A note on the intersection graph of subspaces of a vector space

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Abstract

In this note, we study clique number, chromatic number, domination number and independence number of the intersection graph of subspaces of a finite dimensional vector space over a finite field.

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1 Introduction

Let $G = (V, E)$ be a graph with vertex set V of order n and edge set E. We denote the *open neighborhood* of a vertex v of G by $N_G(v)$, or just $N(v)$, and its *closed neighborhood* by $N_G[v] = N[v]$. For a vertex set $S \subseteq V(G)$, $N(S) = \cup_{v \in S} N(v)$ and $N[S] = \cup_{v \in S} N[v]$. A set of vertices S in G is a *dominating set*, if $N[S] = V(G)$. The *domination number*, $\gamma(G)$, of G is the minimum cardinality of a dominating set of G. A set of vertices S in G is an *independent set*, if $G[S]$ has no edge.

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The *independence number*, $\alpha(G)$, of G is the maximum cardinality of an independent set of G. We denote the *clique number* and *chromatic number* of a graph G by $\omega(G)$ and $\chi(G)$, respectively. A set of pairwise independent edges in a graph G is called a *matching*. A matching is *perfect* if it is incident with every vertex of G. For terminologies of graph theory, in general, we follow [7].

Let $F = \{S_i : i \in I\}$ be an arbitrary family of sets. The *intersection graph* $G(F)$ is the one-dimensional skeleton of the nerve of F , i.e., $G(F)$ is the graph whose vertices are S_i, $i \in I$ and in which the vertices S_i and S_j $(i, j \in I)$ are adjacent if and only if $S_i \neq S_j$ and $S_i \cap S_j \neq \emptyset$ [6].

The study of algebraic structures using the properties of graphs has become an exciting research topic in the last few decades, leading to many fascinating results and questions. It is interesting to study the intersection graphs $G(F)$ when the members of F have an algebraic structure. For the last few decades several mathematicians studied such graphs on various algebraic structures. These interdisciplinary studies allow us to obtain characterizations and representations of special classes of algebraic structures in terms of graphs and vice versa. For references of intersection graphs of algebraic structures see for example [1, 2, 3, 8].

Intersection graph of subspaces of a vector space are studied by Jafari Rad and Jafari in [5]. For a vector space V the *intersection graph of subspaces* of V, denoted by $G(V)$, is the graph whose vertices are in a one-to-one correspondence with proper nontrivial subspaces of V and two distinct vertices are adjacent if and only if the corresponding subspaces of V have a nontrivial (nonzero) intersection. Clearly the set of vertices is empty if $dim(V) = 1$. Jafari Rad and Jafari characterized all finite dimensional vector spaces over a finite field whose intersection graph are connected, bipartite, complete, Eulerian, or planar.

In this note we study clique number, chromatic number, domination number and independence number of $G(V)$, where V is a finite dimensional vector space over a finite field.

Throughout this paper V is a vector space with $\dim(V) = n$ on a finite field F with $|F| = q$. We denote by 0 the zero subspace of a vector space. We also mean by $W < V$ that W is a proper subspace of V.

Let F be a finite field with $|F| = q$, and let V be an n-dimensional vector space over F. For integer $t \in \{1, 2, ..., n\}$, the number of t-dimensional subspaces of V is given in [4] by

$$\begin{bmatrix} n \\ t \end{bmatrix}_q = \prod_{0 \leq i < t} \frac{q^{n-i} - 1}{q^{t-i} - 1}. \tag{1}$$

We suppose that $\begin{bmatrix} n \\ 0 \end{bmatrix}_q = 1$ and $\begin{bmatrix} n \\ t \end{bmatrix}_q = 0$ if $t \notin \{0, 1, 2, ..., n\}$. Note that $\begin{bmatrix} n \\ t \end{bmatrix}_q = \begin{bmatrix} n \\ n-t \end{bmatrix}_q$ for any $t \in \{0, 1, ..., n\}$. Let W be an m-dimensional subspace of V . In [5], we proved that $G(V)$ is connected if and only if $\dim(V) \geq 3$, and

$$|\{W' : \dim(W') = t, W \cap W' = 0\}| = q^{mt} \begin{bmatrix} n-m \\ t \end{bmatrix}_q. \tag{2}$$

2 Main results

We first evaluate clique number and the chromatic number in the intersection graph of subspaces of a vector space. We then determine the domination number and the independence number in the intersection graph of subspaces of a vector space.

Theorem 1 *If n is odd, then $\omega(G(V)) = \chi(G(V)) = \sum_{i=1}^{\lfloor \frac{n}{2} \rfloor} \begin{bmatrix} n \\ i \end{bmatrix}_q$.*

Proof. First notice that if W_1, W_2 are two subspace of V with $dim(W_i) > \frac{n}{2}$ for $i = 1, 2$, then $W_1 \cap W_2 \neq 0$. But for every graph G, $\chi(G) \geq w(G)$. Thus we find that $\chi(G(V)) \geq w(G(V)) \geq |\{W : W < V, dim(W) > \frac{n}{2}\}| = \sum_{i=1}^{\lfloor \frac{n}{2} \rfloor} \begin{bmatrix} n \\ i \end{bmatrix}_q$. Now we show that $w(G(V)) \leq |\{W : W < V, dim(W) > \frac{n}{2}\}|$. Let A_t be the set of all t-dimensional subspaces of V with $1 \leq t \leq n-1$. For $1 \leq t < \frac{n}{2}$, let H_t be the induced bipartite subgraph of $G(V)$ with partite sets A_t and A_{n-t}. Let $\overline{H}_t$ be the complement of H_t in $K_{|A_t|,|A_{n-t}|}$. By (2), $\overline{H}_t$ is a k-regular graph with $k = q^{t(n-t)}$. But for $k > 0$, every k-regular bipartite graph has a perfect matching. Thus, $\overline{H}_t$ has a perfect matching M_t. Now we consider a proper vertex coloring f for $G(V)$ as follows. Let $W < V$ and assume that n is odd. If $dim(W) < \frac{n}{2}$, then $f(W) = W$, and if $dim(W) = n - t > \frac{n}{2}$, then $f(W) = f(W^*)$ where $dim(W^*) = t < \frac{n}{2}$ and $WW^* \in M_t$. We deduce that $\chi(G(V)) \leq |\{W : W < V, dim(W) > \frac{n}{2}\}|$. Since for every graph G, $\chi(G) \geq w(G)$, we obtain that $\omega(G(V)) \leq \chi(G(V)) \leq |\{W : W < V, dim(W) > \frac{n}{2}\}|$. Therefore $\omega(G(V)) \leq \chi(G(V)) \leq \sum_{i=1}^{\lfloor \frac{n-1}{2} \rfloor} \begin{bmatrix} n \\ i \end{bmatrix}_q \leq \omega(G(V))$. But this implies that $\omega(G(V)) = \chi(G(V)) = \sum_{i=1}^{\lfloor \frac{n}{2} \rfloor} \begin{bmatrix} n \\ i \end{bmatrix}_q$ as required.□

Theorem 2 *If n is even then*

$$\sum_{i=1}^{\frac{n}{2}-1} \begin{bmatrix} n \\ i \end{bmatrix}_q + \begin{bmatrix} n-1 \\ \frac{n-2}{2} \end{bmatrix}_q \leq \omega(G(V)) \leq \chi(G(V)) \leq \sum_{i=1}^{\frac{n}{2}-1} \begin{bmatrix} n \\ i \end{bmatrix}_q + \begin{bmatrix} n \\ \frac{n}{2} \end{bmatrix}_q - q^{\frac{n^2}{4}} - 1.$$

Proof. First notice that if W_1, W_2 are two subspace of V with $dim(W_i) > \frac{n}{2}$ for $i = 1, 2$, then $W_1 \cap W_2 \neq 0$. Also if W_1, W_2 are two subspace of V with $dim(W_1) = \frac{n}{2}$ and $dim(W_2) > \frac{n}{2}$, then $W_1 \cap W_2 \neq 0$. But for a 1-dimensional subspace V_1 of V, any subspace of $\frac{V}{V_1}$ of dimension $\frac{n-2}{2}$ is in the form $\frac{W}{V_1}$, where $dim(W) = \frac{n}{2}$. So there are at least $\begin{bmatrix} n-1 \\ \frac{n-2}{2} \end{bmatrix}_q$ subspaces of dimension $\frac{n}{2}$ which contain V_1. We deduce that

$$\chi(G(V)) \geq \omega(G(V)) \geq |\{W : W < V, dim(W) > \frac{n}{2}\}| + \begin{bmatrix} n-1 \\ \frac{n-2}{2} \end{bmatrix}_q$$
$$\geq \sum_{i=1}^{\frac{n}{2}-1} \begin{bmatrix} n \\ i \end{bmatrix}_q + \begin{bmatrix} n-1 \\ \frac{n-2}{2} \end{bmatrix}_q .$$

Let A_t be the set of all t-dimensional subspaces of V with $1 \leq t \leq n-1$. Let $G_1 = G[A_{\frac{n}{2}}]$ and $G_2 = G - G_1$. As in the proof of Theorem 1, $\chi(G_2) = \sum_{i=1}^{\frac{n}{2}-1} \begin{bmatrix} n \\ i \end{bmatrix}_q$. It is obvious that G_1 is connected. Since G_1 is $(\begin{bmatrix} n \\ \frac{n}{2} \end{bmatrix}_q - q^{\frac{n^2}{4}})$-regular and $|V(G_1)| = \begin{bmatrix} n \\ \frac{n}{2} \end{bmatrix}_q$, G is neither a complete graph nor an odd cycle. Thus we have $\chi(G_1) \leq \Delta(G_1)$. So by (1) we obtain that $\chi(G_1) \leq \begin{bmatrix} n \\ \frac{n}{2} \end{bmatrix}_q - q^{\frac{n^2}{4}} - 1$. Thus $\chi(G(V)) \leq \sum_{i=1}^{\frac{n}{2}-1} \begin{bmatrix} n \\ i \end{bmatrix}_q + \begin{bmatrix} n \\ \frac{n}{2} \end{bmatrix}_q - q^{\frac{n^2}{4}} - 1$. Now the result follows from the fact that $\omega(G(V)) \leq \chi(G(V))$. □

Next we study the domination number and the independence number in the intersection graph of subspaces of a vector space.

Theorem 3 $\gamma(G(V)) = q + 1$.

Proof. Let W be a subspace of V with $dim(W) = n - 2$. It follows that $\frac{V}{W}$ has $q + 1$ subspaces $\frac{W_1}{W}, \frac{W_2}{W}, ..., \frac{W_{q+1}}{W}$ with $dim(\frac{W_i}{W}) = 1$ for $i = 1, 2, ..., q + 1$. It is obvious that

$$\frac{V}{W} = \frac{W_1}{W} \cup \frac{W_2}{W} \cup ... \cup \frac{W_{q+1}}{W}.$$

Now we can see that $V = W_1 \cup W_2 \cup ... \cup W_{q+1}$. This means that $\{W_1, W_2, ..., W_{q+1}\}$ is a dominating set for $G(V)$ and so $\gamma(G(V)) \leq q + 1$. On the other hand suppose that $S = \{V_1, V_2, ..., V_t\}$ is a minimum dominating set for $G(V)$. If there is a vector $x \notin V_1 \cup V_2 \cup ... \cup V_t$, then $\langle x \rangle$ is not dominated by S. This contradiction implies that $V_1 \cup V_2 \cup ... \cup V_t = V$. Then

$$q^n = |\cup_{i=1}^t V_i| < \sum_{i=1}^t |V_i| \leq \sum_{i=1}^t q^{n-1} = tq^{n-1}.$$

We deduce that $t \geq q+1$, and so $\gamma(G(V)) \geq q+1$. This completes the proof. □

Theorem 4 $\alpha(G(V)) = \frac{q^n-1}{q-1}$.

Proof. First notice that the set of all 1-dimensional subspaces of V form an independent set. Since V has $\frac{q^n-1}{q-1}$ 1-dimensional subspaces, we obtain $\alpha(G(V)) \geq \frac{q^n-1}{q-1}$. On the other hand let $A = \{V_1, V_2, ..., V_k\}$ be a maximum independent set. Since $V_1 \cup V_2 \cup ... \cup V_k \subseteq V$, we obtain $\sum_{i=1}^t(|V_i|-1) \leq q^n - 1$. But $|V_i| \geq q$ for any $i = 1, 2, ..., k$. So $k(q-1) \leq q^n - 1$. This implies that $k \leq \frac{q^n-1}{q-1}$. We conclude that $\alpha(G(V)) = \frac{q^n-1}{q-1}$. □

We close with the following problem.

Problem 5 *What is the exact value of* $w(G(V))$ *for even* n*?*

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