{2} \rceil \leq n-5$, for $n \geq 10$. When $n = 9$, by Theorem 2.3, we know $rvc(G) \leq \lceil \frac{n}{2} \rceil - 2 \leq n-5$.
If $k = n-6$, then $n \geq 10$, thus $rvc(G) \leq \lceil \frac{n}{2} \rceil \leq n-6$, for $n \geq 12$. When $n = 10, 11$, by Theorem 2.3, we know $rvc(G) \leq \lceil \frac{n}{2} \rceil - 1 \leq n-6$. ∎

Theorem 3.2 *Let G be a connected graph but not 2-connected graph of order n, if $|E(G)| \geq \binom{n-k}{2} + k$, then $rvc(G) \leq k$ for $n-6 \leq k \leq n-4$.*

Proof. Let $k = n-t$, then $g(n,k) = g(n, n-t) = \binom{t}{2} + n - t = n + \binom{t-1}{2} - 1$. The proof will be by induction on the order n.

For $n = t+1, g(t+1, 1) = \binom{t}{2} + 1$. We may assume that the result is true for the graphs with order less n. Let G be the graph of order $n+1$ and $|E(G)| = n+1+\binom{t-1}{2} - 1$.

If G contains a bridge, say e, then $|E(G/e)| = |E(G)| - 1 = n + \binom{t-1}{2} - 1$. Hence by induction hypothesis, $rvc(G/e) \leq n-t$ and therefor $rvc(G) \leq n+1-t$.

If G contains no bridges, then G has a cut vertex, say w, thus G can be divided into two subgraphs G_1, G_2 such that $V(G_1) \cup V(G_2) = V(G)$ and $V(G_1) \cap V(G_2) = \{w\}$. Let $v(G_i) = n_i$ and $E(G_i) = n_i + s_i$ for $i = 1, 2$.

Since G has no bridge we conclude that $n_i \geq 3$ and $s_i \geq 0$. For each s_i, there is an integer $t_i \geq 3$ such that $\binom{t_i-1}{2} - 1 \leq s_i < \binom{t_i}{2} - 1$.

If G_i is 2-connected, then $rvc(G_i) \leq n_i - t_i$ by Theorem 3.1. If G_i is not 2-connected and $4 \leq t_i \leq 6$, by induction hypothesis, we have $rvc(G_i) \leq n_i - t_i$, this is also true for $t_i = 3$ by Theorem 2.5. Hence we have $rvc(G) \leq rvc(G_1) + rvc(G_2) \leq n_1 - t_1 + n_2 - t_2 = n + 2 - (t_1 + t_2)$.

We claimed that $t_1 + t_2 \geq t + 1$. Suppose that $t_1 + t_2 \leq t$. From $|E(G)| = n + 1 + \binom{t-1}{2} - 1 = n_1 + s_1 + n_2 + s_2 = n + 2 + (s_1 + s_2)$, we have

$$\begin{aligned} s_1 + s_2 + 2 &= \binom{t-1}{2} = \sum_{j=1}^{t-2} j \geq \sum_{j=1}^{t_1+t_2-2} j \\ &= \sum_{j=1}^{t_1-1} j + \sum_{j=t_1}^{t_1+t_2-2} j > \binom{t_1}{2} + \sum_{j=1}^{t_2-1} j \\ &= \binom{t_1}{2} + \binom{t_2}{2} \\ &> s_1 + s_2 + 2. \end{aligned}$$

a contradiction. So we get $rvc(G) \leq n+1-t$, which complete the proof. ∎

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