On metric dimension of convex polytopes with pendant edges*

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Abstract. A family $\mathcal{G}$ of connected graphs is said to be a family with constant metric dimension if $dim(G)$ does not depend upon the choice of G in $\mathcal{G}$.
In this paper we study the metric dimension of some plane graphs which are obtained from some convex polytopes by attaching a pendant edge to each vertex of the outer cycle in a plane representation of these convex polytopes. We prove that the metric dimension of these plane graphs is constant and only three vertices appropriately chosen suffice to resolve all the vertices of these classes of graphs. It is natural to ask for the characterization of graphs G which are plane representations of convex polytopes having the property that $dim(G) = dim(G')$, where G' is obtained from G by attaching a pendant edge to each vertex of the outer cycle of G.

Keywords: *Metric dimension, basis, resolving set, plane graph, convex polytope*

1 Notation and preliminary results

A basic problem in chemistry is to provide mathematical representations for a set of chemical compounds in a way that gives distinct representations

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to distinct compounds. As described in [6], the structure of a chemical compound can be represented by a labeled graph whose vertex and edge labels specify the atom and bond types, respectively. Thus, a graph-theoretic interpretation of this problem is to provide representations for the vertices of a graph in such a way that distinct vertices have distinct representations. This is the subject of the papers [3, 6, 17, 19, 20].

If G is a connected graph, the *distance* $d(u,v)$ between two vertices $u, v \in V(G)$ is the length of a shortest path between them. Let $W = \{w_1, w_2, ..., w_k\}$ be an ordered set of vertices of G and let v be a vertex of G. The *representation* $r(v|W)$ of v with respect to W is the k-tuple $(d(v,w_1), d(v,w_2), ..., d(v,w_k))$. If distinct vertices of G have distinct representations with respect to W, then W is called a *resolving set* or *locating set* for G [3]. A resolving set of minimum cardinality is called a *metric basis* for G and this cardinality is the *metric dimension* of G, denoted by $dim(G)$. The concepts of resolving set and metric basis have previously appeared in the literature (see [3-7, 10-22]).

For a given ordered set of vertices $W = \{w_1, w_2, ..., w_k\}$ of a graph G, the *ith* component of $r(v|W)$ is 0 if and only if $v = w_i$. Thus, to show that W is a resolving set it suffices to verify that $r(x|W) \neq r(y|W)$ for each pair of distinct vertices $x, y \in V(G) \backslash W$.

A useful property in finding $dim(G)$ is the following lemma [21]:

Lemma 1. *Let W be a resolving set for a connected graph G and $u, v \in V(G)$. If $d(u,w) = d(v,w)$ for all vertices $w \in V(G) \setminus \{u,v\}$, then $\{u,v\} \cap W \neq \emptyset$.*

Motivated by the problem of uniquely determining the location of an intruder in a network, the concept of metric dimension was introduced by Slater in [19, 20] and studied independently by Harary and Melter in [10]. Applications of this invariant to the navigation of robots in networks are discussed in [16] and applications to chemistry in [6] while applications to problem of pattern recognition and image processing, some of which involve the use of hierarchical data structures are given in [17].

By denoting $G + H$ the join of G and H, a *wheel* W_n is defined as $W_n = K_1 + C_n$, for $n \geq 3$, a *fan* is $f_n = K_1 + P_n$ for $n \geq 1$ and the *gear graph* $J_{2n}, (n \geq 2)$ is obtained from a wheel W_{2n} by alternately deleting n spokes. Buczkowski *et al.* [3] determined the metric dimension of the wheel W_n, Caceres *et al.* [5] the metric dimension of a fan f_n and Tomescu and Javaid [22] the metric dimension of the gear graph J_{2n}.

Theorem 1. *([3], [5], [22]) We have*
(i) $dim(W_n) = \lfloor \frac{2n+2}{5} \rfloor$ *for* $n \geq 7$;
(ii) $dim(f_n) = \lfloor \frac{2n+2}{5} \rfloor$ *for* $n \geq 7$;
(iii) $dim(J_{2n}) = \lfloor \frac{2n}{3} \rfloor$ *for* $n \geq 4$.

The metric dimension of all these *plane graphs* depends upon the number of vertices in the graph.

On the other hand, we say that a family $\mathcal{G}$ of connected graphs is a family with constant metric dimension if $dim(G)$ does not depend upon the choice of G in $\mathcal{G}$. In [6] it was shown that a graph has metric dimension 1 if and only if it is a *path*, hence paths on n vertices constitute a family of graphs with constant metric dimension. Similarly, *cycles* with $n(\geq 3)$ vertices also constitute such a family of graphs as their metric dimension is 2 and does not depend upon the number n of vertices . A *Cartesian product* of two graphs G and H, denoted by $G\square H$, is the graph with vertex set $V(G) \times V(H)$, where two vertices (x, x') and (y, y') are adjacent if and only if $x = y$ and $x'y' \in E(H)$ or $x' = y'$ and $xy \in E(G)$. The metric dimension of the Cartesian product of graphs has been studied in [4] and [18]. In [4] it was proved that

$$dim(P_m\square C_n)=\begin{cases} 2, & \text{if } n \text{ is odd;} \\ 3, & \text{if } n \text{ is even.} \end{cases}$$

The antiprism A_n [1], $n \geq 3$, is a 4-regular graph and consists of an outer n-cycle $y_1y_2...y_n$, an inner n-cycle $x_1x_2...x_n$, and a set of n spokes x_iy_i and $x_{i+1}y_i, i = 1, ..., n$ where $x_{n+1} = x_1$. For $n = 3$ it is the octahedron.

Also Javaid *et al.* proved in [12] that the *antiprisms* A_n constitute a family of regular graphs with constant metric dimension as $dim(A_n) = 3$ for every $n \geq 5$. For more details on convex polytopes, we refer the readers to [9].

It was shown in [11] that some families of plane graphs generated by convex polytopes constitute families of plane graphs with constant metric dimension. Note that the problem of determining whether $dim(G) < k$ is an NP-complete problem [8]. Some bounds for this invariant, in terms of the diameter of the graph, are given in [16] and it was shown in [6, 16–18] that the metric dimension of trees can be determined efficiently. It appears unlikely that significant progress can be made in determining the metric dimension of a graph unless it belongs to a class for which the distances between vertices can be described in some systematic manner.

A fundamental question in graph theory concerns how the value of a parameter is affected by making a small change in the graph. If G' is a graph obtained by adding a pendant edge to a nontrivial connected graph G, then it is easy to verify that

$$dim(G) \leq dim(G') \leq dim(G) + 1$$

A *helm* H_n, $n \geq 3$ is a graph obtained from a wheel W_n by attaching a pendant vertex to each rim vertex. Javaid [12] proved that $dim(H_n) = dim(W_n)$. In this paper we extend this study by considering some classes of convex polytopes. We prove that the metric dimension of these classes of plane graphs which are obtained from plane representation of these convex ploytopes by attaching a pendant edge to each vertex of the outer cycle is constant and only three vertices appropriately chosen suffice to resolve all the vertices of these classes of graphs. Consequently, by attaching a pendant edge to each vertex of the outer cycle of these graphs their metric dimension is not affected. Thus, it is natural to ask for the characterization of graphs G which are plane representation of convex polytopes having the property that $dim(G) = dim(G')$, where G' is obtained from G by attaching a pendant edge to each vertex of the outer cycle of G.

2 Plane graph R_n^p

For $n \geq 5$, let R_n^p be the plane graph represented in Fig. 1. By deleting pendant edges $z_1w_1, ..., z_nw_n$ the resulting graph is denoted by R_n and it is a plane representation of a convex polytope defined in [2].
It follows that

$$V(R_n^p) = V(R_n) \cup \{w_i : 1 \leq i \leq n\}$$

and

$$E(R_n^p) = E(R_n) \cup \{z_iw_i : 1 \leq i \leq n\}$$

The metric dimension of R_n has been determined in [11] and it has been

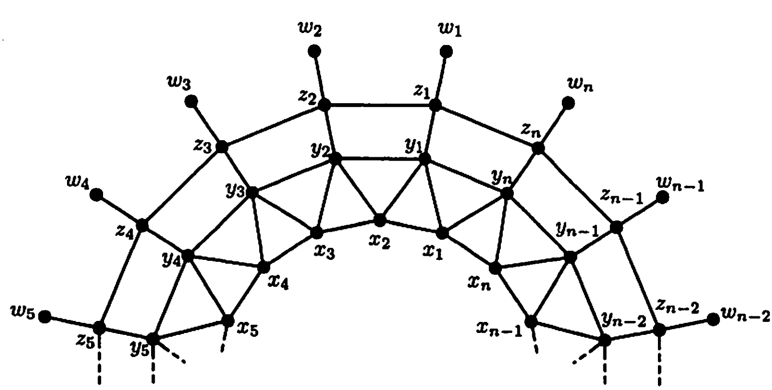

Fig. 1. The plane graph R_n^p

shown that it has metric dimension equal to 3. In the next theorem we

prove that by attaching a pendant edge at each vertex of the outer cycle of R_n is not affected its metric dimension.
For our purpose, we call the cycle induced by $\{x_i : 1 \leq i \leq n\}$, the inner cycle, the cycle induced by $\{y_i : 1 \leq i \leq n\}$, the middle cycle, the cycle induced by $\{z_i : 1 \leq i \leq n\}$, the outer cycle and the set of vertices $\{w_i : 1 \leq i \leq n\}$, the set of outer vertices. Note that the choice of appropriate metric basis vertices (also refereed to as landmarks in [15]) is the core of the problem.

Theorem 2. *Let R_n^p be the graph defined above; we have $dim(R_n^p) = 3$ for every $n \geq 6$.*

Proof. We will prove the above equality by double inequality.
Case 1. When n is even.
In this case, we can write $n = 2k$, $k \geq 3$, $k \in \mathbb{Z}^+$. Let $W = \{x_1, x_2, x_{k+1}\} \subset V(R_n^p)$, we show that W is a resolving set for R_n^p in this case. For this we give the representation of any vertex of $V(R_n^p)\backslash W$ with respect to W.
The representation of the vertices on the inner cycle are

$$r(x_i|W) = \begin{cases} (i-1, i-2, k-i+1), & 3 \leq i \leq k; \\ (2k-i+1, 2k-i+2, i-k-1), & k+2 \leq i \leq 2k. \end{cases}$$

The representations of the vertices on the middle cycle are

$$r(y_i|W) = \begin{cases} (1, 1, k), & i = 1; \\ (i, i-1, k-i+1), & 2 \leq i \leq k; \\ (k, k, 1), & i = k+1; \\ (2k-i+1, 2k-i+2, i-k), & k+2 \leq i \leq 2k. \end{cases}$$

The representations of the vertices on the outer cycle are

$$r(z_i|W) = \begin{cases} (2, 2, k+1), & i = 1; \\ (i+1, i, k-i+2), & 2 \leq i \leq k+1; \\ (k+1, k+1, 2), & i = k+2; \\ (2k-i+2, 2k-i+3, i-k+1), & k+3 \leq i \leq 2k. \end{cases}$$

The representations of the set of outer vertices are

$$r(w_i|W) = \begin{cases} (3, 3, k+2), & i = 1; \\ (i+2, i+1, k-i+3), & 2 \leq i \leq k; \\ (k+2, k+2, 3), & i = k+1; \\ (2k-i+3, 2k-i+4, i-k+2), & k+2 \leq i \leq 2k. \end{cases}$$

We note that there are no two vertices having the same representations implying that $dim(R_n^p) \leq 3$.
On the other hand, we show that $dim(R_n^p) \geq 3$. Suppose on contrary that

$dim(R_n^p) = 2$, hence there exists a resolving set consisting of two vertices. If both vertices are in the inner cycle, without loss of generality we suppose that one resolving vertex is x_1. Suppose that the second resolving vertex is x_t $(2 \le t \le k+1)$. Then for $2 \le t \le k$, we have $r(x_n|\{x_1, x_t\}) = r(y_n|\{x_1, x_t\}) = (1, t)$ and when $t = k+1$, $r(x_2|\{x_1, x_{k+1}\}) = r(x_n|\{x_1, x_{k+1}\}) = (1, k-1)$, a contradiction.
If one vertex is in the inner cycle and the other in the middle cycle, without loss of generality we suppose that one resolving vertex is x_1. Suppose that the second resolving vertex is y_t $(1 \le t \le k+1)$. Then for $t = 1$, we have $r(x_2|\{x_1, y_1\}) = r(y_n|\{x_1, y_1\}) = (1, 1)$ and when $2 \le t \le k+1$, $r(x_2|\{x_1, y_t\}) = r(y_1|\{x_1, y_t\}) = (1, t-1)$, a contradiction. The remaining cases may be solved in a similar manner.
It follows that there is no resolving set with two vertices for $V(R_n^p)$, implying that $dim(R_n^p) = 3$ in this case.
Case 2. When n is odd.
In this case we can write $n = 2k+1$, $k \ge 3$, $k \in \mathbf{Z}^+$. Let $W = \{x_1, x_2, x_{k+1}\} \subset V(R_n^p)$, again we show that W is a resolving set for R_n^p in this case also. For this we give the representations of the vertices of $V(R_n^p)\backslash W$ with respect to W.
The representations of the vertices on the inner cycle are

$$r(x_i|W) = \begin{cases} (i-1, i-2, k-i+1), & 3 \le i \le k; \\ (k, k, 1), & i = k+2; \\ (2k-i+2, 2k-i+3, i-k-1), & k+3 \le i \le 2k+1. \end{cases}$$

The representations of the vertices on the middle cycle are

$$r(y_i|W) = \begin{cases} (1, 1, k), & i = 1; \\ (i, i-1, k-i+1), & 2 \le i \le k; \\ (k+1, k, 1), & i = k+1; \\ (2k-i+2, 2k-i+3, i-k), & k+2 \le i \le 2k+1. \end{cases}$$

The representations of the vertices on the outer cycle are

$$r(z_i|W) = \begin{cases} (2, 2, k+1), & i = 1; \\ (i+1, i, k-i+2), & 2 \le i \le k; \\ (k+2, k+1, 2), & i = k+1; \\ (2k-i+3, 2k-i+4, i-k+1), & k+2 \le i \le 2k+1. \end{cases}$$

The representations of the set of the outer vertices are

$$r(w_i|W) = \begin{cases} (3, 3, k+2), & i = 1; \\ (i+2, i+1, k-i+3), & 2 \le i \le k; \\ (k+3, k+2, 3), & i = k+1; \\ (2k-i+4, 2k-i+5, i-k+2), & k+2 \le i \le 2k+1. \end{cases}$$

Again we see that there are no two vertices having the same representations which implies that $dim(R_n^p) \leq 3$.
On the other hand, suppose that $dim(R_n^p) = 2$, then there are the same possibilities as in case (1) and a contradiction can be deduced analogously. This implies that $dim(R_n^p) = 3$ in this case, which completes the proof.

3 Plane graph D_n^p

For $n \geq 6$ we consider the plane graph represented in Fig. 2 which is denoted by D_n^p. Deleting pendant edges $d_1e_1, ..., d_ne_n$ yields the graph denoted by D_n^*, which is a plane representation of a convex polytope defined in [1]. We have

$$V(D_n^p) = V(D_n^*) \cup \{e_i : 1 \leq i \leq n\}$$

and

$$E(D_n^p) = E(D_n^*) \cup \{d_ie_i : 1 \leq i \leq n\}$$

The metric dimension of the graph D_n^* has been studied in [11] and it has

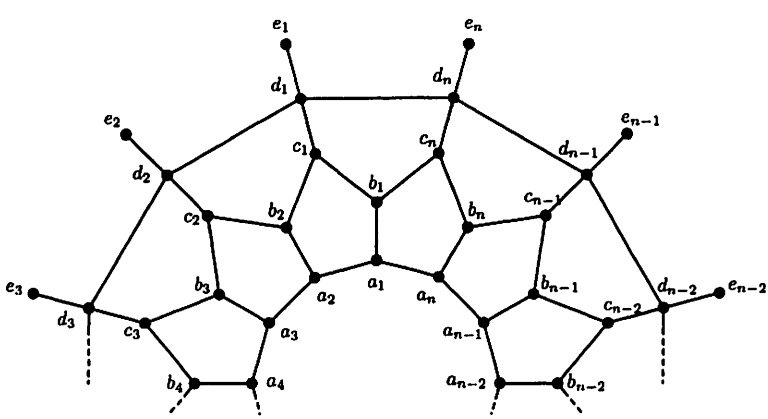

Fig. 2. The plane graph D_n^p

been proved that this graph has metric dimension 3. In the next theorem we prove that the metric dimension of D_n^* is the same as the metric dimension of the graph D_n^p.
For our purpose, we call the cycle induced by $\{a_i : 1 \leq i \leq n\}$, the inner cycle, the cycle induced by $\{b_i : 1 \leq i \leq n\} \cup \{c_i : 1 \leq i \leq n\}$, the middle

cycle, the cycle induced by $\{d_i : 1 \leq i \leq n\}$, the outer cycle and the set of vertices $\{e_i : 1 \leq i \leq n\}$, the set of outer vertices. Again, the choice of appropriate metric basis vertices is crucial.

Theorem 3. *We have $dim(D_n^p) = 3$ for every $n \geq 6$.*

Proof. We will prove the above equality by double inequality.
Case 1. When n is even.
In this case, we can write $n = 2k$, $k \geq 3$, $k \in \mathbf{Z}^+$. Let $W = \{x_1, x_2, x_{k+1}\} \subset V(D_n^p)$, we show that W is a resolving set for D_n^p in this case. For this we give the representation of any vertex of $V(D_n^p) \backslash W$ with respect to W.
The representations of the vertices on the inner cycle are

$$r(a_i|W) = \begin{cases} (i-1, i-2, k-i+1), & 3 \leq i \leq k; \\ (2k-i+1, 2k-i+2, i-k-1), & k+2 \leq i \leq 2k. \end{cases}$$

The representations of the vertices on the middle cycle are

$$r(b_i|W) = \begin{cases} (1, 2, k+1), & i = 1; \\ (i, i-1, k-i+2), & 2 \leq i \leq k; \\ (2k-i+2, 2k-i+3, i-k), & k+2 \leq i \leq 2k. \end{cases}$$

and

$$r(c_i|W) = \begin{cases} (2, 2, k+1), & i = 1; \\ (i+1, i, k-i+2), & 2 \leq i \leq k; \\ (k+1, k+1, 2), & i = k+1; \\ (2k-i+2, 2k-i+3, i-k+1), & k+2 \leq i \leq 2k. \end{cases}$$

The representations of the vertices on the outer cycle are

$$r(d_i|W) = \begin{cases} (3, 3, k+2), & i = 1; \\ (i+2, i+1, k-i+3), & 2 \leq i \leq k; \\ (k+2, k+2, 3), & i = k+1; \\ (2k-i+3, 2k-i+4, i-k+2), & k+2 \leq i \leq 2k. \end{cases}$$

The representations of the set of outer vertices are

$$r(e_i|W) = \begin{cases} (4, 4, k+3), & i = 1; \\ (i+3, i+2, k-i+4), & 2 \leq i \leq k; \\ (k+3, k+3, 4), & i = k+1; \\ (2k-i+4, 2k-i+5, i-k+3), & k+2 \leq i \leq 2k. \end{cases}$$

We note that there are no two vertices having the same representations, implying that $dim(D_n^p) \leq 3$.
On the other hand, we show that $dim(D_n^p) \geq 3$. Suppose on contrary that

$dim(D_n^p) = 2$. It follows that there exists a resolving set containing two vertices.
If both vertices are in the inner cycle, without loss of generality we suppose that one resolving vertex is a_1. Suppose that the second resolving vertex is a_t $(2 \leq t \leq k+1)$. Then for $2 \leq t \leq k$, we have $r(a_n|\{a_1, a_t\}) = r(b_1|\{a_1, a_t\}) = (1, t)$ and when $t = k+1$, $r(a_2|\{a_1, a_{k+1}\}) = r(a_n|\{a_1, a_{k+1}\}) = (1, k-1)$, a contradiction.
If one vertex is in the set of vertices $\{b_i : 1 \leq i \leq n\}$ and the other in the set $\{c_i : 1 \leq i \leq n\}$, we can suppose that one resolving vertex is b_1. Suppose that the second resolving vertex is c_t $(2 \leq t \leq k+1)$. For $2 \leq t \leq k-1$, we have $r(a_1|\{b_1, c_t\}) = r(c_n|\{b_1, c_t\}) = (1, t+1)$. If $t = k$, $r(c_{n-2}|\{b_1, c_k\}) = r(d_n|\{b_1, c_k\}) = (3, k+1)$ and when $t = k+1$, $r(c_{n-2}|\{b_1, c_{k+1}\}) = r(d_n|\{b_1, c_{k+1}\}) = (3, k)$, a contradiction. The proof of the remaining cases follows the same lines and are therefore omitted.
We deduce that there is no resolving set with two vertices for $V(D_n^p)$, implying that $dim(D_n^p) = 3$ in this case.
Case 2. When n is odd.
In this case, we can write $n = 2k+1$, $k \geq 3$, $k \in \mathbf{Z}^+$. Let $W = \{x_1, x_2, x_{k+1}\} \subset V(D_n^p)$, again we show that W is a resolving set for D_n^p in this case. For this we give the representation of any vertex of $V(D_n^p)\backslash W$ with respect to W.
The representations of the vertices on the inner cycle are

$$r(a_i|W) = \begin{cases} (i-1, i-2, k-i+1), & 3 \leq i \leq k; \\ (k, k, 1), & i = k+2; \\ (2k-i+1, 2k-i+2, i-k-1), & k+3 \leq i \leq 2k+1. \end{cases}$$

The representations of the vertices on the middle cycle are

$$r(b_i|W) = \begin{cases} (1, 2, k+1), & i = 1; \\ (i, i-1, k-i+2), & 2 \leq i \leq k+1; \\ (k+1, k+1, 2), & i = k+2; \\ (2k-i+2, 2k-i+3, i-k), & k+3 \leq i \leq 2k+1. \end{cases}$$

and

$$r(c_i|W) = \begin{cases} (2, 2, k+1), & i = 1; \\ (i+1, i, k-i+2), & 2 \leq i \leq k; \\ (k+2, k+1, 2), & i = k+1; \\ (2k-i+3, 2k-i+4, i-k+1), & k+2 \leq i \leq 2k+1. \end{cases}$$

The representations of the vertices on the outer cycle are

$$r(d_i|W) = \begin{cases} (3, 3, k+2), & i = 1; \\ (i+2, i+1, k-i+3), & 2 \leq i \leq k; \\ (k+2, k+2, 3), & i = k+1; \\ (2k-i+4, 2k-i+5, i-k+2), & k+2 \leq i \leq 2k+1. \end{cases}$$

The representations of the set of the outer vertices are

$$r(e_i|W) = \begin{cases} (4,4,k+3), & i=1; \\ (i+3,i+2,k-i+4), & 2 \le i \le k; \\ (k+4,k+3,4), & i=k+1; \\ (2k-i+5,2k-i+6,i-k+3), & k+2 \le i \le 2k+1. \end{cases}$$

Again we see that there are no two vertices having the same representations, which implies that $dim(D_n^p) \le 3$.
On the other hand, suppose that $dim(D_n^p) = 2$, then there are the same possibilities as in case (**1**) and a contradiction can be deduced analogously. This implies that $dim(D_n^p) = 3$ in this case, which completes the proof.

4 Plane graph Q_n^p

The plane graph denoted by Q_n is defined by using D_n^* as follows: $V(Q_n) = V(D_n^*)$ and $E(Q_n) = E(D_n^*) \cup \{b_i b_{i+1} : 1 \le i \le n\}$ (by convention $b_{n+1} = b_1$).
It has 3-,4-,5- and n-sided faces and is a plane representation of a convex polytope defined in [2].
The plane graph Q_n^p (Fig. 3) is obtained from the graph Q_n by attaching a pendant edge at each vertex of the outer cycle of Q_n. We have

$$V(Q_n^p) = V(Q_n) \cup \{e_i : 1 \le i \le n\}$$

and

$$E(Q_n^p) = E(Q_n) \cup \{d_i e_i : 1 \le i \le n\}$$

The metric dimension of the graph Q_n has been investigated in [11]. In the next theorem we prove that the metric dimension of Q_n is the same as the metric dimension of Q_n^p.
For our purpose, we call the cycle induced by $\{a_i : 1 \le i \le n\}$, the a-cycle, the cycle induced by $\{b_i : 1 \le i \le n\}$, the b-cycle, the set of vertices $\{c_i : 1 \le i \le n\}$, the set of inner vertices, the cycle induced by $\{d_i : 1 \le i \le n\}$, the d-cycle and the set of vertices $\{e_i : 1 \le i \le n\}$, the set of outer vertices. Once again, the choice of appropriate metric basis vertices is very important.

Theorem 4. *For every $n \ge 6$, $dim(Q_n^p) = 3$ holds.*

Proof. We will also prove the above equality by double inequality.
Case 1. When n is even.
In this case, we can write $n = 2k$, $k \ge 3$, $k \in \mathbf{Z}^+$. Let $W = \{x_1, x_2, x_{k+1}\} \subset V(Q_n^p)$, we show that W is a resolving set for Q_n^p in this case. For this we

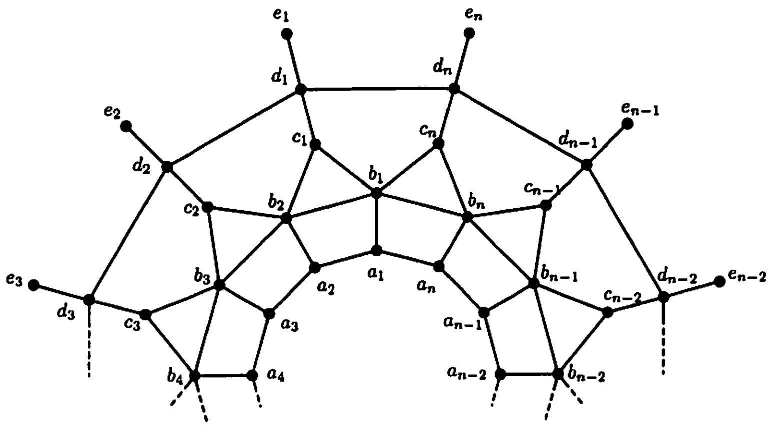

Fig. 3. The plane graph Q_n^p

give the representation of any vertex of $V(Q_n^p)\backslash W$ with respect to W. The representations of the vertices on the a-cycle are

$$r(a_i|W) = \begin{cases} (i-1, i-2, k-i+1), & 3 \leq i \leq k; \\ (2k-i+1, 2k-i+2, i-k-1), & k+2 \leq i \leq 2k. \end{cases}$$

The representations of the vertices on the b-cycle are

$$r(b_i|W) = \begin{cases} (1, 2, k+1), & i = 1; \\ (i, i-1, k-i+2), & 2 \leq i \leq k; \\ (2k-i+2, 2k-i+3, i-k), & k+2 \leq i \leq 2k. \end{cases}$$

The representations of the set of inner vertices are

$$r(c_i|W) = \begin{cases} (2, 2, k+1), & i = 1; \\ (i+1, i, k-i+2), & 2 \leq i \leq k; \\ (k+1, k+1, 2), & i = k+1; \\ (2k-i+2, 2k-i+3, i-k+1), & k+2 \leq i \leq 2k. \end{cases}$$

The representations of the vertices on the d-cycle are

$$r(d_i|W) = \begin{cases} (3, 3, k+2), & i = 1; \\ (i+2, i+1, k-i+3), & 2 \leq i \leq k; \\ (k+2, k+2, 3), & i = k+1; \\ (2k-i+3, 2k-i+4, i-k+2), & k+2 \leq i \leq 2k. \end{cases}$$

The representations of the set of outer vertices are

$$r(e_i|W) = \begin{cases} (4,4,k+3), & i=1; \\ (i+3,i+2,k-i+4), & 2 \le i \le k; \\ (k+3,k+3,4), & i=k+1; \\ (2k-i+4,2k-i+5,i-k+3), & k+2 \le i \le 2k. \end{cases}$$

We note that there are no two vertices having the same representations, implying that $dim(Q_n^p) \le 3$.

On the other hand, we show that $dim(Q_n^p) \ge 3$. Suppose on contrary that $dim(Q_n^p) = 2$, i.e., there exists a resolving set including exactly two vertices.

If both vertices are in the a-cycle, without loss of generality we suppose that one resolving vertex is a_1. Suppose that the second resolving vertex is a_t $(2 \le t \le k+1)$. Then for $2 \le t \le k$, we have $r(a_n|\{a_1,a_t\}) = r(b_1|\{a_1,a_t\}) = (1,t)$ and when $t = k+1$, $r(a_2|\{a_1,a_{k+1}\}) = r(a_n|\{a_1,a_{k+1}\}) = (1,k-1)$, a contradiction.

If one vertex is in the a-cycle and the other in the b- cycle, without loss of generality we can suppose that one resolving vertex is a_1. Suppose that the second resolving vertex is b_t $(1 \le t \le k+1)$. Then for $1 \le t \le k$, $r(b_n|\{a_1,b_t\}) = r(c_n|\{a_1,b_t\}) = (2,t)$ and when $t = k+1$ $r(a_2|\{a_1,b_{k+1}\}) = r(a_n|\{a_1,b_{k+1}\}) = (1,k)$, a contradiction. The remaining cases can be treated in a similar way.

It follows that there is no resolving set with two vertices for $V(Q_n^p)$, implying that $dim(Q_n^p) = 3$ in this case.

Case 2. When n is odd.

In this case, we can write $n = 2k+1$, $k \ge 3$, $k \in \mathbf{Z}^+$. Let $W = \{x_1, x_2, x_{k+1}\} \subset V(Q_n^p)$, again we show that W is a resolving set for Q_n^p in this case. For this we give the representations of the vertices of $V(Q_n^p)\backslash W$ with respect to W.

The representations of the vertices on the a-cycle are

$$r(a_i|W) = \begin{cases} (i-1,i-2,k-i+1), & 3 \le i \le k; \\ (k,k,1), & i=k+2; \\ (2k-i+1,2k-i+2,i-k-1), & k+3 \le i \le 2k+1. \end{cases}$$

The representations of the vertices on the b-cycle are

$$r(b_i|W) = \begin{cases} (1,2,k+1), & i=1; \\ (i,i-1,k-i+2), & 2 \le i \le k+1; \\ (k+1,k+1,2), & i=k+2; \\ (2k-i+2,2k-i+3,i-k), & k+3 \le i \le 2k+1. \end{cases}$$

The representations of the set of inner vertices are

$$r(c_i|W) = \begin{cases} (2,2,k+1), & i=1; \\ (i+1,i,k-i+2), & 2 \le i \le k; \\ (k+2,k+1,2), & i=k+1; \\ (2k-i+3,2k-i+4,i-k+1), & k+2 \le i \le 2k+1. \end{cases}$$

The representations of the vertices on the d-cycle are

$$r(d_i|W) = \begin{cases} (3,3,k+2), & i=1; \\ (i+2,i+1,k-i+3), & 2 \le i \le k; \\ (k+2,k+2,3), & i=k+1; \\ (2k-i+4,2k-i+5,i-k+2), & k+2 \le i \le 2k+1. \end{cases}$$

The representations of the set of outer vertices are

$$r(e_i|W) = \begin{cases} (4,4,k+3), & i=1; \\ (i+3,i+2,k-i+4), & 2 \le i \le k; \\ (k+4,k+3,4), & i=k+1; \\ (2k-i+5,2k-i+6,i-k+3), & k+2 \le i \le 2k+1. \end{cases}$$

Again we see that there are no two vertices having the same representations which implies that $dim(Q_n^p) \le 3$.
On the other hand, suppose that $dim(Q_n^p) = 2$, then there are the same possibilities as in case (1) and a contradiction can be deduced analogously. This implies that $dim(Q_n^p) = 3$ in this case, which completes the proof.

5 Concluding remarks

In this paper we have studied the metric dimension of some plane graphs which are obtained from convex polytopes by attaching a pendant edge to each vertex of the outer cycle in a plane representation of these convex polytopes. We proved that the metric dimension of these plane graphs does not depend upon the number of vertices in these graphs and only three vertices appropriately chosen suffice to resolve all the vertices of these plane graphs. It can be proved infact that for these graphs if we attach a path $P_t (t \ge 1)$ at each vertex of the outer cycle, the metric dimension will not be affected. It is natural to ask for the characterization of graphs G which are plane representation of convex polytopes having the property that $dim(G) = dim(G')$, where G' is obtained from G by attaching a pendant edge to each vertex of the outer cycle of G.

Note that in [17] Melter and Tomescu gave an example of infinite regular graphs (namely the digital plane endowed with city-block and chessboard distances, respectively) having no finite metric basis. We close this section by raising a question that naturally arises from the text.

Open Problem: *Let G' be a graph obtained from a plane representation G of a convex polytope by attaching a pendant edge to each vertex of the outer cycle of G. Is it the case that $dim(G') = dim(G)$ always holds?*

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