Enumeration of a class of rooted planar unicyclic maps*

Shude Long[a] **Junliang Cai**[b]

[a] *Department of Mathematics, Chongqing University of Arts and Sciences, Chongqing 402160, P.R.China*

[b] *School of Mathematical Sciences, Beijing Normal University, Beijing 100875, P.R.China*

(longshude@163.com; caijunliang@bnu.edu.cn)

Abstract

A unicyclic map is a rooted planar map such that there is only one cycle which is the boundary of the unique inner face (the inner face contains no trees) and the root-vertex is on the cycle. In this paper we investigate the number of unicyclic maps and present some formulae for such maps with up to three parameters: the number of edges and the valencies of the root-vertex and the root-face.

MSC: O5C45; 05C30

Keywords: Unicyclic map; Enumerating function; Functional equation; Lagrangian inversion

1. Introduction

Throughout this paper we consider the rooted maps on the plane. Definitions of terms not given here may be found in [14].

The concept of a rooted map was first introduced by W.T. Tutte. His series of census papers [19–22] laid the foundation for the theory. Since then, the theory has been developed by many scholars such as Arquès [1], Brown [7,8], Mullin et al. [18], Tutte [23], Bender et al. [2–6], Liskovets et al. [12,13], Gao [9,10] and Liu [14–17].

In this paper we investigate the number of unicyclic maps and also present explicit formulae for such maps with up to three parameters: the number of edges and the valencies of the root-vertex and the root-face. Furthermore, all of them are summation-free. Before stating our main results we have to define some basic concepts and terms.

A *map* is a connected graph cellularly embedded on a surface. A map is *rooted* if an edge, a direction along the edge, and a side of the edge are

*Supported by the NNSFC under Grant No. 10271017 and Chongqing Municipal Education Commission under Grant No. KJ101204.

all distinguished. If the root is the distinguished edge from u to v, then u is the root-vertex while the face on the distinguished side of the edge is defined as the root-face. A unicyclic map is a rooted planar map such that there is only one cycle which is the boundary of the unique inner face (the inner face contains no trees) and the root-vertex is on the cycle. Although the enumeration of unicyclic maps had been investigated by some scholars such as Liu [14], the case of unicyclic maps with the root-vertex being on the cycle has not been solved. In addition, the problem is also motivated by the classification of Belyi functions, which are in correspondence with planar (hyper) maps.

For any map $M \in \mathcal{M}$, let $M - e_r(M)$ and $M \bullet e_r(M)$ be the maps obtained by deleting $e_r(M)$, the root-edge, from M and contracting $e_r(M)$ into a vertex as the new root-vertex, respectively.

Given two maps M_1 and M_2 with roots $r_1 = r(M_1)$ and $r_2 = r(M_2)$, respectively, we define $M = M_1 \dot{+} M_2$ to be the map obtained by identifying the root-vertices and the root-faces of M_1 and M_2 and rooting M at r_1. The operation for getting M from M_1 and M_2 is called the $1v$-*addition*. Further, for two sets of maps $\mathcal{M}_1$ and $\mathcal{M}_2$, the set of maps

$$\mathcal{M}_1 \odot \mathcal{M}_2 = \{M_1 \dot{+} M_2 \mid M_i \in \mathcal{M}_i, i = 1, 2\}$$

is said to be the $1v$-*production* of $\mathcal{M}_1$ and $\mathcal{M}_2$.

For a set of some maps $\mathscr{M}$, the enumerating function discussed in this paper is defined as

$$f_{\mathscr{M}}(x, y, z) = \sum_{M \in \mathscr{M}} x^{m(M)} y^{n(M)} z^{l(M)}, \tag{1}$$

where $m(M), n(M)$ and $l(M)$ are, respectively, the root-vertex valency, the number of edges and the root-face valency of M and we write that

$$g_{\mathscr{M}}(x, y) = f_{\mathscr{M}}(x, y, 1), \quad h_{\mathscr{M}}(y, z) = f_{\mathscr{M}}(1, y, z),$$
$$H_{\mathscr{M}}(y) = f_{\mathscr{M}}(1, y, 1) = g_{\mathscr{M}}(1, y) = h_{\mathscr{M}}(y, 1). \tag{2}$$

In addition, for the power series $f(x)$, $f(x, y)$ and $f(x, y, z)$, we employ the following notations:

$$\partial_x^m f(x), \quad \partial_{(x,y)}^{(m,n)} f(x, y) \quad \text{and} \quad \partial_{(x,y,z)}^{(m,n,l)} f(x, y, z)$$

to represent the coefficients of x^m in $f(x)$, $x^m y^n$ in $f(x, y)$ and $x^m y^n z^l$ in $f(x, y, z)$, respectively.

Let $\mathscr{T}$ be the set of all rooted plane trees. Obviously, $\mathscr{T}$ can be divided into two parts as

$$\mathscr{T} = \mathscr{T}_1 + \mathscr{T}_2 \tag{3}$$

such that $\mathscr{T}_1$ consists of a one-vertex map ϑ.

Lemma 1 (Liu [14]). Let $\mathscr{T}_{<2>} = \{T - e_r(T) \mid T \in \mathscr{T}_2\}$. Then,

$$\mathscr{T}_{<2>} = \mathscr{T} \times \mathscr{T}, \tag{4}$$

where $\times$ is the Cartesian product.

Proof. Because any $T \in \mathcal{T}_2$ the root-edge $e_r(T)$ of T is a cut edge, $T - e_r(T) = T_1 + T_2, T_1, T_2 \in \mathcal{T}$. So that, $T - e_r(T) \in \mathcal{T} \times \mathcal{T}$. Thus, $\mathcal{T}_{<2>} \subseteq \mathcal{T} \times \mathcal{T}$.

Conversely, for $T \in \mathcal{T} \times \mathcal{T}$ we have $T = T_1 + T_2, T_1, T_2 \in \mathcal{T}$. The map T' obtained by adding an edge a connecting the two root-vertices $v_r(T_1)$ and $v_r(T_2)$ is a member of $\mathcal{T}_2$. Hence, $T = T' - a \in \mathcal{T}_{<2>}$. This means that $\mathcal{T} \times \mathcal{T} \subseteq \mathcal{T}_{<2>}$. □

Lemma 2. The enumerating function $f_{\mathcal{T}} = f_{\mathcal{T}}(x, y, z)$ satisfies the following equation:

$$f_{\mathcal{T}} = \frac{1}{1 - xyz^2 h_{\mathcal{T}}}, \tag{5}$$

where $h_{\mathcal{T}} = h_{\mathcal{T}}(y, z) = f_{\mathcal{T}}(1, y, z)$.

Proof. By (3) and (4), we have

$$f_{\mathcal{T}} = 1 + xyz^2 f_{\mathcal{T}} h_{\mathcal{T}}$$

which is equivalent to Eq. (5). □

If $z = 1$, then we have:

Lemma 3. The enumerating function $g_{\mathcal{T}} = g_{\mathcal{T}}(x, y)$ satisfies the following equation:

$$g_{\mathcal{T}} = \frac{1}{1 - xyH_{\mathcal{T}}}, \tag{6}$$

where $H_{\mathcal{T}} = H_{\mathcal{T}}(y) = g_{\mathcal{T}}(1, y)$.

Let $x = 1$ in (5). Then we have

$$h_{\mathcal{T}} = \frac{1}{1 - yz^2 h_{\mathcal{T}}}.$$

Now, let $yz = \eta(1 - \eta)$ and $z = \frac{\lambda(1-\lambda\eta)}{1-\eta}$. Then we get

$$h_{\mathcal{T}} = \frac{1}{1 - \lambda\eta}.$$

Again let $x = \frac{\xi}{1+\xi\eta\lambda}$. Then one may find that Eq. (5) has a parametric solution as follows:

$$\begin{cases} x = \dfrac{\xi}{1 + \xi\eta\lambda}, \quad yz = \eta(1 - \eta), \quad z = \dfrac{\lambda(1 - \lambda\eta)}{1 - \eta}; \\ h_{\mathcal{T}} = \dfrac{1}{1 - \lambda\eta}, \quad f_{\mathcal{T}} = 1 + \xi\eta\lambda. \end{cases} \tag{7}$$

Further, by (7) one may find a parametric solution to Eq. (6) as follows:

$$\begin{cases} x = \dfrac{\xi}{1 + \xi\eta}, \quad y = \eta(1 - \eta); \\ H_{\mathcal{T}} = \dfrac{1}{1 - \eta}, \quad g_{\mathcal{T}} = 1 + \xi\eta. \end{cases} \tag{8}$$

2. The case of the root-edge on the cycle

In this section we will discuss the enumeration of unicyclic maps having the root-edge on the cycle.

Let $\mathscr{U}$ be the set of all unicyclic maps having the root-edge on the cycle. The set $\mathscr{U}$ can be partitioned into two parts as

$$\mathscr{U} = \mathscr{U}_1 + \mathscr{U}_2, \tag{9}$$

where $\mathscr{U}_1 = \{M \in \mathscr{U} \mid e_r(M) \text{ is a loop}\}$.

Lemma 4. Let $\mathscr{U}_{<1>} = \{M - e_r(M) \mid M \in \mathscr{U}_1\}$. Then,

$$\mathscr{U}_{<1>} = \mathscr{T}. \tag{10}$$

Proof. First, it is clear that any $M \in \mathscr{U}_{<1>}$ is a tree. Then, any tree can be seen as resulting from deleting the root-edge which is a loop in the map obtained by adding a loop which is chosen to be the root-edge at the root-vertex of the tree, and hence an element of the set on the left side of (10). □

Lemma 5. For $M \in \mathscr{U}$, let $M_{(i,m(M)-i+2)}$ be the map obtained by splitting the root-vertex into a root-edge $< o_1, o_2 >$ so that the valency of o_1 is i and the valency of o_2 is $m(M) - i + 2$ for $1 \leq i \leq m(M) + 1$. Then, we have

$$\mathscr{U}_2 = \sum_{M \in \mathscr{U}} \{M_{(i,m(M)-i+2)} \mid 2 \leq i \leq m(M)\}. \tag{11}$$

Proof. Because for any $M \in \mathscr{U}$, among all the maps resulting from splitting the root-vertex of M, only those for $2 \leq i \leq m(M)$ are members of $\mathscr{U}_2$. This proves the lemma. □

By Lemma 4, the enumerating function of $\mathscr{U}_1$ is

$$f_{\mathscr{U}_1} = x^2 yz f_{\mathscr{T}}. \tag{12}$$

Further, by Lemma 5, the enumerating function of $\mathscr{U}_2$ is

$$f_{\mathscr{U}_2} = \frac{xyz(xh_{\mathscr{U}} - f_{\mathscr{U}})}{1 - x}, \tag{13}$$

where $h_{\mathscr{U}} = h_{\mathscr{U}}(y, z) = f_{\mathscr{U}}(1, y, z)$.

Combining (9) with (12–13) yields

$$f_{\mathscr{U}} = x^2 yz f_{\mathscr{T}} + \frac{xyz(xh_{\mathscr{U}} - f_{\mathscr{U}})}{1 - x}. \tag{14}$$

After rearranging the items in the above equation, we have our first main result.

Theorem 1. The enumerating function $f_{\mathscr{U}} = f_{\mathscr{U}}(x, y, z)$ satisfies the following equation:

$$(1 - x + xyz) f_{\mathscr{U}} = (1 - x)x^2 yz f_{\mathscr{T}} + x^2 yz h_{\mathscr{U}}, \tag{15}$$

where $h_{\mathscr{U}} = h_{\mathscr{U}}(y,z) = f_{\mathscr{U}}(1,y,z)$.

If $z = 1$, then we have:

Corollary 1. The enumerating function $g_{\mathscr{U}} = g_{\mathscr{U}}(x,y)$ satisfies the following equation:

$$(1-x+xy)g_{\mathscr{U}} = (1-x)x^2yg_{\mathscr{T}} + x^2yH_{\mathscr{U}}, \tag{16}$$

where $H_{\mathscr{U}} = H_{\mathscr{U}}(y) = g_{\mathscr{U}}(1,y)$.

Let θ be the root of characteristic equation of (16). Then we get

$$\begin{cases} 1-\theta+\theta y = 0; \\ (1-\theta)\theta^2 y g_{\mathscr{T}}(\theta,y) + \theta^2 y H_{\mathscr{U}} = 0. \end{cases} \tag{17}$$

By (17) we have

$$y = \frac{\theta-1}{\theta}, \quad H_{\mathscr{U}} = (\theta-1)g_{\mathscr{T}}(\theta,y). \tag{18}$$

Further, let $\theta = \frac{1}{1-\eta+\eta^2}$. Then (18) becomes

$$y = \eta(1-\eta), \quad H_{\mathscr{U}} = \frac{\eta(1-\eta)}{1-\eta+\eta^2}g_{\mathscr{T}}(\frac{1}{1-\eta+\eta^2},y). \tag{19}$$

By (8) and (19), one may find the parametric expression of $H_{\mathscr{U}} = H_{\mathscr{U}}(y)$ as follows:

$$y = \eta(1-\eta), \quad H_{\mathscr{U}} = \frac{\eta}{1-\eta}. \tag{20}$$

Applying Lagrangian inversion [14] to (20), we obtain

$$\begin{aligned} H_{\mathscr{U}}(y) &= \sum_{n\geq 1} \frac{y^n}{n!}\frac{d^{n-1}}{d\eta^{n-1}}\left\{(1-\eta)^{-(n+2)}\right\}\Big|_{\eta=0} \\ &= \sum_{n\geq 1}\frac{(2n)!}{n!(n+1)!}y^n, \end{aligned} \tag{21}$$

which proves

Theorem 2. The number of unicyclic maps having the root-edge on the cycle with n edges is

$$\frac{(2n)!}{n!(n+1)!} \tag{22}$$

for $n \geq 1$. □

By substituting (8) and (20) into Eq. (16) and regrouping the terms, we may find the following parametric expression of the function $g_{\mathscr{U}} = g_{\mathscr{U}}(x,y)$:

$$x = \frac{\xi}{1+\xi\eta}, \quad y = \eta(1-\eta), \quad x^{-2}y^{-1}g_{\mathscr{U}} = \frac{1+\xi\eta}{1-\eta}, \tag{23}$$

from which we obtain

$$\Delta_{(\xi,\eta)} = \begin{vmatrix} \frac{1}{1+\xi\eta} & * \\ 0 & \frac{1-2\eta}{1-\eta} \end{vmatrix} = \frac{1-2\eta}{(1+\xi\eta)(1-\eta)}. \tag{24}$$

By employing Lagrangian theorem with two parameters [14], from (23) and (24) one may find that

$$\begin{aligned} g_{\mathscr{U}}(x,y) &= \sum_{m,n\geq 0} \partial^{(m,n)}_{(\xi,\eta)} \frac{(1+\xi\eta)^m(1-2\eta)}{(1-\eta)^{n+2}} x^{m+2}y^{n+1} \\ &= \sum_{m\geq 2}\sum_{n\geq 1} \partial^{(m-2,n-1)}_{(\xi,\eta)} \frac{(1+\xi\eta)^{m-2}(1-2\eta)}{(1-\eta)^{n+1}} x^m y^n \\ &= \sum_{n\geq 1}\sum_{m=2}^{n+1} \partial^{n-m+1}_{\eta} \frac{1-2\eta}{(1-\eta)^{n+1}} x^m y^n \\ &= \sum_{n\geq 1}\sum_{m=2}^{n+1} \frac{(2n-m)!(m-1)}{n!(n-m+1)!} x^m y^n, \end{aligned}$$

which proves

Theorem 3. The number of unicyclic maps having the root-edge on the cycle with n edges and the root-vertex valency being m is

$$\frac{(2n-m)!(m-1)}{n!(n-m+1)!} \tag{25}$$

for $2 \leq m \leq n+1$. □

Similarly, by (15) one may also find the parametric expression of $h_{\mathscr{U}} = h_{\mathscr{U}}(y,z)$ as follows:

$$yz = \eta(1-\eta), \quad z = \frac{\lambda(1-\lambda\eta)}{1-\eta}, \quad (yz)^{-1}h_{\mathscr{U}} = \frac{1}{1-\eta(1-\eta+\lambda)}. \tag{26}$$

According to (26), we get

$$\Delta_{(\eta,\lambda)} = \begin{vmatrix} \frac{1-2\eta}{1-\eta} & 0 \\ * & \frac{1-2\lambda\eta}{1-\lambda\eta} \end{vmatrix} = \frac{(1-2\eta)(1-2\lambda\eta)}{(1-\eta)(1-\lambda\eta)}. \tag{27}$$

Theorem 4. The enumerating function $h_{\mathscr{U}} = h_{\mathscr{U}}(y,z)$ has the following explicit expression:

$$h_{\mathscr{U}}(y,z) = \sum_{l\geq 1}\sum_{n=\lceil \frac{l+1}{2}\rceil}^{l} \frac{(l-1)!(2n-l)}{n!(l-n)!} y^n z^l. \tag{28}$$

Proof. (26) and (27) allow us to employ Lagrangian theorem with two variables [14] for finding

$$h_{\mathscr{U}}(y,z) = \sum_{n,l\geq 0} \partial_{(\eta,\lambda)}^{(n,l)} \frac{(1-2\eta)(1-2\lambda\eta)}{(1-\eta)^{n-l+1}(1-\lambda\eta)^{l+1}[1-\eta(1-\eta+\lambda)]} y^{n+1} z^{n+l+1}$$

$$= \sum_{l\geq 1}\sum_{n=1}^{l} \partial_{(\eta,\lambda)}^{(n-1,l-n)} \frac{(1-2\eta)(1-2\lambda\eta)}{(1-\eta)^{2n-l}(1-\lambda\eta)^{l-n+1}[1-\eta(1-\eta+\lambda)]} y^n z^l$$

$$= \sum_{l\geq 1}\sum_{n=1}^{l} \partial_{(\eta,\lambda)}^{(n-1,l-n)} \frac{(1-2\eta)(1-2\lambda\eta)}{(1-\eta)^{2n-l}(1-\lambda\eta)^{l-n+2}(1-\frac{1-\eta}{1-\lambda\eta}\eta)} y^n z^l$$

$$= \sum_{l\geq 1}\sum_{n=1}^{l}\sum_{k=0}^{n-1} \partial_{(\eta,\lambda)}^{(n-k-1,l-n)} \frac{(1-2\eta)(1-2\lambda\eta)}{(1-\eta)^{2n-l-k}(1-\lambda\eta)^{l-n+k+2}} y^n z^l$$

$$= \sum_{l\geq 1}\sum_{n=\lceil\frac{l+1}{2}\rceil 1}^{l}\sum_{k=0}^{2n-l-1} \frac{(2l-2n+k)!(k+1)}{(l-n)!(l-n+k+1)!} \partial_{\eta}^{2n-l-k-1} \frac{1-2\eta}{(1-\eta)^{2n-l-k}} y^n z^l$$

$$= \sum_{l\geq 1}\sum_{n=\lceil\frac{l+1}{2}\rceil}^{l} \frac{(l-1)!(2n-l)}{n!(l-n)!} y^n z^l.$$

This completes the proof of Theorem 4. □

In what follows we present a useful corollary of Theorem 4.

Corollary 2. The number of unicyclic maps having the root-edge on the cycle with q edges on the cycle and p edges not on it is

$$\frac{(2p+q-1)!q}{p!(p+q)!} \tag{29}$$

for $p \geq 0, q \geq 1$.

Proof. It follows immediately from (28) with $n = p+q$ and $l = 2p+q$. □

By substituting (7) and (26) into Eq. (15) and regrouping the terms, one may find the following parametric expression of the function $f_{\mathscr{U}} = f_{\mathscr{U}}(x,y,z)$:

$$x = \frac{\xi}{1+\xi\eta\lambda}, \quad yz = \eta(1-\eta), \quad z = \frac{\lambda(1-\lambda\eta)}{1-\eta},$$
$$x^{-2}(yz)^{-1} f_{\mathscr{U}} = \frac{(1-\eta\lambda)(1+\xi\eta\lambda)}{1-\eta(1-\eta+\lambda)}. \tag{30}$$

By (30) we have

$$\Delta_{(\xi,\eta,\lambda)} = \begin{vmatrix} \frac{1}{1+\xi\eta\lambda} & * & * \\ 0 & \frac{1-2\eta}{1-\eta} & 0 \\ 0 & * & \frac{1-2\lambda\eta}{1-\lambda\eta} \end{vmatrix} = \frac{(1-2\eta)(1-2\lambda\eta)}{(1+\xi\eta\lambda)(1-\eta)(1-\lambda\eta)}. \tag{31}$$

Theorem 5. The enumerating function $f_{\mathscr{U}} = f_{\mathscr{U}}(x, y, z)$ has the following explicit expression:

$$f_{\mathscr{U}}(x,y,z) = \sum_{l\geq 2} \sum_{n=\lceil \frac{l+1}{2} \rceil}^{l} \sum_{m=2}^{l-n+2} \frac{(l-m)!(2n+m-l-3)}{(n-1)!(l-n-m+2)!} x^m y^n z^l + x^2yz. \tag{32}$$

Proof. By using Lagrangian theorem with three parameters [14], from (30) and (31) one may find that

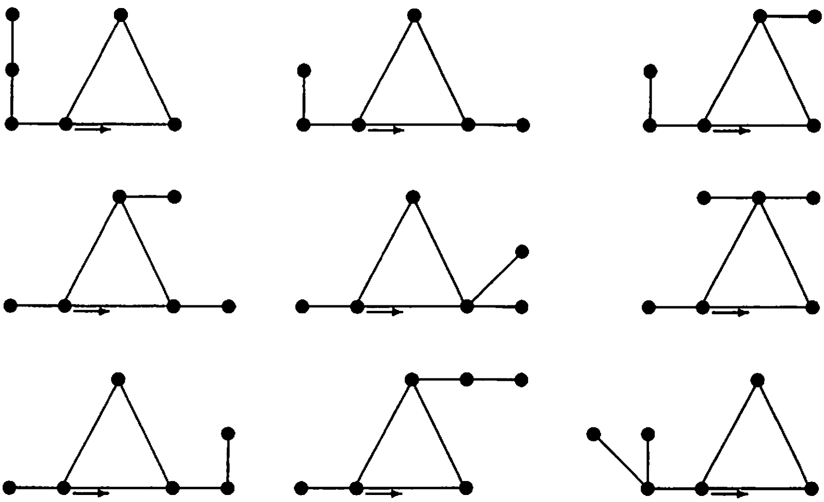
$$\begin{aligned}
f_{\mathscr{U}}(x,y,z) &= \sum_{m,n,l\geq 0} \partial_{(\xi,\eta,\lambda)}^{(m,n,l)} \frac{(1-2\eta)(1-2\lambda\eta)(1+\xi\eta\lambda)^m x^{m+2} y^{n+1} z^{n+l+1}}{[1-\eta(1-\eta+\lambda)](1-\eta)^{n-l+1}(1-\lambda\eta)^l} \\
&= \sum_{m\geq 2} \sum_{l\geq 1} \sum_{n=1}^{l} \partial_{(\xi,\eta,\lambda)}^{(m-2,n-1,l-n)} \frac{(1-2\eta)(1-2\lambda\eta)(1+\xi\eta\lambda)^{m-2} x^m y^n z^l}{[1-\eta(1-\eta+\lambda)](1-\eta)^{2n-l}(1-\lambda\eta)^{l-n}} \\
&= \sum_{l\geq 1} \sum_{n=1}^{l} \sum_{m=2}^{\min\{n+1,l-n+2\}} \partial_{(\eta,\lambda)}^{(n-m+1,l-n-m+2)} \frac{(1-2\eta)}{[1-\eta(1-\eta+\lambda)]} \\
&\times \frac{(1-2\lambda\eta)}{(1-\eta)^{2n-l}(1-\lambda\eta)^{l-n}} x^m y^n z^l \\
&= x^2yz + \sum_{l\geq 2} \sum_{n=2}^{l} \sum_{m=2}^{\min\{n+1,l-n+2\}} \partial_{(\eta,\lambda)}^{(n-m+1,l-n-m+2)} \frac{(1-2\eta)}{(1-\frac{1-\eta}{1-\lambda\eta}\eta)} \\
&\times \frac{(1-2\lambda\eta)}{(1-\eta)^{2n-l}(1-\lambda\eta)^{l-n+1}} x^m y^n z^l \\
&= x^2yz + \sum_{l\geq 2} \sum_{n=2}^{l} \sum_{m=2}^{\min\{n+1,l-n+2\}} \sum_{k=0}^{n-m+1} \partial_{(\eta,\lambda)}^{(n-m-k+1,l-n-m+2)} \frac{(1-2\eta)}{(1-\eta)^{2n-l-k}} \\
&\times \frac{(1-2\lambda\eta)}{(1-\lambda\eta)^{l-n+k+1}} x^m y^n z^l \\
&= x^2yz + \sum_{l\geq 2} \sum_{n=\lceil \frac{l+1}{2} \rceil}^{l} \sum_{m=2}^{l-n+2} \sum_{k=0}^{2n-l-1} \frac{(2l-2n-m+k+1)!(m+k-2)}{(l-n-m+2)!(l-n+k)!} \\
&\times \partial_\eta^{2n-l-k-1} \frac{1-2\eta}{(1-\eta)^{2n-l-k}} x^m y^n z^l \\
&= x^2yz + \sum_{l\geq 2} \sum_{n=\lceil \frac{l+1}{2} \rceil}^{l} \sum_{m=2}^{l-n+2} \frac{(l-m)!(2n+m-l-3)}{(n-1)!(l-n-m+2)!} x^m y^n z^l.
\end{aligned}$$

This completes the proof of Theorem 5. □

By (32) the following table of numbers can be obtained:

(m,n,l)	$(2,3,3)$	$(2,3,4)$	$(3,5,8)$	$(3,6,9)$	$\cdots$
$f_{\mathscr{U}}(m,n,l)$	1	1	5	9	$\cdots$

With the table above, there are 9 such maps with the root-vertex valency being 3, 6 edges and the root-face valency being 9 as shown in Fig. 1.



$(m,n,l)=(3,6,9)$

Fig. 1

Now, we give a useful corollary of Theorem 5.

Corollary 3. The number of unicyclic maps having the root-edge on the cycle with q edges on the cycle, p edges not on it and the root-vertex valency being m is

$$\frac{(2p+q-m)!(m+q-3)}{(p-m+2)!(p+q-1)!} \tag{33}$$

for $p\geq 0, q\geq 2, 2\leq m\leq p+2$; 1 for $p=0, q=1, m=2$.

Proof. It follows easily from (32) by putting $n=p+q$ and $l=2p+q$. □

3. The case of the root-edge not on the cycle

In this section we will concentrate on the enumeration of unicyclic maps having the root-edge not on the cycle.

Let $\widetilde{\mathscr{U}}$ be the set of all unicyclic maps having the root-edge not on the cycle. Then, we obtain the following result:

Lemma 6. For $\widetilde{\mathscr{U}}$, we have

$$\widetilde{\mathscr{U}} = \mathscr{T}_2 \odot \mathscr{U}, \tag{34}$$

where $\odot$ denotes the $1v$-production.

Proof. For any $M \in \widetilde{\mathscr{U}}$, it is clear that there exist two maps $T \in \mathscr{T}_2$ and $U \in \mathscr{U}$ such that $M = T \dotplus U$. Thus, M is a member of the set on the right side of (34).

Conversely, For $T \in \mathscr{T}_2$ and $U \in \mathscr{U}$, Since $M = T \dotplus U$ is a unicyclic map having the root-edge not on the cycle, we see that $M \in \widetilde{\mathscr{U}}$.

In consequence, the lemma holds. □

By Lemma 6, we have

$$\begin{aligned} f_{\widetilde{\mathscr{U}}}(x,y,z) =& \Big(\sum_{T\in\mathscr{T}_2} x^{m(T)}y^{n(T)}z^{l(T)}\Big)\Big(\sum_{U\in\mathscr{U}} x^{m(U)}y^{n(U)}z^{l(U)}\Big) \\ =& f_{\mathscr{T}_2}(x,y,z) f_{\mathscr{U}}(x,y,z). \end{aligned} \tag{35}$$

By (35) one may find that

$$\begin{aligned} g_{\widetilde{\mathscr{U}}}(x,y) =& g_{\mathscr{T}_2}(x,y) g_{\mathscr{U}}(x,y), \quad h_{\widetilde{\mathscr{U}}}(y,z) = h_{\mathscr{T}_2}(y,z) h_{\mathscr{U}}(y,z), \\ H_{\widetilde{\mathscr{U}}}(y) =& H_{\mathscr{T}_2}(y) H_{\mathscr{U}}(y). \end{aligned} \tag{36}$$

According to (3), (8), (20) and the last part of (36), one may find the following parametric expression of the function $H_{\widetilde{\mathscr{U}}} = H_{\widetilde{\mathscr{U}}}(y)$:

$$y = \eta(1-\eta), \quad H_{\widetilde{\mathscr{U}}} = \frac{\eta^2}{(1-\eta)^2}. \tag{37}$$

Applying Lagrangian inversion to (37), we get

$$\begin{aligned} H_{\widetilde{\mathscr{U}}}(y) =& \sum_{n\geq 0} \frac{2y^n}{n!} \frac{d^{n-1}}{d\eta^{n-1}} \left\{ \frac{\eta}{(1-\eta)^{n+3}} \right\} \Big|_{\eta=0} \\ =& \sum_{n\geq 2} \frac{4\cdot(2n-1)!}{(n-2)!(n+2)!} y^n, \end{aligned} \tag{38}$$

which proves

Theorem 6. The number of unicyclic maps having the root-edge not on the cycle with n edges is

$$\frac{4\cdot(2n-1)!}{(n-2)!(n+2)!}, \tag{39}$$

for $n \geq 2$. □

By (3),(8),(23) and the first part of (36), we may find the parametric expression of $g_{\widetilde{\mathscr{U}}} = g_{\widetilde{\mathscr{U}}}(x, y)$ as follows:

$$x = \frac{\xi}{1+\xi\eta}, \quad y = \eta(1-\eta), \quad x^{-2}y^{-1}g_{\widetilde{\mathscr{U}}} = \frac{\xi\eta(1+\xi\eta)}{1-\eta}, \tag{40}$$

from which we get

$$\Delta_{(\xi,\eta)} = \begin{vmatrix} \frac{1}{1+\xi\eta} & * \\ 0 & \frac{1-2\eta}{1-\eta} \end{vmatrix} = \frac{1-2\eta}{(1+\xi\eta)(1-\eta)}. \tag{41}$$

By using Lagrangian theorem with two variables, from (40) and (41) one may find that

$$\begin{aligned} g_{\widetilde{\mathscr{U}}}(x,y) &= \sum_{m,n\geq 0} \partial^{(m,n)}_{(\xi,\eta)} \frac{\xi\eta(1-2\eta)(1+\xi\eta)^m}{(1-\eta)^{n+2}} x^{m+2}y^{n+1} \\ &= \sum_{m\geq 3}\sum_{n\geq 2} \partial^{(m-3,n-2)}_{(\xi,\eta)} \frac{(1-2\eta)(1+\xi\eta)^{m-2}}{(1-\eta)^{n+1}} x^m y^n \\ &= \sum_{n\geq 2}\sum_{m=3}^{n+1} (m-2)\partial_\eta^{n-m+1} \frac{1-2\eta}{(1-\eta)^{n+1}} x^m y^n \\ &= \sum_{n\geq 2}\sum_{m=3}^{n+1} \frac{(2n-m)!(m-1)(m-2)}{n!(n-m+1)!} x^m y^n, \end{aligned}$$

which proves

Theorem 7. The number of unicyclic maps having the root-edge not on the cycle with n edges and the root-vertex valency being m is

$$\frac{(2n-m)!(m-1)(m-2)}{n!(n-m+1)!} \tag{42}$$

for $3 \leq m \leq n+1$. □

Combining (3),(7),(26) with the second part of (36), one may find the parametric expression of $h_{\widetilde{\mathscr{U}}} = h_{\widetilde{\mathscr{U}}}(y, z)$ as follows:

$$\begin{aligned} &yz = \eta(1-\eta), \quad z = \frac{\lambda(1-\lambda\eta)}{1-\eta}, \\ &(yz)^{-1}h_{\widetilde{\mathscr{U}}} = \frac{\lambda\eta}{(1-\lambda\eta)[1-\eta(1-\eta+\lambda)]}. \end{aligned} \tag{43}$$

According to (43), we obtain

$$\Delta_{(\eta,\lambda)} = \begin{vmatrix} \frac{1-2\eta}{1-\eta} & 0 \\ * & \frac{1-2\lambda\eta}{1-\lambda\eta} \end{vmatrix} = \frac{(1-2\eta)(1-2\lambda\eta)}{(1-\eta)(1-\lambda\eta)}. \tag{44}$$

Theorem 8. The enumerating function $h_{\widetilde{\mathscr{U}}} = h_{\widetilde{\mathscr{U}}}(y,z)$ has the following explicit expression:

$$h_{\widetilde{\mathscr{U}}}(y,z) = \sum_{l\geq 3} \sum_{n=\lceil \frac{l+1}{2} \rceil}^{l-1} \frac{(l-1)!(2n-l+2)}{(n+1)!(l-n-1)!} y^n z^l. \tag{45}$$

Proof. By employing Lagrangian theorem with two parameters, from (43) and (44) one may find that

$$\begin{aligned}
h_{\widetilde{\mathscr{U}}}(y,z) &= \sum_{n,l\geq 1} \partial_{(\eta,\lambda)}^{(n-1,l-1)} \frac{(1-2\eta)(1-2\lambda\eta) y^{n+1} z^{n+l+1}}{[1-\eta(1-\eta+\lambda)](1-\eta)^{n-l+1}(1-\lambda\eta)^{l+2}} \\
&= \sum_{l\geq 3} \sum_{n=2}^{l-1} \partial_{(\eta,\lambda)}^{(n-2,l-n-1)} \frac{(1-2\eta)(1-2\lambda\eta)}{[1-\eta(1-\eta+\lambda)](1-\eta)^{2n-l}(1-\lambda\eta)^{l-n+2}} y^n z^l \\
&= \sum_{l\geq 3} \sum_{n=2}^{l-1} \partial_{(\eta,\lambda)}^{(n-2,l-n-1)} \frac{(1-2\eta)(1-2\lambda\eta)}{(1-\frac{1-\eta}{1-\lambda\eta}\eta)(1-\eta)^{2n-l}(1-\lambda\eta)^{l-n+3}} y^n z^l \\
&= \sum_{l\geq 3} \sum_{n=2}^{l-1} \sum_{k=0}^{n-2} \partial_{(\eta,\lambda)}^{(n-k-2,l-n-1)} \frac{(1-2\eta)(1-2\lambda\eta)}{(1-\eta)^{2n-l-k}(1-\lambda\eta)^{l-n+k+3}} y^n z^l \\
&= \sum_{l\geq 3} \sum_{n=\lceil \frac{l+1}{2} \rceil}^{l-1} \sum_{k=0}^{2n-l-1} \frac{(2l-2n+k)!(k+3)}{(l-n-1)!(l-n+k+2)!} \\
&\quad \times \partial_{\eta}^{2n-l-k-1} \frac{1-2\eta}{(1-\eta)^{2n-l-k}} y^n z^l \\
&= \sum_{l\geq 3} \sum_{n=\lceil \frac{l+1}{2} \rceil}^{l-1} \frac{(l-1)!(2n-l+2)}{(n+1)!(l-n-1)!} y^n z^l.
\end{aligned}$$

This completes the proof of Theorem 8. □

If $l = 2p+q$ and $n = p+q$, then we have:

Corollary 4. The number of unicyclic maps having the root-edge not on the cycle with q edges on the cycle and p edges not on it is

$$\frac{(2p+q-1)!(q+2)}{(p-1)!(p+q+1)!} \tag{46}$$

for $p \geq 1, q \geq 1$.

By (3),(7),(30) and (35), we may also find the following parametric expression of the function $f_{\widetilde{\mathscr{U}}} = f_{\widetilde{\mathscr{U}}}(x, y, z)$:

$$x = \frac{\xi}{1+\xi\eta\lambda}, \quad yz = \eta(1-\eta), \quad z = \frac{\lambda(1-\lambda\eta)}{1-\eta},$$
$$(xyz)^{-1} f_{\widetilde{\mathscr{U}}} = \frac{\xi^2\eta\lambda(1-\eta\lambda)}{1-\eta(1-\eta+\lambda)}, \tag{47}$$

from which we have

$$\Delta_{(\xi,\eta,\lambda)} = \begin{vmatrix} \frac{1}{1+\xi\eta\lambda} & * & * \\ 0 & \frac{1-2\eta}{1-\eta} & 0 \\ 0 & * & \frac{1-2\lambda\eta}{1-\lambda\eta} \end{vmatrix} = \frac{(1-2\eta)(1-2\lambda\eta)}{(1+\xi\eta\lambda)(1-\eta)(1-\lambda\eta)}. \tag{48}$$

Theorem 9. The enumerating function $f_{\widetilde{\mathscr{U}}} = f_{\widetilde{\mathscr{U}}}(x, y, z)$ has the following explicit expression:

$$f_{\widetilde{\mathscr{U}}}(x,y,z) = \sum_{l\geq 3} \sum_{n=\lceil \frac{l+1}{2} \rceil}^{l-1} \sum_{m=3}^{l-n+2} \frac{(l-m)!(2n+m-l-3)(m-2)}{(n-1)!(l-n-m+2)!} x^m y^n z^l. \tag{49}$$

Proof. By using Lagrangian theorem with three variables, from (47) and (48) one may find that

$$f_{\widetilde{\mathscr{U}}}(x,y,z) = \sum_{m\geq 2} \sum_{n,l\geq 1} \partial_{(\xi,\eta,\lambda)}^{(m-2,n-1,l-1)} \frac{(1-2\eta)(1-2\lambda\eta)}{[1-\eta(1-\eta+\lambda)](1-\eta)^{n-l+1}}$$
$$\times \frac{(1+\xi\eta\lambda)^{m-1}}{(1-\lambda\eta)^l} x^{m+1} y^{n+1} z^{n+l+1}$$
$$= \sum_{m\geq 3} \sum_{l\geq 3} \sum_{n=2}^{l-1} \partial_{(\xi,\eta,\lambda)}^{(m-3,n-2,l-n-1)} \frac{(1-2\eta)(1-2\lambda\eta)}{[1-\eta(1-\eta+\lambda)](1-\eta)^{2n-l}}$$
$$\times \frac{(1+\xi\eta\lambda)^{m-2}}{(1-\lambda\eta)^{l-n}} x^m y^n z^l$$
$$= \sum_{l\geq 3} \sum_{n=2}^{l-1} \sum_{m=3}^{\min\{n+1,l-n+2\}} (m-2)$$
$$\times \partial_{(\eta,\lambda)}^{(n-m+1,l-n-m+2)} \frac{(1-2\eta)(1-2\lambda\eta) x^m y^n z^l}{[1-\eta(1-\eta+\lambda)](1-\eta)^{2n-l}(1-\lambda\eta)^{l-n}}$$
$$= \sum_{l\geq 3} \sum_{n=2}^{l-1} \sum_{m=3}^{\min\{n+1,l-n+2\}} (m-2) \partial_{(\eta,\lambda)}^{(n-m+1,l-n-m+2)} \frac{(1-2\eta)}{(1-\frac{1-\eta}{1-\lambda\eta}\eta)}$$
$$\times \frac{(1-2\lambda\eta)}{(1-\eta)^{2n-l}(1-\lambda\eta)^{l-n+1}} x^m y^n z^l$$

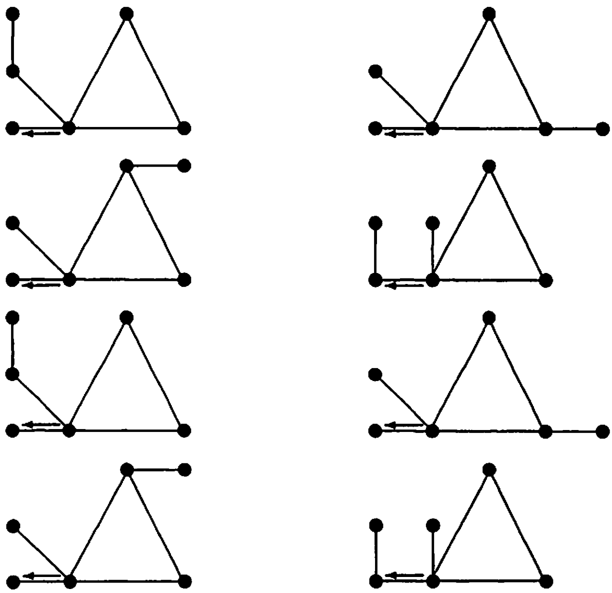
$$=\sum_{l\geq 3}\sum_{n=2}^{l-1}\sum_{m=3}^{\min\{n+1,l-n+2\}}\sum_{k=0}^{n-m+1}(m-2)\partial_{(\eta,\lambda)}^{(n-m-k+1,l-n-m+2)}\frac{(1-2\eta)}{(1-\eta)^{2n-l-k}}$$
$$\times\frac{(1-2\lambda\eta)}{(1-\lambda\eta)^{l-n+k+1}}x^m y^n z^l$$
$$=\sum_{l\geq 3}\sum_{n=\lceil\frac{l+1}{2}\rceil}^{l-1}\sum_{m=3}^{l-n+2}\sum_{k=0}^{2n-l-1}\frac{(2l-2n-m+k+1)!(m+k-2)(m-2)}{(l-n-m+2)!(l-n+k)!}$$
$$\times\,\partial_{\eta}^{2n-l-k-1}\frac{1-2\eta}{(1-\eta)^{2n-l-k}}x^m y^n z^l$$
$$=\sum_{l\geq 3}\sum_{n=\lceil\frac{l+1}{2}\rceil}^{l-1}\sum_{m=3}^{l-n+2}\frac{(l-m)!(2n+m-l-3)(m-2)}{(n-1)!(l-n-m+2)!}x^m y^n z^l.$$

This completes the proof of Theorem 9. □

By (49) the following table of numbers can be obtained:

(m,n,l)	$(3,4,5)$	$(3,4,6)$	$(4,6,9)$	$(5,8,12)$	$\cdots$
$f_{\widetilde{\mathscr{U}}}(m,n,l)$	1	2	8	18	$\cdots$

From the above table, there are 8 such maps with 6 edges, root-vertex valency 4 and root-face valency 9, as shown in Fig. 2.



$(m,n,l)=(4,6,9)$

Fig. 2

Finally, let $l = 2p + q$ and $n = p + q$. Then we get the following result:
Corollary 5. The number of unicyclic maps having the root-edge not on the cycle with q edges on the cycle, p edges not on it and the root-vertex valency being m is

$$\frac{(2p+q-m)!(m+q-3)(m-2)}{(p-m+2)!(p+q-1)!} \tag{50}$$

for $p \geq 1, q \geq 1, 3 \leq m \leq p+2$.

Acknowledgements

The authors are grateful to the referees for their careful reading of this paper. Their constructive suggestions enable us to take some major revisions which make the paper more readable and concrete.

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