Bases of primitive nonpowerful zero-symmetric sign pattern matrices without nonzero diagonal entry*

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Abstract

It is well known that the properties about the power sequences of different classes of sign pattern matrices may be very different. In this paper, we consider the base of primitive nonpowerful zero-symmetric square sign pattern matrices without nonzero diagonal entry. The base set is shown to be $\{2, 3, \cdots, 2n-1\}$; the extremal sign pattern matrices with base $2n-1$ are characterized. As well, for the sign patterns with order 3, the sign patterns with bases 3, 4, 5 are characterized, respectively.

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1 Introduction

We adopt the standard conventions, notations and definitions for sign patterns and generalized sign patterns, their entries, arithmetics and pow-

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ers, and in particular, for walks in the corresponding signed digraphs. The reader who is not familiar with these matters is referred to [5], [9] or [12].

The sign pattern of a real matrix A, denoted by $\mathrm{sgn}(A)$, is the $(0, 1, -1)$-matrix obtained from A by replacing each entry by its sign. Notice that in the computation of the entries of the power A^k, an "*ambiguous sign*" may arise when we add a positive sign to a negative sign. So a new symbol "#" has been introduced to denote the ambiguous sign.

For convenience, we call the set $\Gamma = \{0,\ 1,\ -1,\ \#\}$ the generalized sign set and define the addition and multiplication involving the symbol # as follows (the addition and multiplication which do not involve # are obvious):

$$(-1) + 1 = 1 + (-1) = \#,\ \ a + \# = \# + a = \#\ \text{(for all } a \in \Gamma),$$

$$0 \cdot \# = \# \cdot 0 = 0,\ \ b \cdot \# = \# \cdot b = \#\ \text{(for all } b \in \Gamma \backslash \{0\}).$$

It is straightforward to check that the addition and multiplication in Γ defined in this way are commutative and associative, and the multiplication is distributive with respect to addition. It is easy to see that a $(0, 1)$-Boolean matrix is a nonnegative sign pattern matrix.

Definition 1.1 *Let A be a square sign pattern matrix of order n with power sequence $A, A^2, \cdots$. Because there are only 4^{n^2} different generalized sign pattern matrices of order n, there must be repetitions in the power sequence of A. Suppose $A^l = A^{l+p}$ is the first pair of powers that are repeated in the sequence. Then l is called the generalized base (or simply base) of A, and is denoted by $l(A)$. The least positive integer p such that $A^l = A^{l+p}$ holds for $l = l(A)$ is called the generalized period (or simply period) of A, and is denoted by $p(A)$. For a square (0, 1)-Boolean matrix A, $l(A)$ is also known as the convergence index of A, denoted by $k(A)$.*

In 1994, Z. Li, F. Hall and C. Eschenbach [5] extended the concept of the base (or convergence index) and period from nonnegative matrices to sign pattern matrices. They defined powerful and nonpowerful for sign pattern matrices, gave a sufficient and necessary condition that an irreducible sign pattern matrix is powerful and also gave a condition for the nonpowerful case.

Definition 1.2 *A square sign pattern matrix A (whose entries are $+1$, -1 or 0) is powerful if all the powers A^1, A^2, A^3, $\cdots$ are unambiguously defined, namely there is no # in A^k $(k = 1, 2, \cdots)$. Otherwise, A is called nonpowerful.*

In this paper, for a sign pattern matrix A, we denote by $|A|$ the nonnegative matrix obtained from A by replacing a_{ij} with $|a_{ij}|$.

Definition 1.3 *An irreducible* $(0, 1)$*-Boolean matrix* A *is primitive if there exists a positive integer* k *such that all the entries of* A^k *are nonzero; the least such* k *is called the primitive index of* A*, denoted by* $\exp(A) = k$*. A square sign pattern matrix* A *is called primitive if* $|A|$ *is primitive and the primitive index of* A*, denoted by* $\exp(A)$*, equals* $\exp(|A|)$.

It is well known that graph theoretical methods are often useful in the study of the powers of square matrices, so we now introduce some graph theoretical concepts.

Definition 1.4 *Let* A *be a square sign pattern matrix of order* n*. The associated digraph of* A*, denoted by* $D(A)$*, has vertex set* $V = \{1, 2, \cdots, n\}$ *and arc set* $E = \{(i, j) | a_{ij} \neq 0\}$*. The associated signed digraph of* A*, denoted by* $S(A)$*, is obtained from* $D(A)$ *by assigning the sign of* a_{ij} *to arc* (i, j) *for all* i *and* j*. Let* S *be a signed digraph of order* n *and let* A *be a square sign pattern matrix of order* n*;* A *is called associated sign pattern matrix of* S *if* $S(A)=S$*. The associated sign pattern matrix of a signed digraph* S *is always denoted by* $A(S)$*. Note that* $D(A) = D(|A|)$*, so* $D(A)$ *is also called the underlying digraph of the associated signed digraph of* A *or is simply called the underlying digraph of* A*. We always denote by* $D(A(S))$ *or* $|S|$ *the underlying digraph of a signed digraph* S*. Sometimes,* $|A(S)|$ *is called the associated or underlying matrix of signed digraph* S.

In this paper, we permit no loop and no multiple arcs in a signed digraph. Denote by $V(S)$ the vertex set and denoted by $E(S)$ the arc set for a signed digraph S. For $T \subseteq V(S)$, the (vertex) induced subgraph $S[T]$ is the subgraph induced by T. Let $W = v_0 e_1 v_1 e_2 \cdots e_k v_k$ ($e_i = (v_{i-1}, v_i)$, $1 \leq i \leq k$) be a directed walk of signed digraph S. The sign of W, denoted by $\operatorname{sgn}(W)$, is $\prod_{i=1}^{k} \operatorname{sgn}(e_i)$. Sometimes a directed walk can be denoted simply by $W = v_0 v_1 \cdots v_k$, $W = (v_0, v_1, \cdots, v_k)$ or $W = e_1 e_2 \cdots e_k$ if there is no ambiguity. Positive integer k is called the length of the directed walk W, denoted by $L(W)$. The length of the shortest directed path form v_i to v_j is called the distance from v_i to v_j in signed digraph S, denoted by $d(v_i, v_j)$. A cycle with length k is always called a *k-cycle*, a cycle with even length is called a *even cycle* and a *odd cycle* is similarly defined. When there is no ambiguity, a directed walk, a directed path, a direct circuit or a directed cycle will be called a walk, a path, a circuit or a cycle. A walk is called a *positive walk* if its sign is positive, and a walk is called a *negative walk* if its sign is negative. The union of digraphs H and G is the digraph $G \bigcup H$ with vertex set $V(G) \bigcup V(H)$ and arc set $E(G) \bigcup E(H)$. The intersection $G \bigcap H$ of digraphs H and G is defined analogously. If p is a positive integer and if C is a cycle, then pC denotes the walk obtained by traversing C p times. If a cycle C passes through the end vertex of W, $W \bigcup pC$ denotes the the walk obtained by going along W and then going around the cycle C p times; $pC \bigcup W$ is similarly defined.

Definition 1.5 *Assume that W_1, W_2 are two directed walks in signed digraph S. They are called a pair of $SSSD$ walks if they have the same initial vertex, the same terminal vertex and and the same length, but they have different sign.*

From [5] or [9], we know that a signed digraph S is powerful if and only if there is no pair of $SSSD$ walks in S. Otherwise, S is nonpowerful.

Definition 1.6 *A strongly connected digraph S is primitive if there exists a positive integer k such that for all vertices $v_i, v_j \in V(S)$ (not necessarily distinct), there exists a directed walk of length k from v_i to v_j. The least such k is called the primitive index of S, and is denoted by $\exp(S)$. Let S be a primitive digraph. The least l such that there is a directed walk of length t from v_i to v_j for any integer $t \geq l$ is called the local primitive index from v_i to v_j, denoted by $\exp_S(v_i, v_j) = l$. Similarly, $\exp_S(v_i) = \max_{v_j \in V(S)} \{\exp_S(v_i, v_j)\}$ is called the local primitive index at v_i, so $\exp(S) = \max_{v_i \in V(S)} \{\exp_S(v_i)\}$.*

For a square sign pattern A, let $W_k(i,j)$ denote the set of walks of length k from vertex i to vertex j in $S(A)$. Notice that the entry $(A^k)_{ij}$ of A^k satisfies $(A^k)_{ij} = \sum_{W \in W_k(i,j)} \mathrm{sgn}(W)$; then we have

(1) $(A^k)_{ij} = 0$ if and only if there is no walk of length k from i to j in $S(A)$ (i.e., $W_k(i,j) = \phi$);

(2) $(A^k)_{ij} = 1$ (or -1) if and only if $W_k(i,j) \neq \phi$ and all walks in $W_k(i,j)$ have the same sign 1 (or -1);

(3) $(A^k)_{ij} = \#$ if and only if there is a pair of $SSSD$ walks of length k from i to j.

So the associated signed digraph can be used to study the properties of the power sequence of a sign pattern matrix, and the signed digraph is taken as the tool in this paper. In matrix theory, a primitive matrix must be a nonnegative real matrix. From the relation between sign pattern matrices and signed digraphs, for a primitive signed digraph S, we have $\exp(S) = \exp(|A(S)|)$. Hence it is logical to define a sign pattern A to be primitive if $|A|$ is primitive, and to define $\exp(A) = \exp(D(A)) = \exp(|A|)$ if A is primitive.

Definition 1.7 *A signed digraph S is primitive and nonpowerful if there exists a positive integer l such that for any integer $t \geq l$, there is a pair of SSSD walks of length t from any vertex v_i to any vertex v_j $(v_i, v_j \in V(S))$. The least such l is called the base of S, denoted by $l(S)$. Let S be a primitive nonpowerful signed digraph of order n. For $u, v \in V(S)$, the local base from u to v, denoted by $l_S(u, v)$, is defined to be the least integer k such that there are SSSD walks of length t from u to v for any integer $t \geq k$. The local base at a vertex $u \in V(S)$ is defined to be $l_S(u) = \max_{v \in V(S)} \{l_S(u, v)\}$. So*

$$l(S) = \max_{u \in V(S)} l_S(u) = \max_{u,v \in V(S)} l_S(u, v).$$

Therefore, a sign pattern A is primitive nonpowerful if and only if $S(A)$ is primitive nonpowerful, and the base $l(A) = l(S(A))$ is the least positive integer l such that every entry of A^l is #.

Definition 1.8 *A sign pattern matrix A is called zero-symmetric if $|A|$ is symmetric. A signed digraph S is called zero-symmetric if $A(S)$ is zero-symmetric. So, for a zero-symmetric digraph S, $(v_j, v_i) \in E(S)$ if $(v_i, v_j) \in E(S)$.*

In a primitive nonpowerful signed zero-symmetric digraph S, we denote by $W^{-1} = v_k v_{k-1} \cdots v_2 v_1$ for directed walk $W = v_1 v_2 \cdots v_{k-1} v_k$ if no edge is a loop in W, and denote by $C^{-1} = (v_k, v_{k-1}, \cdots, v_2, v_1, v_k)$ for directed cycle $C = (v_1, v_2, \cdots, v_{k-1}, v_k, v_1)$ if C is not a loop.

Primitivity, base, local base, extremal patterns and other properties of power sequence of a square sign pattern matrix are of great significance. The bases of sign patterns are closely related to many other problems in various areas of pure and applied mathematics (see [2], [4], [6], [7], [10], [12]).

In 2008, B. Cheng [1] studied the base set of the primitive nonpowerful zero-symmetric sign pattern matrices. Some interesting questions are that what about the base set of the primitive nonpowerful zero-symmetric sign pattern matrices with or without nonzero diagonal entry, whether there is gap in the base set and what about the extremal sign patterns with the maximum base. Motivated by this and that the properties about the power sequences of different classes of sign pattern matrices may be very different, in this paper, we consider the base of primitive nonpowerful zero-symmetric square sign pattern matrices without nonzero diagonal entry. The base set is shown to be $\{2, 3, \cdots, 2n-1\}$ which is different from the base set of zero-symmetric sign pattern matrices obtained by B. Cheng in [1]; the extremal sign pattern matrices with base $2n-1$ are characterized. As well, for the sign patterns with order 3, the sign patterns with base 3, 4, 5 are characterized, respectively.

2 Preliminaries

Lemma 2.1 ([3]), ([6]) *Let A be an irreducible matrix, then A is cogredient to a matrix of the form*

$$A = \begin{pmatrix} 0 & A_1 & 0 & \cdots & 0 \\ 0 & 0 & A_2 & \cdots & 0 \\ & \vdots & \vdots & \vdots & \\ 0 & 0 & 0 & \cdots & A_{h-1} \\ A_h & 0 & 0 & \cdots & 0 \end{pmatrix}$$

where the zero blocks along the main diagonal are square, there is no zero row or zero column in A_i ($i = 1, 2, \cdots, h$) and $\prod_1^h A_i$ is a primitive matrix.

Such h in Lemma 2.1 is called the imprimitivity index of irreducible matrix A, denoted by $h(A)$ ($h(A)$ is equal to the period of $|A|$, see [5]).

Let S be a strongly connected digraph of order n and $C(S)$ denote the set of all cycle lengths in S. For a strongly connected digraph S of order n, suppose $C(S) = \{p_1, p_2, \ldots, p_u\}$ and $\gcd(p_1, p_2, \cdots, p_u) = p$. From [6], we know that $p = h(|A(S)|)$.

Lemma 2.2 ([5]) *An irreducible sign pattern matrix A with imprimitivity index h is powerful if and only if all cycles of $S(A)$ with lengths odd multiples of h have the same sign and all cycles (if any) of $S(A)$ with length even multiples of h are positive.*

Definition 2.3 *Let $\{s_1, s_2, \cdots, s_\lambda\}$ be a set of distinct positive integers with $gcd(s_1, s_2, \cdots, s_\lambda) = 1$. The Frobenius number of $s_1, s_2, \cdots, s_\lambda$, denoted by $\phi(s_1, s_2, \cdots, s_\lambda)$, is the smallest positive integer m such that for all positive integers $k \geq m$, there are nonnegative integers a_i ($i = 1, 2, \cdots, \lambda$) such that $k = \sum_{i=1}^{\lambda} a_i s_i$.*

It is well known that

Lemma 2.4 ([6]) *If $gcd(s_1, s_2) = 1$, then $\phi(s_1, s_2) = (s_1 - 1)(s_2 - 1)$.*

From Definition 2.3, it is easy to see that $\phi(s_1, s_2, \cdots, s_\lambda) \leq \phi(s_i, s_j)$ if there exist $s_i, s_j \in \{s_1, s_2, \cdots, s_\lambda\}$ such that $\gcd(s_i, s_j) = 1$. So if $\min\{s_i : 1 \leq i \leq \lambda\} = 1$, then $\phi(s_1, s_2, \cdots, s_\lambda) = 0$.

Lemma 2.5 ([4]) *Boolean matrix A is primitive if and only if $D(A)$ is strongly connected and* $\gcd(p_1, p_2, \cdots, p_t) = 1$ *where $C(D(A)) = \{p_1, p_2, \cdots, p_t\}$.*

Lemma 2.6 ([9]) *Let S be a primitive nonpowerful signed digraph. Then S must contain a p_1-cycle C_1 and a p_2-cycle C_2 satisfying one of the following two conditions:*

(1) p_i is odd, p_j is even and $sgnC_j = -1$ $(i, j = 1, 2; i \neq j)$.

(2) p_1 and p_2 are both odd and $sgnC_1 = -sgnC_2$.

Definition 2.7 *In a primitive nonpowerful signed digraph, a pair of cycles C_1, C_2 satisfying conditions (1) or (2) of Lemma 2.6 are called a distinguished cycle pair.*

It is easy to prove that $W_1 = p_2C_1$ and $W_2 = p_1C_2$ have the same length p_1p_2 but different sign if p_1-cycle C_1 and p_2-cycle C_2 are a distinguished cycle pair, namely $(\text{sgn}C_1)^{p_2} = -((\text{sgn}C_2)^{p_1})$.

Lemma 2.8 ([12])*Let S be a primitive signed digraph. Then S is nonpowerful if and only if S contains a distinguished cycle pair.*

Lemma 2.9 ([9]) *Let S be a primitive nonpowerful signed digraph of order n. If there are SSSD walks of length r from v_i to v_j, then $l_S(v_i) \leq r +$* $\exp_S(v_j)$.

Lemma 2.10 ([11]) *Let S be a primitive nonpowerful signed digraph of order n. Then $l_S(k) \le l_S(k-1)+1$ for $2 \le k \le n$.*

Lemma 2.11 ([12]) *Let S be a primitive nonpowerful signed digraph of order n. Then $l_S(v_i) \le d(v_i, v_j) + l_S(v_j)$ for $v_i,\ v_j \in V(S)$.*

Lemma 2.12 *Let S be a zero-symmetric digraph without loop consisting of odd length cycle C and C^{-1}. Then $exp(S) \le L(C) - 1$.*

Proof. It is easy to see that S is primitive by Lemma 2.5. Note that $L(C)-1$ is even and one of the two paths obtained by going respectively along C and C^{-1} from v_i to v_j for $v_i, v_j \in V(S)$ is even and its length is at most $L(C)-1$, so there exists a directed walk of length k (with $k \ge L(C)-1$) from v_i to v_j, so $\exp_S(v_i, v_j) \le L(C)-1$ by Definition 1.6. Note that v_i, v_j are arbitrary, so $\exp(S) \le L(C)-1$. □

Definition 2.13 *For a primitive digraph S, suppose $C(S) = \{p_1, p_2, \ldots, p_u\}$. Let $d_{C(S)}(v_i,\ v_j)$ denote the length of the shortest walk from v_i to v_j which meets at least one p_i-cycle for each i, $i = 1, 2, \cdots, u$. Such a shortest directed walk is called a $C(S)$-walk from v_i to v_j. Further, $d_{C(S)}(v_i)$, $d_i(C(S))$ and $d(C(S))$ are defined as follows: $d_{C(S)}(v_i) = \max\{d_{C(S)}(v_i,\ v_j)\colon v_j \in V(S)\}$, $d(C(S)) = \max\{d_{C(S)}(v_i,\ v_j)\colon v_i,\ v_j \in V(S)\}$, $d_i(C(S))$ $(1 \le i \le n)$ is the ith smallest one in $\{d_{C(S)}(v_i) \mid 1 \le i \le n\}$, $d_n(C(S)) = d(C(S))$. In particular, if $C(S) = \{p, q\}$, $d(C(S))$ can be simply denoted by $d\{p, q\}$.*

Definition 2.14 *Let S be a strongly connected digraph of order n, $C^* = \{C_1, C_2, \cdots, C_m\}$ be a cycles set, $d_{C^*}(v_i, v_j)$ denote the length of the shortest walk from v_i to v_j which meets all C_i $(i = 1, 2, \cdots, m)$. Such shortest walk is called C^*-walk from v_i to v_j. Define $d_{C^*}(v_i) = \max\limits_{v_j \in V(S)} \{d_{C^*}(v_i,\ v_j)\}$ and $d(C^*) = \max\limits_{v_i, v_j \in V(S)} \{d_{C^*}(v_i, v_j)\}$.*

Lemma 2.15 ([12]) *Let S be a primitive nonpowerful signed digraph of order n and $C(S) = \{p_1,\ p_2,\ \ldots,\ p_m\}$. If the cycles in S with the same length have the same sign, p_1-cycle C_1 and p_2-cycle C_2 are a distinguished cycle pair, then*

(i) $l_S(v_i, v_j) \le d_{C(S)}(v_i, v_j) + \phi(p_1, p_2, \ldots, p_m) + p_1p_2,\ \ v_i, v_j \in V(S)$.

(ii) $l_S(v_i) \le d_{C(S)}(v_i) + \phi(p_1, p_2, \ldots, p_m) + p_1p_2$.

(iii) $l(S) \le d(C(S)) + \phi(p_1, p_2, \ldots, p_m) + p_1p_2$.

Lemma 2.16 *Let S be a zero-symmetric signed digraph of order $n \geq 3$ without loop. C is an odd length cycle of S. If there is a pair of SSSD walks of length r from u to itself for $u \in V(C)$, then $l_S(u) \leq r + L(C) - 1 + \max_{v \in V(S)}\{d(u,v)\}$ and $l(S) \leq r + L(C) - 1 + 2\max_{v \in V(S)}\{d(u,v)\}$.*

Proof. It is easy to see that S_1 is primitive by Lemma 2.5 and nonpowerful by Definition 1.2. Let C_2^u denote a 2-cycle at vertex $u(u \in V(C_2^u))$ and $C^* = \{C, C_2^u\}$. So $d_{C^*}(u) \leq \max_{v \in V(S)}\{d(u,v)\}$ and $\exp(u) \leq \phi(2, L(C)) + d_{C^*}(u) \leq L(C) - 1 + \max_{v \in V(S)}\{d(u,v)\}$. Thus, by Lemma 2.9, $l_S(u) \leq r + \exp(u) \leq r + L(C) - 1 + \max_{v \in V(S)}\{d(u,v)\}$. Note that S is zero-symmetric, so $\max_{v \in V(S)}\{d(u,v)\} = \max_{v \in V(S)}\{d(v,u)\}$, and by Lemma 2.11, then $l(S) \leq l_S(u) + \max_{v \in V(S)}\{d(v,u)\} \leq r + L(C) - 1 + 2\max_{v \in V(S)}\{d(u,v)\}$. □

Lemma 2.17 ([8])*Let S be a primitive nonpowerful signed digraph of order n. If there are SSSD walks of length r from from v_i to v_j, then $l_S(v_i) \leq r + \exp_S(v_j)$.*

Lemma 2.18 *Let S be a zero-symmetric primitive nonpowerful signed digraph of order $n \geq 3$ without loop. If there are SSSD walks of length r from u to itself for $u \in V(S)$, then for any $v \in V(S)$, $l_S(u,v) \leq r + exp_S(u,v)$.*

Proof. Let W_1, W_2 denote the a pair of *SSSD* walks of length r from u to itself and W denote a directed walk with length t (with $t \geq \exp_S(u,v)$) from u to v. Then there are *SSSD* walks of length $r+t$ (with $r + t \geq r + \exp_S(u,v)$) from u to v which are $W_1 \bigcup W$ and $W_2 \bigcup W$. So for any $v \in V(S)$, $l_S(u,v) \leq r + \exp_S(u,v)$ by Definition 1.7. □

3 The base set

Lemma 3.1 *Let $E_n = \{l(A) | A$ be a primitive nonpowerful zero-symmetric $n \times n$ sign pattern matrix without nonzero diagonal entry $\}$. Then $E_{n-1} \subseteq E_n$.*

Proof. Let A, B be zero-symmetric primitive non-powerful sign pattern matrices without nonzero diagonal entry. We say $A \sim B$ if $a_{i,j}$ and $b_{i,j}$ have the same sign. So $A \sim A$, $A + B \sim A$ and $A + B \sim B$ if $A \sim B$.

Now, we can construct a map $\varphi: A_n \mapsto A'_{n+1}$ by defining $\varphi(A)$ to be the unique $(n+1) \times (n+1)$ sign pattern matrix satisfying the following conditions:

(1) The upper left $n \times n$ principal submatrix of $\varphi(A)$ is A;

(2) The last two rows of $\varphi(A)$ are equal and the last two columns of $\varphi(A)$ are equal.

Thus $\varphi(A)\varphi(B) = \varphi(AB + C_n(A)R_n(B))$ where $C_n(A)$ is the last column of A and $R_n(B)$ is the last row of B.

Note that $2a_{i,n}b_{n,j} = a_{i,n}b_{n,j} + a_{i,n}b_{n,j} = a_{i,n}b_{n,j}$ in sign pattern matrices computations, so

$$\text{sgn}((AB)_{i,j}) = \text{sgn}(a_{i,1}b_{1,j} + \cdots + a_{i,n-1}b_{n-1,j} + a_{i,n}b_{n,j}) = \text{sgn}(a_{i,1}b_{1,j} + \cdots + a_{i,n-1}b_{n-1,j} + 2a_{i,n}b_{n,j}) = \text{sgn}((AB + C_n(A)R_n(B))_{ij})$$

and $\varphi(AB + C_n(A)R_n(B)) \sim \varphi(AB)$. So $\varphi(AB) \sim \varphi(A)\varphi(B)$ and $(\varphi(A))^k \sim \varphi(A^k)$ for $k \in Z^+$. Furthermore $\varphi(A^{l(A)}) \sim \varphi(A)^{l(A)}$, $E_{n-1} \subseteq E_n$. □

Lemma 3.2 *Let S be a primitive nonpowerful signed zero-symmetric digraph of order $n \geq 3$ without loop. If there is no negative 2−cycle in S but there is negative even length $h \geq 4$ cycle, then $l(S) \leq 2n - 2$.*

Proof. Suppose C_1 is a shortest negative even cycle with length s_1 and C_2 is a shortest odd cycle with length s_2. Then C_1 and C_2 form a distinguished cycle pair by Lemma 2.8, and $s_1 \geq 4, s_2 \geq 3$. Suppose the directions of C_1 and C_2 are both clockwise.

Case 1 C_1 and C_2 have common vertex [see Fig. 3.1].

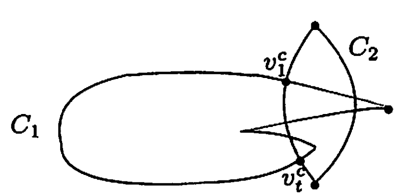

Fig. 3.1. C_1 and C_2 have common vertex

Let $V(C_1) \bigcap V(C_2) = \{v_1^c, v_2^c, \cdots, v_t^c\}$ and all the common vertices are between v_1^c and v_t^c along C_1. Let $S_0 = C_2 \bigcup C_2^{-1}$, and we get $\exp(S_0) \leq s_2 - 1$ by Lemma 2.12. Note that any 2-cycle is positive and $\text{sgn}(C_1) = -1$, so there is a pair of $SSSD$ walks of length s_1 from v_i^c $(1 \le i \le t)$ to itself, and so $l_S(v_i^c) \le s_1 + s_2 - 1 + \max_{v\in V(S)}\{d(v_i, v)\}$ for $v_i \in \{v_1^c, \cdots, v_t^c\}$. By Lemma 2.16, we have

$$l(S) \le \max_{v\in V(S)}\{d(v, v_1^c)\} + l_S(v_1^c) \le s_1 + s_2 - 1 + 2\max_{v\in V(S)}\{d(v_1^c, v)\}$$

$$\le s_1 + s_2 - 1 + 2\max\{n - s_1 - s_2 + \frac{s_1}{2}, n - s_1 - s_2 + [\frac{s_2}{2}]\}$$

$$\le \max\{2n - s_2 - 1, 2n - s_1 - 2\} \le \max\{2n - 4, 2n - 6\} < 2n - 2.$$

Case 2 C_1 and C_2 have no common vertex, then $s_1 + s_2 \le n$.

Let Q denote the shortest path from C_1 to C_2, $Q \bigcap C_1 = v_a, Q \bigcap C_2 = v_b$, $C^* = \{C_1, C_2\}$.

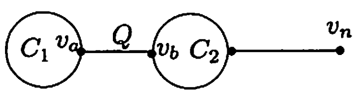

Fig. 3.2. $s_1 \le s_2 - 1$

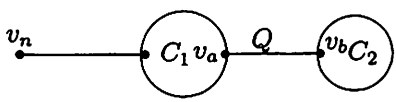

Fig. 3.3. $s_1 \ge s_2 + 1$

(i) $s_1 \le s_2 - 1$ [see Fig. 3.2].

Then $d(C^*) \le 2L(Q) + 2 \cdot \lfloor\frac{s_2}{2}\rfloor + 2(n - s_1 - s_2 - (L(Q) - 1)) = 2n - 2s_1 - s_2 + 1$. Note that there is a pair of $SSSD$ walks of length s_1 from any vertex of C_1 to itself because there is no negative 2-cycle in S and $s_1 \ge 4$, so $l(S) \le d(C^*) + \phi(2, s_2) + s_1 \le 2n - s_1 \le 2n - 4$.

(ii) $s_1 \ge s_2 + 1$ [see Fig. 3.3].

Then $d(C^*) \le 2L(Q) + 2 \cdot \frac{s_1}{2} + 2(n - s_1 - s_2 - (L(Q) - 1)) = 2n - s_1 - 2s_2 + 2$. Note that there is a pair of $SSSD$ walks of length s_1 from any vertex of C_1 to itself because there is no negative 2-cycle in S and $s_2 \ge 3$, so $l(S) \le d(C^*) + \phi(2, s_2) + s_1 \le 2n - s_2 + 1 \le 2n - 2$.

By (i) and (ii), we get $l(S) \le 2n - 2$ if C_1 and C_2 have no common vertex.

By Case 1 and Case 2, theorem is proved. □

Lemma 3.3 *Let S be a primitive nonpowerful zero-symmetric signed digraph of order $n \ge 3$ without loop. If there is no negative even cycle in S, then we have $l(S) \le 2n - 2$.*

Proof. Because S is primitive and nonpowerful, there must exists a distinguished cycle pair C_1, C_2 with lengths s_1, s_2 satisfying that $s_1 + s_2 = \min\{L(C') + L(C'')|C'$ and C'' form a distinguished cycle pair of $S\}$. Because there is no negative even cycle in S, so s_1, s_2 are both odd.

We can suppose that $3 \le s_1 \le s_2$ and the directions of C_1 and C_2 are both clockwise.

Case 1 C_1 and C_2 have common vertex. We assert $|V(C_1) \bigcap V(C_2)| = 1$.

Otherwise, suppose $V(C_1) \bigcap V(C_2) = \{v_1^c, v_2^c, \cdots, v_t^c\}$ $(t \ge 2)$ and all the common vertices are between v_1^c and v_t^c along C_1. C_1 is partitioned into two paths W_1 and W_3 $(C_1 = W_1 \bigcup W_3)$ by v_1^c and v_t^c, C_2 is partitioned into two paths W_2 and W_4 $(C_2 = W_2 \bigcup W_4)$ by v_1^c and v_t^c [see Fig. 3.4]. Suppose $L(W_2) = \min\{L(W_2), L(W_4)\}$ and $L(W_3) = \min\{L(W_1), L(W_3)\}$, then $L(W_2), L(W_3) \ge 1$.

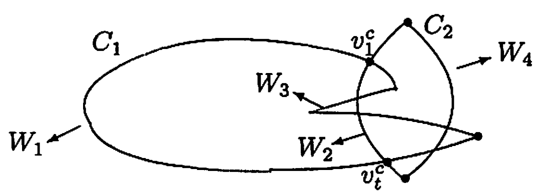

Fig. 3.4. $V(C_1) \bigcap V(C_2) = \{v_1^c, v_2^c, \cdots, v_t^c\}$ $(t \ge 2)$

(i) $L(W_2) + L(W_3)$ is even.

It is clear that $L(W_2) + L(W_3) \ge 2$ and easy to know that $L(W_1) + L(W_4)$ is also even. Note that $\text{sgn}C_1 = -\text{sgn}C_2$, so $\text{sgn}(W_2 \bigcup W_3) = -\text{sgn}(W_1 \bigcup W_4)$ now.

Suppose $\text{sgn}(W_2 \bigcup W_3) = -1$, which cause at least two odd cycles C_1' and C_2' in $W_2 \bigcup W_3$ such that $\text{sgn}(C_1') = -\text{sgn}(C_2')$ because there is no negative even cycle in S, so C_1' and C_2' form a distinguished cycle pair whose length sum is smaller than $s_1 + s_2$, which contradicts the choice of C_1, C_2.

(ii) $L(W_2) + L(W_3)$ is odd.

It is clear that $L(W_1)+L(W_4)$ is also odd. Note that $L(W_1)+L(W_3)$ and $L(W_2) + L(W_4)$ are both odd, so $L(W_1) \equiv L(W_2) \pmod 2$ and $L(W_3) \equiv L(W_4) \pmod 2$, and so $W_1 \bigcup W_2^{-1}$ and $W_3^{-1} \bigcup W_4$ are both even circuits and $\text{sgn}(W_1 \bigcup W_2^{-1}) = -\text{sgn}(W_3^{-1} \bigcup W_4)$.

Suppose $\operatorname{sgn}(W_1 \bigcup W_2^{-1}) = -1$, as the proof of (i), there exists a distinguished cycle pair C_1' and C_2' whose length sum is smaller than $s_1 + s_2$ in $W_1 \bigcup W_2^{-1}$, which contradicts the choice of C_1, C_2. So $L(W_i) = 0$ $(i = 2, 3)$ and $|V(C_1) \bigcap V(C_2)| = 1$ by (i), (ii), and so our assertion holds.

Suppose $V(C_1) \bigcap V(C_2) = \{v_1\}$. Let $S_0 = S[V(C_1) \bigcup V(C_2)]$. Then S_0 is also primitive and nonpowerful by Lemmas 2.5, 2.8.

We assert $l(S_0) \leq 2s_2 + s_1 - 1$. Let $v_i, v_j \in V(S_0)$.

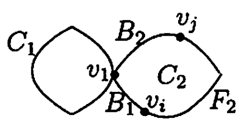

Fig. 3.5. $v_1 \in V(F_1)$

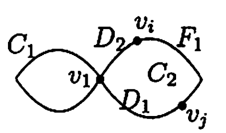

Fig. 3.6. $v_1 \notin V(F_1)$

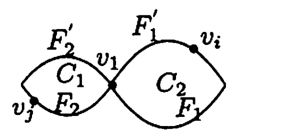

Fig. 3.7. $v_i \in V(C_2)$, $v_j \in V(C_1)$, $v_j \neq v_1$

Subcase 1.1 $v_i, v_j \in V(C_2)$. Let C_2^i denote a 2−cycle at vertex v_i, F_1 with length t_1 denote the path from v_i to v_j along C_2 and F_2 with length t_2 denote the path from v_i to v_j along C_2^{-1}. Then $t_1 + t_2 = s_2$. Note that s_2 is odd, so we can suppose t_1 is even , then $t_1 \leq s_2 - 1$ and t_2 is odd.

If $v_1 \in V(F_1)$ [see Fig. 3.5], let B_1 denote the path from v_i to v_1 along C_2 and B_2 denote the path from v_1 to v_j along C_2, then $F_1 = B_1 \bigcup B_2$. Note that the sign of any 2-cycle in S is positive, so there is a pair of $SSSD$ walks of length $2s_2 + s_1 - 1$ from v_i to v_j which are $\frac{2s_2 + s_1 - 1 - t_1}{2} C_2^i \bigcup F_1$ and $\frac{s_2 - 1 - t_1}{2} C_2^i \bigcup B_1 \bigcup C_1 \bigcup C_2 \bigcup B_2$.

If $v_1 \notin V(F_1)$ [see Fig. 3.6], let D_1 denote the path from v_j to v_1 along C_2 and D_2 denote the path from v_i to v_1 along C_2^{-1}. Let $A_1 = F_1 \bigcup D_1 \bigcup D_1^{-1}, A_2 = D_2 \bigcup D_2^{-1} \bigcup F_1$, then $L(D_2) \equiv L(D_1) + 1 \pmod 2$ because $D_2 \bigcup D_1^{-1} = F_2$, and then $\min\{L(A_i)|\ i = 1, 2\} \leq s_2 - 1$. For convenience, suppose $L(A_1) \leq s_2 - 1$, note that the sign of any 2−cycle in S is positive, so there is a pair of $SSSD$ walks of length $2s_2 + s_1 - 1$ from v_i to v_j which are $\frac{2s_2 + s_1 - 1 - L(A_1)}{2} C_2^i \bigcup A_1$ and $\frac{s_2 - 1 - L(A_1)}{2} C_2^i \bigcup F_1 \bigcup D_1 \bigcup C_1 \bigcup C_2 \bigcup D_1^{-1}$.

Subcase 1.2 $v_i \in V(C_2)$, $v_j \in V(C_1), v_j \neq v_1$ [see Fig. 3.7]. Let F_1 with length t_1 denote the path from v_i to v_1 along C_2 and F_1' with length t_1' denote the path from v_i to v_1 along C_2^{-1}. Then $t_1 + t_1' = s_2$ and $t_1 \equiv t_1' + 1$ (mod 2) because s_2 is odd. Suppose t_1 is even. Then $t_1 \leq s_2 - 1$. Let F_2 with length p_1 denote the path from v_1 to v_j along C_1 and F_2' with length p_1' denote the path from v_1 to v_j along C_1^{-1}. Then $p_1 + p_1' = s_1$ and $p_1 \equiv p_1' + 1 \pmod 2$ because s_1 is odd. Suppose p_1 is even, then $p_1 \le s_1 - 1$. Let $A_1 = F_1 \bigcup F_2, A_2 = F_1' \bigcup F_2'$, then $\mathrm{sgn}(A_1) = -\mathrm{sgn}(A_2)$. Note that there is no negative 2-cycle in S, so there is a pair of $SSSD$ walks of length $2s_2 + s_1 - 1$ from v_i to v_j which are $\frac{2s_2 + s_1 - 1 - L(A_1)}{2} C_2^i \bigcup A_1$ and $\frac{2s_2 + s_1 - 1 - L(A_2)}{2} C_2^i \bigcup A_2$.

In a same way, we can prove that there is a pair of $SSSD$ walks of length $2s_2 + s_1 - 1$ from v_i to v_j for the case that $v_i, v_j \in V(C_1)$ and the case that $v_i \in V(C_1)$, $v_j \in V(C_2)$.

Because v_i and v_j are arbitrary, so $l(S_0) \le 2s_2 + s_1 - 1$ by Definition 1.7. Therefore, our assertion holds.

For any $v_l, v_k \in V(S)$, let P_l denote the shortest path from v_l to S_0 and $P_l \bigcap S_0 = v_a$, let P_k denote the shortest path from v_k to S_0 and $P_k \bigcap S_0 = v_b$. Note that there is a pair of $SSSD$ walks of length $2s_2+s_1-1$ from v_a to v_b and

$$2s_2+s_1-1+L(P_l)+L(P_k) \le 2s_2+s_1-1+2(n-s_2-s_1+1) = 2n-s_1+1 \le 2n-2,$$

so there is a pair of $SSSD$ walks of length $2n - 2$ from v_l to v_k. Because $v_l, v_k \in V(S)$ are arbitrary, so $l(S) \le 2n - 2$ by Definition 1.7.

Case 2 C_1 and C_2 have no common vertex.

Let Q denote the shortest path from C_1 to C_2, $Q \bigcap C_1 = v_a, Q \bigcap C_2 = v_b$, $C^* = \{C_1, C_2\}$ [see Fig. 3.2]. Then $d(C^*) \le 2L(Q) + 2 \cdot \lfloor \frac{s_2}{2} \rfloor + 2(n - s_1 - s_2 - (L(Q) - 1)) = 2n - 2s_1 - s_2 + 1$. Let C_2^a denote a 2−cycle at v_a. Note that any 2-cycle is positive in S, then $\mathrm{sgn}(\frac{s_2 - s_1}{2} C_2^a \bigcup C_1) = -\mathrm{sgn}(C_2)$. Because $s_1 \ge 3$, so $l(S) \le d(C^*) + \phi(2, s_1) + s_2 \le 2n - s_1 \le 2n - 3$.

To sum up, Lemma is proved. □

Lemma 3.4 *Let S be a primitive nonpowerful zero-symmetric signed digraph of order $n \ge 3$ without loop. If the 2-cycles in S have different sign, then $l(S) \le 2n - 2$.*

Proof. There must exist an odd cycle $C_k = (v_1, v_2, \cdots, v_k, v_1)$ in S because S is primitive. Let $d^*(v_i, v_j)$ denote the length of the shortest directed walk meeting at least one positive 2-cycle, at least one negative 2-

cycle and C_k from vertex v_i to vertex v_j and let $d^* = \max_{v_i, v_j \in V(S)} \{d^*(v_i, v_j)\}$. For convenience, suppose C_k is oriented clockwise.

Case 1 There are at least two negative 2-cycles. Then it is easy to check that $d^* \le 2(n - k - 1 + \frac{k-1}{2})$. So there is a pair of $SSSD$ walks with length $d^* + \phi(2, k) + 2$ from any vertex v_i to any vertex v_j. Note that $d^* + \phi(2, k) + 2 \le 2n - 2$, so $l(S) \le 2n - 2$ by Definition 1.7.

Case 2 There is only one negative 2−cycle in S. Let $S_0 = C_k \bigcup C_k^{-1}$.

Subcase 2.1 The unique negative 2−cycle is not in S_0, then it is easy to check that $d^* \le 2(n - k - 1 + \frac{k-1}{2})$. So $l(S) \le 2n - 2$ follows as Case 1.

Subcase 2.2 The unique negative 2−cycle is in S_0.

It is easy to see that S_0 is primitive and nonpowerful by Lemmas 2.5, 2.8.

We assert $l(S_0) \le k + 1$.

Let $C_0 = (v_a, v_{a+1}, v_a)$ $(v_a, v_{a+1} \in V(S_0))$ denote the unique negative 2-cycle in S. For any vertices $v_i, v_j \in V(S_0)$, let $C_2^i = (v_i, v_{i+1}, v_i)$ $(i \ne a, C_2^k = (v_k, v_1, v_k))$ denote a positive 2-cycle at v_i. Let (v_a, v_{a-1}, v_a) denote a positive 2-cycle at v_a, P_1, P_2 denote the path from v_i to v_j along C_k and C_k^{-1}. It is easy to know $L(P_1) \equiv L(P_2) + 1 \pmod 2$ because $L(P_1) + L(P_2) = k$. For convenience, suppose $L(P_1)$ $(0 \le L(P_1) \le k - 1)$ is even.

1° P_1 meets C_0 [see Fig. 3.8]. Let F_1 denote the path from v_i to v_a and F_2 denote the path from v_{a+1} to v_j along C_k. Then $P_1 = F_1 \bigcup (v_a, v_{a+1}) \bigcup F_2$. So there is a pair of $SSSD$ walks of length $k + 1$ from v_i to v_j which are $\frac{k+1-L(P_1)}{2} C_2^i \bigcup P_1$ and $\frac{k+1-L(P_1)-2}{2} C_2^i \bigcup F_1 \bigcup C_0 \bigcup (v_a, v_{a+1}) \bigcup F_2$.

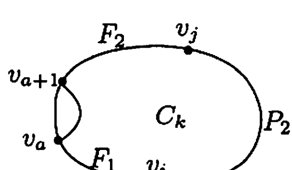

Fig. 3.8. P_1 meets C_0

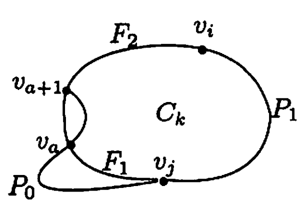

Fig. 3.9. P_1 dose not meets C_0

2° P_1 dose not meets C_0 [see Fig. 3.9]. Then $L(P_1) \leq k-3$.

Let F_1 denote the path from v_j to v_a and F_2 denote the path from v_{a+1} to v_i along C_k. Then $P_2 = F_2^{-1} \bigcup (v_{a+1}, v_a) \bigcup F_1^{-1}$. Let $d' = \min\{d(v_i, v_a), d(v_i, v_{a+1}), d(v_j, v_a), d(v_j, v_{a+1})\}$. Then $d' \leq \min\{L(F_1), L(F_2)\}$, and so $d' \leq \frac{k-1-L(P_1)}{2}$. For convenience, suppose $d' = d(v_j, v_a)$ and let P_0 denote the shortest path from v_j to v_a, namely $L(P_0) = d'$. Note that the directed walk $W = P_1 \bigcup P_0 \bigcup P_0^{-1}$ meets C_0 and $L(W) \leq L(P_1) + 2\frac{k-1-L(P_1)}{2} = k-1$, so there is a pair of $SSSD$ walks of length $k+1$ from v_i to v_j which are $\frac{k+1-L(W)}{2} C_2^i \bigcup W$ and $\frac{k+1-L(W)-2}{2} C_2^i \bigcup P_1 \bigcup P_0 \bigcup C_0 \bigcup P_0^{-1}$.

Because v_i, v_j are arbitrary, so $l(S_0) \leq k+1$ and our assertion holds.

For any $v_l, v_m \in V(S)$, let P_l denote the shortest path from v_l to S_0 and $P_l \bigcap S_0 = v_t$, let P_m denote the shortest path from v_m to S_0 and $P_m \bigcap S_0 = v_z$. Note that there is a pair of $SSSD$ walks of length h (with $h \geq k+1$) from v_t to v_z, then there is a pair of $SSSD$ walks of length l (with $l \geq k+1+L(P_l)+L(P_m)$) from v_l to v_m. Note that

$$k+1+L(P_l)+L(P_m) \leq k+1+2(n-k) = 2n-k+1 \leq 2n-2 \ (k \geq 3)$$

and $v_l, v_m \in V(S)$ are arbitrary, so $l(S) \leq 2n-2$ by Definition 1.7.

To sum up, $l(S) \leq 2n-2$, the theorem is proved. □

Lemma 3.5 *Let S be a primitive nonpowerful zero-symmetric signed digraph of order $n \geq 3$ without loop. If each 2−cycle has negative sign in S, then $l(S) \leq 2n-1$.*

Proof. For any odd directed cycle C in S, there must exist one in $\{C, C^{-1}\}$ is positive cycle and the other one is negative because there are $L(C)$ negative arcs in $C \bigcup C^{-1}$.

Let C be an odd cycle with length s and $S_1 = C \bigcup C^{-1}$. It is easy to see that S_1 is primitive and nonpowerful by lemmas 2.5, 2.8. By lemma 2.12, we get $\exp(S_1) \leq s-1$.

For any vertex v in S, let P_v denote the shortest path from v to S_1 and $P_v \bigcap S_1 = v_a$.

Note that $\exp_S(v_i, v_a) \leq \exp_{S_1}(v_i, v_a) \leq \exp(S_1)$ for any vertex $v_i \in V(S_1)$ by Definition 1.6, and there exists a pair of $SSSD$ walks of length h

(with $h \geq s$) from v_i to itself, then $l_S(v_i, v_a) \leq s + \exp_S(v_i, v_a)$ by Lemma 2.18. So $l_S(v_i, v) \leq l_S(v_i, v_a) + L(P_v)$ and

$$l_S(v_i) \leq l_S(v_i, v_a) + \max_{v \in V(S)} \{L(P_v)\}$$
$$\leq l_S(v_i, v_a) + n - s \leq s + s - 1 + n - s = n + s - 1$$

for $v_i \in V(S_1)$.

For any vertex $v' \in V(S)$, let $P_{v'}$ denote the shortest path from v' to S_1 and $P_{v'} \bigcap S_1 = v_b$. Then $l_S(v') \leq l_S(v_b) + L(P_{v'}) \leq n+s-1+n-s = 2n-1$ by Lemma 2.11. So $l(S) \leq 2n - 1$. □

Theorem 3.6 *Let S be a primitive nonpowerful zero-symmetric signed digraph of order $n \geq 3$ without loop. Then $2 \leq l(S) \leq 2n - 1$.*

Proof. There is no pair of $SSSD$ walks of length 1 from v_i to itself because there are no loop in S. So $2 \leq l(S)$. By Lemmas 3.2–3.5, theorem is proved. □

Theorem 3.7 *Let S_1 [see Fig. 3.10] consists of 3-cycles $C_3 = v_1 e_5 v_2 e_6 v_3 e_4 v_1$ and $C_3^{-1} = v_1 e_1 v_3 e_3 v_2 e_2 v_1$. Let S be a primitive nonpowerful zero-symmetric signed digraph of order 3 without loop. Then $E_3 = \{3, 4, 5\}$ and $|S| \cong S_1$.*

Especially,

(i) there is just one negative 2–cycle in S if and only if $l(S) = 3$;

(ii) there is just two negative 2–cycles in S if and only if $l(S) = 4$;

(iii) each 2–cycle have negative sign in S if and only if $l(S) = 5$.

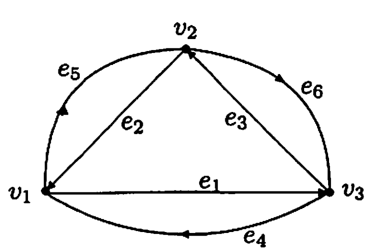

Fig. 3.10. S_1

Proof. By theorem 3.6, we get $l(S) \leq 5$.

Because S is primitive, so there must exist an odd cycle. Because there is no loop in S, so there must exist a 3-cycle in S. Because S is zero-symmetric and there is no multiple arcs in S, so $|S| \cong S_1$.

We assert there must be negative 2–cycle in S. Otherwise, C_3 and C_3^{-1} have the same sign if there is no negative 2–cycle. So S is powerful by Lemma 2.2, which contradicts the condition that S is nonpowerful.

Note that there is no loop in S, so there is no pair of $SSSD$ walks of length 1 from vertex v_i $(i = 1, 2, 3)$ to itself. Note that there is only directed walk of length 2 e_1e_3 from vertex v_1 to vertex v_2, so there is no pair of $SSSD$ walks of length 1 from vertex v_1 to vertex v_2. So $l(S) \geq 3$ by Definition 1.7.

For convenience, suppose $|S| = S_1$.

Case 1 There is just one negative 2–cycle in S.

Clearly, S is primitive and nonpowerful by Lemmas 2.5, 2.8. Suppose $\mathrm{sgn}e_2 = -1$, each of other arcs has positive sign. Then, from vertex v_1 to itself, there is a pair of $SSSD$ walks of length 3 obtained by going along C_3 and C_3^{-1} respectively. $e_5e_2e_5$ and $e_1e_4e_5$ are $SSSD$ walks of length 3 from vertex v_1 to v_2. $e_1e_4e_1$ and $e_5e_2e_1$ are a pair of $SSSD$ walks of length 3 from vertex v_1 to v_3. So $l_S(v_1) = 3$. In a similar way, we get $l_S(v_2) = 3$. From vertex v_3 to itself, there is a pair of $SSSD$ walks of length 3 obtained by going along C_3 and C_3^{-1} respectively. $e_4e_1e_4$ and $e_4e_5e_2$ are a pair of $SSSD$ walks of length 3 from vertex v_3 to v_1. $e_3e_6e_3$ and $e_3e_2e_5$ are a pair of $SSSD$ walks of length 3 from vertex v_3 to v_2. So $l_S(v_3) = 3$. So $l(S) = 3$.

Case 2 There are just two negative 2–cycles in S.

Clearly, S is primitive and nonpowerful by Lemmas 2.5, 2.8. Suppose $\mathrm{sgn}(e_1e_4) = -1 = \mathrm{sgn}(e_5e_2)$, each of other arcs has positive sign. It is easy to check that $\mathrm{sgn}C_3 = \mathrm{sgn}C_3^{-1}$ because there are 2 negative arcs in S.

There are no pair of $SSSD$ walks of length 3 from vertex v_i $(i = 1, 2, 3)$ to itself because there are just two directed walk of length 3 along C_3 and along C_3^{-1} respectively from vertex v_i to itself. So $l_S(v_i) \geq 4$.

$e_1e_4e_1e_4$ and $e_1e_3e_6e_4$ are a pair of $SSSD$ walks of length 4 from vertex v_1 to itself. $e_5e_6e_3e_6$ and $e_5e_6e_4e_1$ are a pair of $SSSD$ walks of length 4 from vertex v_1 to v_3. $e_1e_3e_6e_3$ and $e_1e_3e_2e_5$ are a pair of $SSSD$ walks of length 4 from vertex v_1 to v_2. So $l_S(v_1) = 4$. $e_4e_1e_3e_6$ and $e_3e_6e_3e_6$ are a pair of $SSSD$ walks of length 4 from vertex v_3 to itself. $e_3e_2e_5e_2$ and $e_3e_6e_3e_2$ are a pair of $SSSD$ walks of length 4 from vertex v_3 to v_1.

$e_4e_5e_2e_5$ and $e_4e_5e_6e_3$ are a pair of $SSSD$ walks of length 4 from vertex v_3 to v_2. So $l_S(v_3) = 4$. Similar to proof of $l_S(v_3) = 4$, we can prove $l_S(v_2) = 4$. So $l(S) = 4$.

Case 3 Each 2–cycle has negative sign in S. Clearly, S is primitive and nonpowerful by Lemmas 2.5, 2.8.

We assert there is no pair of $SSSD$ walks of length 4 from vertex v_i $(i = 1, 2, 3)$ to itself because the directed walk of length 4 from vertex v_i to itself is just composed of two 2-cycles (may repeated). So $l_S(v_i) \geq 5$.

C_3 and C_3^{-1} have different sign because there are 3 negative arcs in S. Thus $e_5e_6e_4e_1e_4$ and $e_1e_3e_2e_5e_2$ are a pair of $SSSD$ walks of length 5 from vertex v_1 to itself. $e_1e_3e_2e_1e_3$ and $e_1e_3e_6e_4e_5$ are a pair of $SSSD$ walks of length 5 from vertex v_1 to v_2. $e_5e_6e_4e_5e_6$ and $e_5e_6e_3e_2e_1$ are a pair of $SSSD$ walks of length 5 from vertex v_1 to v_3. So $l_S(v_3) = 5$. Similar to the proof of $l_S(v_1) = 5$, we can prove $l_S(v_2), l_S(v_3) = 5$. So $l(S) = 5$.

From Case 1, Case 2, Case 3, it is easy to see that all of (i), (ii), (iii) hold.

To sum up, the theorem is proved. □

Theorem 3.8 $E_n = \{2, 3, \cdots, 2n-1\}$ *for* $n \geq 4$.

Proof. 1. $\{3, 4, 5\} \subseteq E_n$ by theorem 3.7 and lemma 3.1.

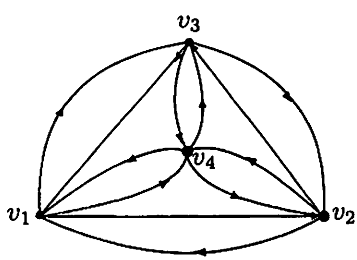

Fig. 3.11. S

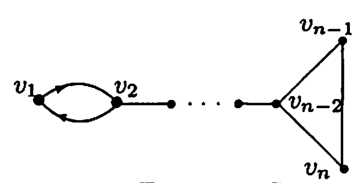

Fig. 3.12. S

2. $2 \in E_n$.

Let S consist of 3–cycles $C_3 = (v_1, v_2, v_3, v_1)$, $C_3^{-1} = (v_1, v_3, v_2, v_1)$ and both arcs (v_i, v_4), (v_4, v_i), $i = 1, 2, 3$ [see Fig. 3.11]. sgn(v_4, v_3) =sgn(v_4, v_2) =sgn(v_2, v_4) =sgn$(v_2, v_1) = -1$, each of other arcs has positive sign. Clearly, S is primitive and nonpowerful by Lemma 2.5, 2.8. Thus $v_1v_3v_1$ and $v_1v_2v_1$ are a pair of $SSSD$ walks of length 2 from vertex v_1 to itself. $v_1v_3v_2$ and $v_1v_4v_2$ are a pair of $SSSD$ walks of length 2 from vertex v_1 to v_2. $v_1v_2v_3$ and $v_1v_4v_3$ are a pair of $SSSD$ walks of length 2 from vertex v_1 to v_3. $v_1v_3v_4$ and $v_1v_2v_4$ are a pair of $SSSD$ walks of length 2 from vertex v_1 to v_4. So $l_S(v_1) = 2$. In a same way, we can prove $l_S(v_2), l_S(v_3), l_S(v_4) = 2$. So $l(S) = 2$ and $2 \in E_n$ by Lemma 3.1.

3. $\{6, 8, \cdots, 2n-2\} \subseteq E_n$ for $n \geq 4$.

Let S consists of paths $P = v_1v_2\cdots v_{n-2}$, P^{-1}, cycle $C_3 = (v_{n-2}, v_{n-1}, v_n)$ and C_3^{-1} [see Fig. 3.12]. $\text{sgn}(v_1, v_2) = -1$ and each of other arcs has positive sign. Clearly, S is primitive and nonpowerful by Lemmas 2.5, 2.8. Let $C_2^1 = (v_1, v_2, v_1)$, $C^* = \{C_2^1, C_3\}$. Then $d(C^*) = d_{C^*}(v_1, v_1) = 2(n-3)$. Note that C_2^1 has different sign from other 2-cycle in S, so $l(S) \leq d(C^*) + \phi(2,3) + 2 \leq 2(n-3) + 4 = 2n-2$.

Now we prove $l(S) = 2n-2$. We prove there is no pair of $SSSD$ walks of length $2n-3$ from vertex v_1 to itself. Otherwise, suppose W_1, W_2 are a pair of $SSSD$ walks of length $2n-3$ from vertex v_1 to itself. Then W_i $(i = 1, 2)$ must be composed of $P \bigcup P^{-1}$, some 2-cycles and some 3-cycles. So $2n-3 = L(W_i) = 2(n-3) + a_i \cdot 2 + b_i \cdot 3$ $(a_i, b_i \geq 0)$. It is easy to see that $a_i \geq 1$ because all 3-cycles have the same sign. So $(a_i - 1) \cdot 2 + b_i \cdot 3 = 1$, which contradicts that $\phi(2,3) = 2$. So there is no pair of $SSSD$ walks of length $2n-3$ from vertex v_1 to itself. Thus $l(S) = 2n-2$.

Because $n \geq 4$, so $\{6, 8, \cdots, 2n-2\} \subseteq E_n$ by Lemma 3.1.

4. $\{7, 9, \cdots, 2n-1\} \subseteq E_n$ for $n \geq 4$.

(i) n is odd.

Let S consists of odd cycle $C_o = (v_1, v_2, \cdots, v_n, v_1)$ and C_o^{-1}. Each 2-cycle of S has negative sign. Clearly, S is primitive and nonpowerful by Lemma 2.5, 2.8. By Theorem 3.6, we know $l(S) \leq 2n-1$. Now we prove $l(S) = 2n-1$. We prove there is no pair of $SSSD$ walks of length $2n-2$ from vertex v_1 to itself. Otherwise, suppose W_1, W_2 are a pair of $SSSD$ walks of length $2n-2$ from vertex v_1 to itself, then W_i $(i = 1, 2)$ must be composed of some 2-cycles and some n-cycles. So $2n-2 = L(W_i) = a_i \cdot 2 + b_i \cdot n$ $(a_i, b_i \geq 0)$. It is easy to see that $b_i \geq 1$ because all 2-cycles have the same sign. So $a_i \cdot 2 + (b_i - 1) \cdot n = n-2$, which contradicts that $\phi(2, n) = n-1$. Thus there is no pair of $SSSD$ walks of length $2n-2$ from vertex v_1 to itself, and so $l_S(v_1) = 2n-1$ and $l(S) = 2n-1$.

(ii) n is even.

Let S consists of odd cycle $C_o = (v_1, v_2, \cdots, v_{n-1}, v_1)$, C_o^{-1} and 2–cycle (v_{n-1}, v_n, v_{n-1}). Each 2-cycle of S has negative sign. $l(S) \leq 2n-1$ by theorem 3.6. Same as (i), we can prove $l_S(v_n) = 2n-1$, so $l(S) = 2n-1$.

Because $n \geq 4$, so $\{7, 9, \cdots, 2n-1\} \subseteq E_n$ by Lemma 3.1.

To sum up, the theorem is proved. □

4 Extremal sign patterns

Definition 4.1 *Let S be a strongly connected digraph of order n and $C = (v_{i_1}, v_{i_2}, \cdots, v_{i_m}, v_{i_1})$ be a cycle in S. If there exists an arc (v_{i_k}, v_{i_j}) $(1 \leq k, j \leq m, |k-j| \geq 2 \pmod{k})$ that v_{i_k} and v_{i_j} are nonconsecutive on cycle C, arc (v_{i_k}, v_{i_j}) $(1 \leq k, j \leq m)$ is called a chord of C.*

Let S consists of cycles $C_k = (v_1, v_2, \cdots, v_k, v_1)$, C^{-1}, paths $P = v_k v_{k+1} \cdots v_n$ and P^{-1}. The connected digraph S is called a k-lollipop [see Fig. 4.1], denoted by l_p^k.

Fig. 4.1. l_p^k

Theorem 4.2 *Let S be a primitive nonpowerful signed zero-symmetric digraph of order $n \geq 3$ without loop. Then $l(S) = 2n-1$ if and only if $|S| \cong l_p^k$ where $k \geq 3$ is odd and each 2-cycle in S has negative sign.*

Proof. By Lemmas 3.2-3.5, it can be known that each 2–cycle in S has negative sign if $l(S) = 2n-1$. There must exist an odd cycle in S because S is primitive. Suppose k–cycle $C_k = (v_1, v_2, \cdots, v_k, v_1)$ $(k \geq 3)$ is a shortest odd cycle in S and C_k is clockwise, then $\text{sgn}(C_k) = -\text{sgn}(C_k^{-1})$ because there are just k negative arcs in $S_0 = C_k \bigcup C_k^{-1}$. We have $\exp(S_0) \leq k-1$ by Lemma 2.12.

We assert there is no chord of C_k. Otherwise, there must cause a shorter odd cycle in S_0, which contradicts the choice of C_k.

Next we prove $|S| \cong l_p^k$ if $l(S) = 2n-1$. Otherwise, suppose $|S| \not\cong l_p^k$. Then $k \leq n-1$ because S is a lollipop if $k = n$.

For any vertices $v_i, v_j \in V(S)$, let P_i denote the shortest path from v_i to S_0, P_j denote the shortest path from v_j to S_0, $P_i \bigcap S_0 = v_c, P_j \bigcap S_0 = v_d$. Then $L(P_i), L(P_j) \leq n-k$.

(i) $v_i = v_j$. Now $L(P_i) = L(P_j)$.

Case 1 $L(P_i) \leq n-k-1$.

Note that $\exp(S_0) \leq k-1$, there is a pair of $SSSD$ walks of length k from v_c to itself, so there are $SSSD$ walks of length l (with $l \geq 2k-1$) from v_c to itself, and there are $SSSD$ walks of length t (with $t \geq 2k-1+2L(P_i)$) from v_i to itself. Note that $L(P_i) \leq n-k-1$, so $2k-1+2L(P_i) \leq 2n-3$.

Case 2 $L(P_i) = n-k$.

Because $|S| \not\cong l_p^k$, so there are at least two different paths from v_i to S_0. Denote by P_i, P_j such two different paths. Then $L(P_i) = L(P_j)$ and all the vertices of P_i, P_j from v_i to S_0 are the same but the ends. Suppose $P_i \bigcap S_0 = v_c, P_j \bigcap S_0 = v_d$ $(v_c \neq v_d)$ and the last common vertex of P_i, P_j along P_i is v_e [see Fig. 4.1].

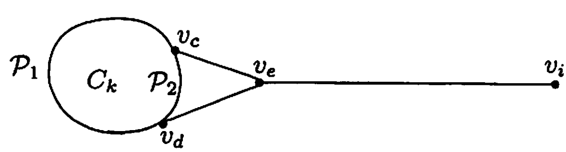

Fig. 4.1. $L(P_i) = n-k$

Suppose C_k is parted into $\mathcal{P}_1$ and $\mathcal{P}_2$ by v_c and v_d, namely $C_k = \mathcal{P}_1 \bigcup \mathcal{P}_2$. Because $k \geq 3$ is odd, so $L(\mathcal{P}_1) \equiv L(\mathcal{P}_2) + 1 \pmod 2$. Suppose $L(\mathcal{P}_1)$ is even, then $C_e = (v_e, v_c) \bigcup \mathcal{P}_2 \bigcup (v_d, v_e)$ is an odd cycle and $\text{sgn}(C_e) = -\text{sgn}(C_e^{-1})$. Let $S_1 = C_e \bigcup C_e^{-1}$. Then $\exp(S_1) \leq L(C_e) - 1$ by Lemma 2.12. So there is a pair of $SSSD$ walks of length l (with $l \geq 2L(C_e) - 1$) from v_e to itself by Lemma 2.18. Let $P_{i,e}$ denote the path along P_i from v_i to v_e, then $L(P_{i,e}) = n-k-1$, and so there is a pair of $SSSD$ walks of length t (with $t \geq 2L(C_e) - 1 + 2L(P_{i,e})$) from v_i to itself. Note that $L(\mathcal{P}_2) \leq k-2$ because $v_c \neq v_d$ and note that $L(C_e) = L(P_2) + 2 \leq k$, so $2L(C_e) - 1 + 2L(P_{i,e}) \leq 2n-3$.

(ii) $v_i \neq v_j$.

Case 1 $L(P_i) > 0, L(P_j) > 0$.

Note that $\exp(S_0) \leq k-1$, so there are $SSSD$ walks of length l (with $l \geq 2k-1$) from v_c to v_d by Lemma 2.18, and so there are $SSSD$ walks of length t (with $t \geq 2k-1+L(P_i)+L(P_j)$) from v_i to v_j. Note that $v_i \neq v_j$, so

$$L(P_i)+L(P_j) \leq n-k+n-k-1 = 2n-2k-1 \text{ and } 2k-1+L(P_i)+L(P_j) \leq 2n-2.$$

Case 2 $L(P_i) = 0, L(P_j) > 0$.

Same as the proof of Case 1, we can prove that there is a pair of $SSSD$ walks of length t (with $t \geq 2k-1+L(P_j)$) from v_i to v_j. Note that $k \leq n-1$ and $L(P_j) \leq n-k$, so $2k-1+L(P_j) \leq 2n-2$.

Case 3 $L(P_i) > 0, L(P_j) = 0$.

Same as the proof of Case 2, we can prove that there is a pair of $SSSD$ walks of length t (with $t \geq 2k-1+L(P_i)$) from v_i to v_j. Note that $k \leq n-1$ and $L(P_i) \leq n-k$, so $2k-1+L(P_i) \leq 2n-2$.

Case 4 $L(P_i) = 0, L(P_j) = 0$.

Note that $\exp(S_0) \leq k-1$, so there is a pair of $SSSD$ walks of length l (with $l \geq 2k-1$) from v_i to v_j by Lemma 2.18.

By (i), (ii), there are $SSSD$ walks of length $2n-2$ from v_i to v_j for any vertices $v_i, v_j \in V(S)$ if $|S| \ncong l_p^k$, so $l(S) \leq 2n-2$ by Definition 1.7, which contradicts $l(S) = 2n-1$. So $|S| \cong l_p^k$ ($k \geq 3$ is odd) if $l(S) = 2n-1$.

It is easy to know that $l(S) \leq 2n-1$ if $|S| \cong l_p^k$ where $k \geq 3$ is odd and there is no positive 2-cycle in S by Theorem 3.6. We prove $l(S) = 2n-1$ next.

We prove there is no pair of $SSSD$ walks of length $2n-2$ from v_n to itself. Otherwise, suppose there is a pair of $SSSD$ walks W_1, W_2 of length $2n-2$ from v_n to itself. Let $P = (v_k, v_{k+1}, \cdots, v_{n-1}, v_n)$. Then W_i ($i = 1,2$) must be composed of P, P^{-1}, some 2−cycles and some k-cycles. So $2n-2 = L(W_i) = 2(n-k)+2a_i+b_ik$ ($a_i, b_i \geq 0$). Because all 2-cycles have the same sign, so $b_i \geq 1$. Then $k-2 = 2a_i+(b_i-1)k$, which contradicts $\phi(2,k) = k-1$. So there exists no pair of $SSSD$ walks of length $2n-2$ from v_n to itself. Thus $l_S(v_n, v_n) = 2n-1$ and $l(S) = 2n-1$. □

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