CASTELNUOVO-MUMFORD REGULARITY OF GRAPH IDEALS

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ABSTRACT. Let G be a simple graph with edge ideal $I(G)$. In this article we study the number of pairwise 3-disjoint edges of cycles and complement of triangle-free graphs. Using that, we determine the Castelnuovo-Mumford regularity of $R/I(G)$ for the above classes of graphs according to the number of pairwise 3-disjoint edges.

1. INTRODUCTION

Let G be a finite simple undirected graph over the vertex set $V = \{x_1, ..., x_n\}$ and let $R = k[x_1, ..., x_n]$ denote the polynomial ring in n variables over the field k. The *edge ideal* of G is the ideal $I(G)$ of R generated by those square-free quadratic monomials x_ix_j such that $\{x_i, x_j\}$ is an edge of G. Edge ideals were first introduced by Villarreal [22]. Fröberg in [6] showed that Stanley-Reisner rings with 2-linear resolutions can be characterized graph-theoretically. Then the edge ideals were studied by many authors in order to examine their algebraic properties in terms of the combinatorial data of graphs, and vice versa. Among the many papers that have studied the properties of edge ideals, we mention [1, 2, 6, 7, 8, 9, 10, 12, 16, 20, 22, 23, 24].

Two edges $\{x, y\}$ and $\{u, v\}$ of a graph G are called *3-disjoint* (or *disconnected*) if the induced subgraph of G on $\{x, y, u, v\}$ consists of exactly two disjoint edges. A set Γ of edges of G is called *pairwise 3-disjoint* if any two edges of Γ are 3-disjoint. The maximum cardinality of all pairwise 3-disjoint sets of edges in G is denoted by $a(G)$.

The *Castelnuovo-Mumford regularity* of a graded R-module M denoted by $\operatorname{reg}(M)$ is defined as follows:

$$\operatorname{reg}(M) = \max\{j - i \mid \beta_{i,j}(M) \neq 0\}.$$

Katzman in [14] showed that for any graph G one has $\operatorname{reg}(R/I(G)) \geq a(G)$ ([14, Lemma 2.2]). There have been several attempts to determine the classes of graphs for which the above inequality is an equality. Zheng [24] proved the equality for trees. Francisco, Hà and Van Tuyl [5] proved equality holds for Cohen-Macaulay bipartite graphs. Van Tuyl [21] generalized this to the family of sequentially Cohen-Macaulay bipartite graphs. Kummini [16] proved equality holds also for unmixed bipartite graphs. In addition, the authors in [17] generalized Kummini's result to the class of very well-covered graphs.

Our first topic in this paper is about the regularity of cycles. In Section 3 we study the regularity of cycles in terms of the number of pairwise 3-disjoint edges. Let C_n denotes the cycle of length n, i.e. $V(C_n) = \{x_1, ..., x_n\}$ and $E(C_n) =$

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$\{\{x_1,x_2\},...,\{x_{n-1},x_n\},\{x_n,x_1\}\}$. We show that $\operatorname{reg}(R/I(C_n)) = a(C_n) = [\frac{n}{3}]$ provided $n = 3k$ or $n = 3k+1$ for some $k \in \mathbb{N}$, otherwise $\operatorname{reg}(R/I(C_n)) = a(C_n)+1 = [\frac{n}{3}]+1$. Our result is not surprising for C_3 and C_4 because the first is chordal and the other is unmixed bipartite but this is an affirmative answer to the Van Tuyl's question [21, Question 3.5] on the regularity of a bipartite graph that is neither unmixed nor sequentially Cohen-Macaulay, namely for cycles of even length. In addition, we show that if $\mathcal{P}_n$ denotes the line graph on $n+1$ vertices, i.e. $V(\mathcal{P}_n) = \{x_1,...,x_{n+1}\}$ and $E(\mathcal{P}_n) = \{\{x_1,x_2\},...,\{x_{n-1},x_n\},\{x_n,x_{n+1}\}\}$, then $a(\mathcal{P}_l) = [\frac{l+2}{3}]$. Combining this together with [24, Theorem 2.18] implies that $\operatorname{reg}(R/I(\mathcal{P}_l)) = [\frac{l+2}{3}]$.

The second topic in this paper is about the regularity of the complement of a triangle-free graph. The *claw graph* is the complete bipartite graph $\mathcal{K}_{1,3}$. A graph G is called *claw-free* if it has no 4 vertices on which the induced graph is a claw. It is easy to see that the complement of a triangle-free graph is claw-free (recall that the complement of a graph G is the graph $\overline{G}$ over the same vertex set of G whose edges are non-edges of G). Note that A graph G is claw-free if the complement of $G_{N(v)}$ (the induced subgraph of G on $N(v)$) is triangle-free for all $v \in V(G)$, where $N(v)$ is the set of vertices which are adjacent to v. Independence complexes of claw-free graphs has been the subject of [3] where Engström gave good bounds on the connectivity of these complexes. Nevo in [18, Theorem 5.1] showed that if G is claw-free without induced C_4 in its complement, then $\operatorname{reg}(R/I(G)) \leq 2$. This does not clarify the regularity of the complement of a triangle-free graph with induced 4-cycle such as the graph obtained by attaching an edge to C_4.

In Section 4 we study the regularity of the complement of a triangle-free graph. Note that the intersection of the set of triangle-free graphs and the set of chordal graphs is precisely the set of forests. The set of triangle-free graphs can be partitioned into two disjoint sets; Those that contain C_4 as induced subgraph and the other comprises of all graphs without induced 4-cycle. According to these two families, we determine the regularity of the complement of triangle-free graphs in terms of their combinatorial data. More precisely, we show that (see Theorem 4.4) for a triangle-free graph the following hold:

(i) If G is forest, then $\operatorname{reg}(R/I(\overline{G})) = a(\overline{G}) = 1$.
(ii) If G has induced 4-cycle, then $\operatorname{reg}(R/I(\overline{G})) = a(\overline{G}) = 2$.
(iii) If G is not forest and has no induced 4-cycle, then $\operatorname{reg}(R/I(\overline{G})) = 2 = a(\overline{G})+1$.

2. Terminology and Preliminaries

For the convenience of the reader we include in this short section the standard terminology and the basic facts which we will use throughout the paper.

Let M be an arbitrary graded R-module, and let

$$0 \to \bigoplus_j R(-j)^{\beta_{t,j}(M)} \to \bigoplus_j R(-j)^{\beta_{t-1,j}(M)} \to \cdots \to \bigoplus_j R(-j)^{\beta_{0,j}(M)} \to M \to 0$$

be a minimal graded free resolution of M over R, where $R(-j)$ is a graded free R-module whose nth graded component is given by R_{n-j}. The number $\beta_{i,j}(M)$ is called the ijth graded Betti number of M and equals the number of generators of degree j in the ith syzygy module. The *Castelnuovo-Mumford regularity* of M

denoted by reg (M) is defined as follows:

$$\operatorname{reg}(M) = \max\{j - i \mid \beta_{i,j}(M) \neq 0\}.$$

Recall that the *projective dimension* of an R-module M, denoted by pd (M), is the length of the minimal free resolution of M, i.e.

$$\operatorname{pd}(M) = \max\{i \mid \beta_{i,j}(M) \neq 0 \text{ for some } j\}.$$

There is a strong connection between the topology of the simplicial complex Δ and the structure of the free resolution of $k[\Delta]$. Let $\beta_{i,j}(\Delta)$ denotes the graded Betti numbers of the Stanley-Reisner ring $k[\Delta]$. One of the most well-known results is the Hochster's formula ([11, Theorem 5.1]), which is a principal tool to study Betti numbers of square-free monomial ideals.

Theorem 2.1. *For $i > 0$ the graded Betti numbers $\beta_{i,j}$ of a simplicial complex Δ are given by*

$$\beta_{i,j}(\Delta) = \sum_{\substack{W \subseteq V(\Delta) \\ |W|=j}} \dim_k \widetilde{H}_{j-i-1}(\Delta_W; k).$$

Let G be a simple graph and $S \subseteq V(G)$. Also let G_S denotes the induced subgraph of G over S and suppose $\Delta(G)$ be the simplicial complex whose faces correspond to the complete subgraphs of G. By specializing the Hochster's formula to edge ideals we obtain the following proposition ([19, Proposition 1.2]).

Proposition 2.2. *Let G be a simple graph with edge ideal $I(G)$. Then*

$$\beta_{i,j}(R/I(G)) = \sum_{\substack{S \subseteq V(G) \\ |S|=j}} \dim_k \widetilde{H}_{j-i-1}(\Delta(\overline{G}_S); k)$$

for all $i, j \geq 0$.

One of the useful invariant of a graph that relates to the regularity is the number of pairwise 3-disjoint edges of G as defined below.

Definition 2.3. Two edges $\{x, y\}$ and $\{u, v\}$ of a graph G are called 3-disjoint if the induced subgraph of G on $\{x, y, u, v\}$ consists of exactly two disjoint edges. A set Γ of edges of G is called Pairwise 3-disjoint set of edges if any two edges of Γ are 3-disjoint. The maximum cardinality of all Pairwise 3-disjoint sets of edges in G is denoted by $a(G)$.

Katzman provided the following result on the regularity of $R/I(G)$.

Lemma 2.4. *([14, Lemma 2.2]) For any graph G,* $\operatorname{reg}(R/I(G)) \geq a(G)$.

3. CYCLE AND LINE GRAPHS

In this section we determine the Castelnuovo-Mumford regularity of $R/I(C_n)$, where C_n is the cycle graph on n vertices, and give its relation with $a(C_n)$. This can be done in both viewpoints of topology and homological algebra. Kozlov in [15, Propositions 4.6 and 5.2] calculated the homotopy type of lines and cycles. Although the regularity of these families can be determined using Kozlov results, we use homological methods to do this. Indeed, we apply Jacques's results on Betti numbers of cycles which can be found in his thesis [12, Section 7]. It should be mentioned that the thesis has never been published, but some its materials such as

Betti numbers of forests and characteristic independence of some Betti numbers of edge ideals are appeared in [13, 14], too.

We begin this section by computing the number $a(C_n)$ in terms of the length of C_n.

Lemma 3.1. *Let C_n denotes the cycle of length n. Then $a(C_n) = [\frac{n}{3}]$.*

Proof. Let $V(C_n) = \{x_1, ..., x_n\}$ with $n = 3k + t$ and $0 \leq t \leq 2$. One can see that

$$\{\{x_1, x_2\}, \{x_4, x_5\}, \{x_7, x_8\}, ...\{x_{3k-2}, x_{3k-1}\}\}$$

is a set of pairwise 3-disjoint edges of C_n with k edges. It follows that $a(C_n) \geq k$. Now assume $a(C_n) > k$ and suppose A is a set of pairwise 3-disjoint edges of C_n with $k + 1$ elements. Also let $e_i = \{x_i, x_{i+1}\}$ be in A and Suppose $E_i = \{\{x_{i-1}, x_i\}, \{x_i, x_{i+1}\}, \{x_{i+1}, x_{i+2}\}\}$. So $E_i \cap A = \{x_i, x_{i+1}\}$. It is easy to see that $E_i \cap E_j = \emptyset$ for any two distinct elements $e_i, e_j \in A$. Therefore

$$|E(C_n)| \geq |\bigcup_{e_i \in A} E_i| = \sum_{e_i \in A} |E_i| = 3|A| = 3k + 3$$

which is impossible. Hence $a(C_n) = k = [\frac{n}{3}]$. □

Using Lemma 3.1 one can determine the regularity of C_n in terms of the number of pairwise 3-disjoint edges.

Theorem 3.2. *Let C_n denotes the cycle of length n. Then*

(i) *If $n \equiv 0 \mod 3$, then $\operatorname{reg}(R/I(C_n)) = a(C_n) = [\frac{n}{3}]$*
(ii) *If $n \equiv 1 \mod 3$, then $\operatorname{reg}(R/I(C_n)) = a(C_n) = [\frac{n}{3}]$*
(iii) *If $n \equiv 2 \mod 3$, then $\operatorname{reg}(R/I(C_n)) = a(C_n) + 1 = [\frac{n}{3}] + 1$.*

Proof. First observe that any $\mathbb{N}$-graded Betti number can be written in the form $\beta_{l,d} = \beta_{i+j,2i+j}$, where $i = d - l$ and $j = 2l - d$. Now we have

$$\begin{aligned}\operatorname{reg}(R/I(C_n)) &= \max\{d - l \mid \beta_{l,d}(R/I(C_n)) \neq 0\} \\ &= \max\{2i + j - (i + j) \mid \beta_{i+j,2i+j}(R/I(C_n)) \neq 0\} \\ &= \max\{\{i \mid \beta_{i+j,2i+j}(R/I(C_n)) \neq 0,\ 2i + j < n\},\ n - \operatorname{pd}(R/I(C_n))\}.\end{aligned}$$

(i) Let $n = 3m$. It follows from [12, Corollary 7.6.30] that $\operatorname{pd}(R/I(C_n)) = 2m$ and hence $n - \operatorname{pd}(R/I(C_n)) = m$. Therefore

$$\operatorname{reg}(R/I(C_n)) = \max\{\{i \mid \beta_{i+j,2i+j}(R/I(C_n)) \neq 0,\ 2i + j < n\},\ m\}.$$

Now we show that for all i with $\beta_{i+j,2i+j}(R/I(C_n)) \neq 0$ and $2i + j < n$ one has $i \leq m$. Suppose the contrary that $i > m$. Using [12, Proposition 7.4.23] we get that $\beta_{i,2i}(R/I(C_n)) \neq 0$. By [14, Lemma 2.2], C_n has an induced subgraph which consists of i disjoint edges. This means that C_n contains at least $3m + 2$ vertices which is impossible. Therefore $i \leq m$ and so

$$\operatorname{reg}(R/I(C_n)) = m = [\frac{n}{3}] = a(C_n).$$

(ii) Let $n = 3m+1$. It follows from [12, Corollary 7.6.30] that $\operatorname{pd}(R/I(C_n)) = 2m+1$ and hence $n - \operatorname{pd}(R/I(C_n)) = m$. The proof of this case is similar to the case (i).
(iii) Let $n = 3m + 2$. Then $\operatorname{pd}(R/I(C_n)) = 2m + 1$ and hence $n - \operatorname{pd}(R/I(C_n)) = m + 1$. We have

$$\operatorname{reg}(R/I(C_n)) = \max\{\{i \mid \beta_{i+j,2i+j}(R/I(C_n)) \neq 0,\ 2i + j < n\},\ m + 1\}.$$

Now we show that for all i with $\beta_{i+j,2i+j}(R/I(C_n)) \neq 0$ and $2i+j < n$ one has $i \leq m+1$. Suppose the contrary that $i > m+1$. A similar argument as in the proof of (i) implies that C_n contains at least $3m+3$ vertices which is impossible. Therefore $i \leq m+1$ and so

$$\operatorname{reg}(R/I(C_n)) = m+1 = [\frac{n}{3}] + 1 = a(C_n) + 1.$$

□

It is known that for any tree T one has $\operatorname{reg}(R/I(T)) = a(T)$, cf. [24, Theorem 2.18]. In the final result of this section we determine the regularity of a line graph in terms of its length.

Theorem 3.3. $\operatorname{reg}(R/I(\mathcal{P}_l)) = [\frac{l+2}{3}]$.

Proof. It is enough to show that $a(\mathcal{P}_l) = [\frac{l+2}{3}]$. Let $V(\mathcal{P}_l) = \{x_1, \dots, x_{l+1}\}$. We prove the assertion in two cases. First suppose $l = 3k$. Since

$$\{\{x_1, x_2\}, \{x_4, x_5\}, \{x_7, x_8\}, \dots, \{x_{3k-2}, x_{3k-1}\}\}$$

is a set of pairwise 3-disjoint edges of $\mathcal{P}_l$ of cardinality k, we get that $a(\mathcal{P}_l) \geq k$. Now suppose $a(\mathcal{P}_l) > k$ and assume that A is a set of pairwise 3-disjoint edges of $\mathcal{P}_l$ of cardinality $k+1$. Let e_i denotes the edge $\{x_i, x_{i+1}\}$ of $\mathcal{P}_l$. A similar argument as in the proof of Lemma 3.1 shows that

- If $|A \cap \{e_1, e_l\}| = 0$, then $|E(\mathcal{P}_l)| \geq 3k+3$,
- If $|A \cap \{e_1, e_l\}| = 1$, then $|E(\mathcal{P}_l)| \geq 3k+2$,
- If $|A \cap \{e_1, e_l\}| = 2$, then $|E(\mathcal{P}_l)| \geq 3k+1$,

which are contradictions. Hence

$$a(\mathcal{P}_l) = a(\mathcal{P}_{3k}) = k = [\frac{3k+2}{3}] = [\frac{l+2}{3}].$$

Now let $l = 3k+t$ with $1 \leq t \leq 2$. Since

$$\{\{x_1, x_2\}, \{x_4, x_5\}, \dots, \{x_{3k-2}, x_{3k-1}\}, \{x_{3k+1}, x_{3k+2}\}\}$$

is a set of pairwise 3-disjoint edges of $\mathcal{P}_l$ with $k+1$ elements, we get that $a(\mathcal{P}_l) \geq k+1$. A similar argument as above shows that if $a(\mathcal{P}_l) > k+1$, then $|E(\mathcal{P}_l)| \geq 3k+4$, a contradiction. Therefore

$$a(\mathcal{P}_l) = a(\mathcal{P}_{3k+t}) = k+1 = [\frac{3k+t+2}{3}] = [\frac{l+2}{3}].$$

□

4. COMPLEMENT OF TRIANGLE-FREE GRAPHS

Let Δ_G denotes the simplicial complex whose faces are independent subsets of the graph G, where a subset is independent if no two vertices in it are adjacent. This simplicial complex called *independence complex* and reflects many nice properties of G. Independence complexes of graphs has been the subject of study in both combinatorics and topology (see [2, 3, 4, 9, 15, 18]). We begin this section with the next general lemma that provides an upper bound for the regularity of the edge ring.

Lemma 4.1. *For any graph* G, $\operatorname{reg}(R/I(G)) \leq \dim\Delta_G + 1$.

Proof. It is clear that $\dim\Delta(\overline{G}) \geq \dim\Delta(\overline{G}_S)$ for all $S \subseteq V(G)$. Note that if $j-i-1 > \dim\Delta(\overline{G})$, then $\tilde{H}_{j-i-1}(\Delta(\overline{G}_S);k) = 0$ for all $S \subseteq V(G)$. It follows from Proposition 2.2 that $\beta_{i,j}(R/I(G)) = 0$ for all $j-i > \dim\Delta(\overline{G})+1$. Now $\Delta(\overline{G}) = \Delta_G$ yields that

$$\operatorname{reg}(R/I(G)) = \max\{j-i \mid \beta_{i,j}(R/I(G)) \neq 0\} \leq \dim\Delta(\overline{G})+1 = \dim\Delta_G + 1.$$

□

In the case where G is triangle-free one has $\dim\Delta_{\overline{G}} = 1$. Hence we have the next result.

Corollary 4.2. *Let G be a triangle-free graph (not necessary connected). Then* $\operatorname{reg}(R/I(\overline{G})) \leq 2$.

Existence of C_4 as induced subgraph in G may affect on $a(\overline{G})$ as we can see in the next remark.

Remark 4.3. Let G be a triangle-free graph. It follows from Lemma 2.4 and Corollary 4.2 that $a(\overline{G}) \leq 2$ and the inequality may be strict (see C_6, the cycle of length 6), but one can determine condition under which the equality holds. In fact, it is an easy observation that $a(\overline{G}) = 2$ if and only if there exist $u, v \in V(G)$ such that $|N_G(u) \cap N_G(v)| \geq 2$ ($N_G(u)$ denotes the neighbor set of u in G). The latter condition holds if and only if G contains C_4 as induced subgraph. Therefore $a(\overline{G}) = 1$ for all cycles G of even length except C_4 and also for any forest G. On the other hand, it is known that a graph has 2-linear resolution if and only if its complement is chordal (see [6, Theorem 1]). Therefore $\operatorname{reg}(R/I(\overline{G})) = 1$ for any chordal graph G. In addition, Lemma 4.1 implies that $\operatorname{reg}(R/I(\overline{G})) = 2$, where G is the complement of a triangle-free graph which is not forest.

It is easy to see that the complement of any forest is claw-free without induced C_4 in its complement. The "complement of a triangle-free graph" without C_4 as induced subgraph in its complement, i.e., the complement of a "triangle-free graph without induced C_4" (which contains the complement of forests) is claw-free without induced C_4 in its complement. The regularity of this family has recently been demonstrated in [18, Theorem 5.1]. Also the complement of a triangle-free graph may contain C_4 as the induced subgraph in its complement (this is the complement of a "triangle-free graph with induced C_4"). This family is a subset of the set of claw-free graphs with C_4 as induced subgraph in their complements. Figure 1 clarifies these relations and provides some examples in each part.

The next Theorem determines the relation between the regularity of $R/I(\overline{G})$ and $a(\overline{G})$ according to the combinatorial data from G, where G is triangle-free. As we explained in Remark 4.3, by [6, Theorem 1] and Lemma 4.1, $\operatorname{reg}(R/I(\overline{G})) = 1$ for any chordal graph G, and $\operatorname{reg}(R/I(\overline{G})) = 2$ where G is the complement of a triangle-free graph which is not forest. Therefore, in view of Remark 4.3, the following result is straightforward.

Theorem 4.4. *Let G be a triangle-free graph. Then*

(i) *If G is forest, then* $\operatorname{reg}(R/I(\overline{G})) = a(\overline{G}) = 1$.
(ii) *If G has induced 4-cycle, then* $\operatorname{reg}(R/I(\overline{G})) = a(\overline{G}) = 2$.
(iii) *If G is not forest and has no induced 4-cycle, then* $\operatorname{reg}(R/I(\overline{G})) = 2$ *but* $a(\overline{G}) = 1$.

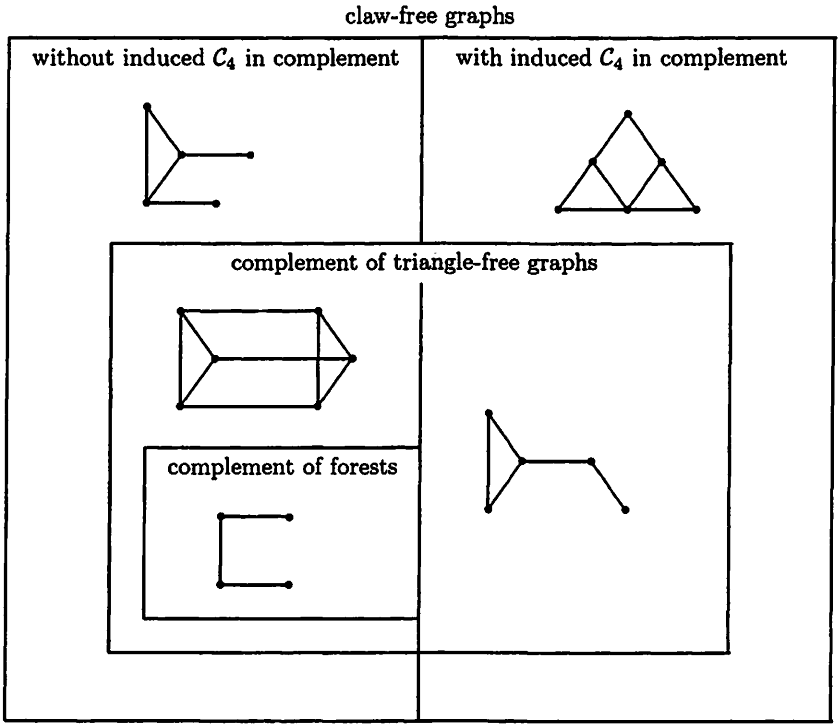
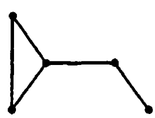


Figure 1

Example 4.5. Let G be the following graph:



G is the complement of a triangle-free graph that contains C_4 as induced subgraph. It follows from Theorem 4.4.(ii) that $\operatorname{reg}(R/I(G)) = a(G) = 2$.

As explained before, the only part of Figure 1 that we do not know about the regularity of $R/I(G))$ is the right side of the Figure that includes the graphs which are not the complement of triangle-free graphs. If an element of the mentioned part is forest, the answer is obvious. So it is interesting to know the answer to the following question:

Question 4.6. *Suppose G is a claw-free graph with induced C_4 in its complement that is neither the complement of a triangle-free graph nor forest. What is* $\operatorname{reg}(R/I(G))$*?*

As we mentioned before, there have been several attempts to determine the classes of graphs for which the regularity of $R/I(G)$ is equal $a(G)$. Zheng [24] proved the equality for trees. Francisco, Hà and Van Tuyl [5] proved equality holds for Cohen-Macaulay bipartite graphs. Van Tuyl [21] generalized this to the family of sequentially Cohen-Macaulay bipartite graphs. Kummini [16] proved equality holds also for unmixed bipartite graphs. In addition, the authors in [17] generalized Kummini's result to the class of very well-covered graphs. So it is natural to ask the following question:

Question 4.7. *Determine the class of bipartite graphs for which the regularity of $R/I(G)$ is equal $a(G)$?*

In this way, by Theorem 3.2, we know that C_n belongs to this class for the following two cases:
(i) $n = 6k$ for any positive integer k.
(ii) $n = 3k + 1$ for any odd integer k.
In addition, for any even positive integer k, if $n = 3k + 2$, then C_n is bipartite but $\operatorname{reg}(R/I(C_n)) = a(C_n) + 1$.

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