The number of maximal independent sets in trees and forests

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Abstract

For a simple undirected graph $G = (V, E)$, a subset I of $V(G)$ is said to be an *independent set* of G if any two vertices in I are not adjacent in G. A *maximal independent set* is an independent set that is not a proper subset of any other independent set. In this paper we survey on the largest to fourth largest numbers of maximal independent sets among all trees and forests. In addition, we further look into the problem of determining the fifth largest number of maximal independent sets among all trees and forests. Extremal graphs achieving these values are also given.

1 Introduction and preliminary

In general we use the standard terminology and notations of graph theory, see [1]. Let $G = (V, E)$ be a simple undirected graph, the *neighborhood* and *closed neighborhood* of v are $N_G(v) = \{u \in V(G) \,|\, uv \in E(G)\}$ and $N_G[v] = \{v\} \cup N_G(v)$, respectively. Two distinct vertices u and v are called *duplicated vertices* if $N_G(u) = N_G(v)$. For a set $A \subseteq V(G)$, the *deletion* of A from G is the graph $G - A$ obtained from G by removing all vertices in A and their incident edges. A subset $I \subseteq V(G)$ is *independent* if there is no edge of G between any two vertices of I. A *maximal independent set* is an independent set that is not a proper subset of any other independent set. The set of all maximal independent sets of G is denoted by $\mathrm{MI}(G)$ and its cardinality by $mi(G)$. For a vertex $x \in V(G)$, let $\mathrm{MI}_{+x}(G) = \{I \in \mathrm{MI}(G) : x \in I\}$ and $\mathrm{MI}_{-x}(G) = \{I \in \mathrm{MI}(G) : x \notin I\}$. The cardinalities of $\mathrm{MI}_{+x}(G)$ and $\mathrm{MI}_{-x}(G)$ are denoted by $mi_{+x}(G)$ and $mi_{-x}(G)$, respectively.

Erdös and Moser raised the problem of determining the maximum value of $mi(G)$ among all graphs of order n and extremal graphs achieving this value.

Shortly after, Moon and Moser [12] solved the problem. It was then studied for various families of graphs, including trees, forests, (connected) graphs with at most one cycle, (connected) triangle-free graphs, (k-)connected graphs, bipartite graphs; for a survey see [7]. Recently, Jin and Li [4] investigated the second largest number of $mi(G)$ among all graphs of order n; Hua and Hou [2] determined the third largest number of $mi(G)$ among all graphs of order n. More recently, Jou and Lin [9, 11] investigated the second largest number and the fourth largest number of $mi(G)$ among all trees and forests of order n, respecitvely. Jin and Yan [5] solved the third largest number of $mi(G)$ among all trees of order n.

In the second section of this paper, we survey on the largest to fourth largest numbers of maximal independent sets among all trees and forests. In the third section, we further look into the problem of determining the fifth largest number of maximal independent sets among all trees and forests. Extremal graphs achieving these values are also given.

2 Survey on the numbers of maximal independent sets among all trees and forests

For integers $i \geq 1$, $j_2 \geq j_1 \geq 0$, a *baton* $B(i; j_1, j_2)$ is the graph obtained from the basic path P of i vertices by attaching j_1 paths of length two to one endpoint of P and j_2 paths of length two to the other endpoint of P. See Figure 1. When $i = 1$, we write $B(1; j)$ for $B(1; 0, j)$. From simple calculation, we have $B(2; 0, j_2) = 2^{j_2} + 1$, $B(3; 0, j_2) = 2^{j_2+1}$ and $B(4; 0, j_2) = 2^{j_2+1} + 1$.

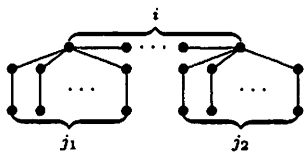


Figure 1: The baton $B(i; j_1, j_2)$

Lemma 2.1. ([3, 6]) *If G is a graph in which x is a leaf adjacent to the vertex y, then $mi(G) = mi(G - N_G[x]) + mi(G - N_G[y])$.*

Lemma 2.2. *For integers $i \geq 5$, $j_2 \geq j_1 \geq 0$, $mi(B(i; j_1, j_2)) = mi(B(i - 3; 0, j_2)) + mi(B(i - 2; 0, j_2)) + (2^{j_1} - 1) \cdot mi(B(i - 1; 0, j_2))$.*

Proof. For the case of $j_1 = 0$, by Lemma 2.1, it is easy to see that $mi(B(i; 0, j_2)) = mi(B(i - 2; 0, j_2)) + mi(B(i - 3; 0, j_2))$. We assume that $j_1 \geq 1$, by repeatedlty applying Lemma 2.1 to the leaves of $B(i; j_1, j_2)$, we have

$$\begin{aligned}
mi(B(i; j_1, j_2)) &= mi(B(i; j_1 - 1, j_2)) + 2^{j_1 - 1} \cdot mi(B(i-1; 0, j_2)) \\
&= mi(B(i; j_1 - 2, j_2)) \\
&\quad + (2^{j_1 - 2} + 2^{j_1 - 1}) \cdot mi(B(i-1; 0, j_2)) \\
&= mi(B(i; j_1 - 3, j_2)) \\
&\quad + (2^{j_1 - 3} + 2^{j_1 - 2} + 2^{j_1 - 1}) \cdot mi(B(i-1; 0, j_2)) \\
&= \qquad \vdots \\
&= mi(B(i; 0, j_2)) \\
&\quad + (2^0 + 2^1 + \cdots + 2^{j_1 - 2} + 2^{j_1 - 1}) \cdot mi(B(i-1; 0, j_2)) \\
&= mi(B(i-3; 0, j_2)) \\
&\quad + mi(B(i-2; 0, j_2)) + (2^{j_1} - 1) \cdot mi(B(i-1; 0, j_2)).
\end{aligned}$$

This completes the proof. □

The results of the largest numbers of maximal independent sets among all trees and forests are described in Theorems 2.3 and 2.4, respectively. Throughout this paper, r denotes $\sqrt{2}$.

Theorem 2.3. ([6, 8]) *If T is a tree with $n \geq 1$ vertices, then $mi(T) \leq t_1(n)$, where*

$$t_1(n) = \begin{cases} r^{n-2} + 1, & \text{if } n \text{ is even}; \\ r^{n-1}, & \text{if } n \text{ is odd}. \end{cases}$$

Furthermore, $mi(T) = t_1(n)$ if and only if $T = T_1(n)$, where

$$T_1(n) = \begin{cases} B(2; j_1, j_2), j_1 + j_2 = \frac{n-2}{2} & \\ \qquad \text{or } B(4; j_3, j_4), j_3 + j_4 = \frac{n-4}{2}, & \text{if } n \text{ is even}; \\ B(1; \frac{n-1}{2}), & \text{if } n \text{ is odd}. \end{cases}$$

For any two graphs G and H, let $G \cup H$ denote the disjoint union of G and H, and for any integer $n \geq 2$, let nG stand for the disjoint union of n copies of G. Denote by P_n a *path* with n vertices.

Theorem 2.4. ([6, 8]) *If F is a forest with $n \geq 1$ vertices, then $mi(F) \leq f_1(n)$, where*

$$f_1(n) = \begin{cases} r^n, & \text{if } n \text{ is even}; \\ r^{n-1}, & \text{if } n \text{ is odd}. \end{cases}$$

Furthermore, $mi(F) = f_1(n)$ if and only if $F = F_1(n)$, where

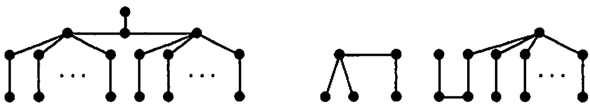
$$F_1(n) = \begin{cases} \frac{n}{2} P_2, & \text{if } n \text{ is even}; \\ B(1; \frac{n-1-2s}{2}) \cup sP_2 \text{ for some } s \text{ with } 0 \leq s \leq \frac{n-1}{2}, & \text{if } n \text{ is odd}. \end{cases}$$

The results of the second largest numbers of maximal independent sets among all trees and forests are described in Theorems 2.5 and 2.6, respectively.

Theorem 2.5. ([9]) *If T is a tree with $n \geq 4$ vertices having $T \neq T_1(n)$, then $mi(T) \leq t_2(n)$, where*

$$t_2(n) = \begin{cases} r^{n-2}, & \text{if } n \text{ is even;} \\ 3, & \text{if } n = 5; \\ 3r^{n-5} + 1, & \text{if } n \text{ is odd.} \end{cases}$$

Furthermore, $mi(T) = t_2(n)$ if and only if $T = T_2'(8), T_2''(8), P_{10}$, or $T_2(n)$, where $T_2(n)$ and $T_2'(8)$, $T_2''(8)$ are shown in Figures 2 and 3, respectively.



$T_{2e}(n)$, $n \geq 4$ is even $\quad$ $T_2(5)$ $\quad$ $T_{2o}(n)$, $n \geq 7$ is odd

Figure 2: The trees $T_2(n)$

$T_2'(8)$ $\quad$ $T_2''(8)$

Figure 3: The trees $T_2'(8)$ and $T_2''(8)$

Theorem 2.6. ([9]) *If F is a forest with $n \geq 4$ vertices having $F \neq F_1(n)$, then $mi(F) \leq f_2(n)$, where*

$$f_2(n) = \begin{cases} 3r^{n-4}, & \text{if } n \text{ is even;} \\ 3, & \text{if } n = 5; \\ 7r^{n-7}, & \text{if } n \text{ is odd.} \end{cases}$$

Furthermore, $mi(F) = f_2(n)$ if and only if $F = F_2(n)$, where

$$F_2(n) = \begin{cases} P_4 \cup \frac{n-4}{2} P_2, & \text{if } n \geq 4 \text{ is even;} \\ T_2(5) \text{ or } P_4 \cup P_1, & \text{if } n = 5; \\ P_7 \cup \frac{n-7}{2} P_2, & \text{if } n \geq 7 \text{ is odd.} \end{cases}$$

The results of the third largest numbers of maximal independent sets among all trees and forests are described in Theorems 2.7 and 2.8, respectively.

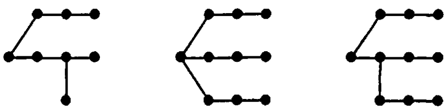
Theorem 2.7. ([5]) *If T is a tree with $n \geq 7$ vertices having $T \neq T_i(n)$, $i = 1, 2$, then $mi(T) \leq t_3(n)$, where*

$$t_3(n) = \begin{cases} 3r^{n-5}, & \text{if } n \geq 7 \text{ is odd;} \\ 7, & \text{if } n = 8; \\ 15, & \text{if } n = 10; \\ 7r^{n-8} + 2, & \text{if } n \geq 12 \text{ is even.} \end{cases}$$

Furthermore, $mi(T) = t_3(n)$ if and only if $T = T_3(8), T_3'(10), T_3''(10)$, or $T_3(n)$, where $T_3(n)$ and $T_3(8), T_3'(10), T_3''(10)$ are shown in Figures 4 and 5, respectively.

$T_{3e}(n)$, $n \geq 12$ is even $\quad$ $T_{3o}(n)$, $n \geq 7$ is odd

Figure 4: The trees $T_3(n)$



$T_3(8)$ $\quad$ $T_3'(10)$ $\quad$ $T_3''(10)$

Figure 5: The trees $T_3(8), T_3'(10)$ and $T_3''(10)$

Theorem 2.8. ([10]) *If F is a forest with $n \geq 8$ vertices having $F \neq F_i(n)$, $i = 1, 2$, then $mi(F) \leq f_3(n)$, where*

$$f_3(n) = \begin{cases} 5r^{n-6}, & \text{if } n \geq 8 \text{ is even;} \\ 13r^{n-9}, & \text{if } n \geq 9 \text{ is odd.} \end{cases}$$

Furthermore, $mi(F) = f_3(n)$ if and only if $F = F_3(n)$, where

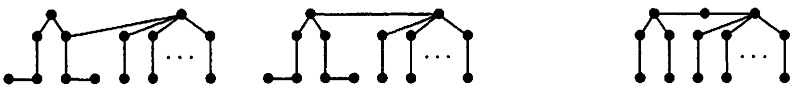
$$F_3(n) = \begin{cases} T_1(6) \cup \frac{n-6}{2} P_2, & \text{if } n \geq 8 \text{ is even;} \\ T_2(9) \cup \frac{n-9}{2} P_2, & \text{if } n \geq 9 \text{ is odd.} \end{cases}$$

The results of the fourth largest numbers of maximal independent sets among all trees and forests are described in Theorems 2.9 and 2.10, respectively.

Theorem 2.9. ([11]) *If T is a tree with $n \geq 11$ vertices having $T \neq T_i(n)$, $i = 1, 2, 3$, then $mi(T) \leq t_4(n)$, where*

$$t_4(n) = \begin{cases} 5r^{n-7} + 3, & \text{if } n \geq 11 \text{ is odd;} \\ 7r^{n-8} + 1, & \text{if } n \geq 12 \text{ is even.} \end{cases}$$

Furthermore, $mi(T) = t_4(n)$ *if and only if* $T = T_4(n)$*, where* $T_4(n)$ *is shown in Figure* 6.



$T_{4e}(n)$, $n \geq 12$ is even $\qquad$ $T_{4o}(n)$, $n \geq 11$ is odd

Figure 6: The trees $T_4(n)$

Theorem 2.10. ([11]) *If* F *is a forest with* $n \geq 10$ *vertices having* $F \neq F_i(n)$, $i = 1, 2, 3$, *then* $mi(F) \leq f_4(n)$, *where*

$$f_4(n) = \begin{cases} 9r^{n-8}, & \text{if } n \geq 10 \text{ is even;} \\ 25r^{n-11}, & \text{if } n \geq 11 \text{ is odd.} \end{cases}$$

Furthermore, $mi(F) = f_4(n)$ *if and only if* $F = F_4(n)$*, where*

$$F_4(n) = \begin{cases} 2P_4 \cup \frac{n-8}{2}P_2 \text{ or } T_1(8) \cup \frac{n-8}{2}P_2, & \text{if } n \geq 10 \text{ is even;} \\ T_2(11) \cup \frac{n-11}{2}P_2, & \text{if } n \geq 11 \text{ is odd.} \end{cases}$$

3 The fifth largest number of maximal independent sets among all trees and forests

Lemma 3.1. ([6]) *If* G *is the union of two disjoint graphs* G_1 *and* G_2*, then* $mi(G) = mi(G_1)mi(G_2)$.

Lemma 3.2. ([6]) *If a graph* G *has duplicated leaves* x_1 *and* x_2*, then* $mi(G) = mi(G - x_2)$.

A vertex v of a graph G is a *support vertex* if it is adjacent to a leaf in G. For an even integer $n \geq 14$, $T_{5e}(n)$ is the tree obtained from $B(1; \frac{n-8}{2})$ by adding a P_7 and a new edge joining the only vertex in the basic path of $B(1; \frac{n-8}{2})$ and the support vertex of P_7. For an odd integer $n \geq 15$, $T_{5o}(n)$ is the tree obtained from $B(1; \frac{n-7}{2})$ by adding a P_6 and a new edge joining the only vertex in the basic path of $B(1; \frac{n-7}{2})$ and the leaf of P_6, see Figure 7.

Lemma 3.3. *If* T *is a tree of even order* $n \geq 14$ *having* $T \neq T_i(n)$, $i = 1, 2, 3, 4$, *then* $mi(T) \leq 7r^{n-8}$. *Furthermore, the equality holds if and only if* $T = T_{5e}(n)$ *or* $B(6; 2, 2)$.

$T_{5e}(n)$, $n \geq 14$ is even $\qquad$ $T_{5o}(n)$, $n \geq 15$ is odd

Figure 7: The trees $T_5(n)$

Proof. It is straightforward to check that $mi(T_{5e}(n)) = 7r^{n-8}$. In addition, by Lemma 2.2, $B(6;2,2) = 56$. Let T be a tree of even order $n \geq 14$ having $T \neq T_i(n), i = 1,2,3,4$ such that $mi(T)$ is as large as possible. By Theorem 2.9, $7r^{n-8} \leq mi(T) \leq t_4(n) - 1 = (7r^{n-8} + 1) - 1 = 7r^{n-8}$, hence $mi(T) = 7r^{n-8}$. Let x be a leaf lying on a longest path P of T, say $P = x, y, z, w, \ldots$ and H the component of $T - N_T[y]$ containing some vertices of P. Since P is a longest path of T, it follows that every component of $T - (N_T[y] \cup V(H))$ is P_1 or P_2. Thus we have that $T - N_T[y] = aP_1 \cup (b-1)P_2 \cup H$ for integers a, b. Suppose that T has duplicated leaves x_1 and x_2, then $T' = T - x_2$ is a tree of odd order $n-1$. Since $T \neq T_2(n)$, this implies that $T' \neq T_1(n-1)$. By Theorem 2.5, we have that $7r^{n-8} = mi(T) = mi(T') \leq t_2(n-1) = 3r^{n-6} + 1 < 7r^{n-8}$, which is a contradiction. Thus T has no duplicated leaves, it follows that $a = 0$ or 1.

Suppose that $a = 1$. Since $T \neq T_1(n)$, this implies that H is a tree of even order $n - 2b - 2 \geq 4$. Thus $2 \leq 2b \leq n-6$. By Theorems 2.3 and 2.4, $mi(T) = mi_{+z}(T) + mi_{-z}(T) \leq mi(H-w) + r^{2b} \cdot mi(H) \leq r^{n-2b-4} + r^{2b} \cdot (r^{n-2b-4} + 1) = r^{n-2b-4} + r^{n-4} + r^{2b}$. Note that $r^{n-2b-4} + r^{2b}$ has a maximum value at $2b = 2$ or $2b = n-6$. Hence, we have that $7r^{n-8} = mi(T) \leq r^{n-4} + r^{n-6} + 2 = 6r^{n-8} + 2 < 7r^{n-8}$. This is a contradiction, hence $a = 0$. It follows that H is a tree of odd order $n - 2b - 1$.

Since $T \neq T_1(n)$ and $T \neq T_2(n)$, these imply that $H \neq T_1(n-2b-1)$ and $H - w \neq F_1(n-2b-2)$. Thus $mi(H) \leq t_2(n-2b-1)$ and $mi(H-w) \leq f_2(n-2b-2)$. We consider the following two cases.

Case 1. $b = 1$.

Since $a = 0$ and $T \neq T_1(n)$, these imply that $T - N_T[x] \neq T_1(n-2)$. Suppose that $T - N_T[x] = T_2(n-2)$, then $mi(H) = mi(T - N_T[y]) = mi(T) - mi(T - N_T[x]) = 7r^{n-8} - r^{n-4} = 3r^{n-8} = t_3(n-3)$. By Theorem 2.7, then $T - N_T[y] = T_3(n-3)$. This means that $T = T_{5e}(n)$. On the other hand, if $T - N_T[x] \neq T_2(n-2)$, then $mi(T - N_T[x]) \leq 7r^{n-10} + 2$. By Lemma 2.1 and Theorem 2.5, we have that $7r^{n-8} \leq mi(T) = mi(T - N_T[x]) + mi(T - N_T[y]) \leq 7r^{n-10} + 2 + 3r^{n-8} + 1 = 13r^{n-10} + 3 < 7r^{n-8}$, which is a contradiction.

Case 2. $b \geq 2$.

Note that $|V(H)| = n - 2b - 1 \geq 5$ is odd. Suppose that $|V(H)| = 5$. Since $T \neq T_1(n)$ and $T \neq T_2(n)$, these imply that $H \neq P_5$. On the other hand, since

T has no duplicated leaves, this implies that $H = T_2(5)$ and z is adjacent to one of the duplicated leaves of $T_2(5)$. See Figure 8.

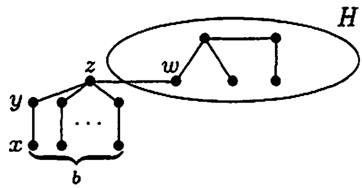


Figure 8: The tree T

By simple calculation, we have that $mi(T) = 3r^{n-6} + 2 < 7r^{n-8}$, which is a contradiction. Hence $|V(H)| = n - 2b - 1 \geq 7$ and $4 \leq 2b \leq n - 8$.

Since $b \geq 2$ and $T \neq T_i(n)$ for $i = 1,2$, these imply that $T - N_T[x]$ is a tree having $T - N_T[x] \neq T_i(n-2)$ for $i = 1,2$. By Theorem 2.7, we have that $7r^{n-8} = mi(T) = mi(T - N_T[x]) + mi(T - N_T[y]) \leq t_3(n-2) + r^{2b-2} \cdot mi(H) \leq 7r^{n-10} + 2 + r^{2b-2} \cdot (3r^{n-2b-6} + 1) = 7r^{n-10} + 2 + 3r^{n-8} + r^{2b-2} = 13r^{n-10} + 2 + r^{2b-2}$. Thus we obtain that $r^{2b-2} + 2 \geq r^{n-10}$. For the case of $2b = n - 10$, then $b = 2$ and $n = 14$. It follows that $T - N_T[x] = T_3(12)$ and $T - N_T[y] = P_2 \cup T_2(9)$. In conclusion, $T = B(6;2,2)$. For the other case of $2b \geq n - 8$, then $2b = n - 8$, $|V(H)| = 7$ and $mi(H) \leq t_2(7) = 7$. Since $2b = n - 8 \geq 6$, this implies that $T - N_T[x] \neq T_i(n-2)$ for $i = 3,4$ and $mi(T - N_T[x]) \leq t_4(n-2) - 1 = 7r^{n-10}$. Thus $r^{n-10} \cdot mi(H) = mi(T - N_T[y]) = mi(T) - mi(T - N_T[x]) \geq 7r^{n-8} - 7r^{n-10} = 7r^{n-10}$, then $mi(H) \geq 7$. Hence $mi(H) = 7$, by Theorem 2.5, $H = T_2(7) = P_7$. Since $T \neq T_3(n)$ and $T \neq T_4(n)$, this implies that $T = T_{5e}(n)$. □

Lemma 3.4. *If T is a tree of odd order $n \geq 15$ having $T \neq T_i(n)$, $i = 1,2,3,4$, then $mi(T) \leq 5r^{n-7} + 2$. Furthermore, the equality holds if and only if $T = T_{5o}(n)$.*

Proof. It is straightforward to check that $mi(T_{5o}(n)) = 5r^{n-7} + 2$. Let T be a tree of odd order $n \geq 15$ having $T \neq T_i(n)$, $i = 1,2,3,4$, such that $mi(T)$ is as large as possible. Then $mi(T) \geq 5r^{n-7} + 2$. Let x be a leaf lying on a longest path P of T, say $P = x, y, z, \ldots$ and H the component of $T - N_T[y]$ containing some vertices of P. Since P is a longest path of T, it follows that every component of $T - (N_T[y] \cup V(H))$ is P_1 or P_2. Thus we have that $T - N_T[y] = aP_1 \cup (b-1)P_2 \cup H$ for integers a, b. Suppose that T has duplicated leaves x_1 and x_2, by Lemma 3.2 and Theorem 2.3, $5r^{n-7} + 2 \leq mi(T) = mi(T - x_2) \leq t_1(n-1) = r^{n-3} + 1 < 5r^{n-7} + 2$, which is a contradiction. Thus T has no duplicated leaves, it follows that $a = 0$ or 1.

Suppose that $a = 1$. Then H is a tree of odd order $n - 2 - 2b \geq 3$. By Theorem 2.3, $mi(H) \leq r^{n-3-2b}$. It follows that $mi(T - N_T[y]) \leq r^{2b-2} \cdot$

$r^{n-3-2b} = r^{n-5}$. Since $T \neq T_3(n)$, this implies that $T - N_T[x] \neq T_1(n-2)$. By Lemma 2.1 and Theorem 2.5, we have that $3r^{n-7} + 1 = t_2(n-2) \geq mi(T - N_T[x]) = mi(T) - mi(T - N_T[y]) \geq (5r^{n-7} + 2) - r^{n-5} = 3r^{n-7} + 2$, which is a contradiction. Hence we obtain that $a = 0$ and H is a tree of even order $n - 2b - 1$. We consider the following two cases.

Case 1. $b = 1$.

Since $T \neq T_1(n)$, this implies that $T - N_T[x] \neq T_1(n-2)$. By Theorems 2.3 and 2.5, we have that $5r^{n-7} + 2 \leq mi(T) = mi(T - N_T[x]) + mi(T - N_T[y]) \leq t_2(n-2) + t_1(n-3) = (3r^{n-7} + 1) + (r^{n-5} + 1) = 5r^{n-7} + 2$. Furthermore, the equalities holding imply that $T - N_T[x] = T_2(n-2)$ and $T - N_T[y] = T_1(n-3)$. In conclusion, $T = T_{5_o}(n)$.

Case 2. $b \geq 2$.

Since $T \neq T_i(n)$ for $i = 1, 2, 3, 4$ and $b \geq 2$, these imply that $|V(H)| = n - 2b - 1 \geq 6$ and $T - N_T[x] \neq T_i(n-2)$ for $i = 1, 2, 3$. By Theorem 2.9, we have that $5r^{n-7} + 2 = mi(T) = mi(T - N_T[x]) + mi(T - N_T[y]) \leq t_4(n-2) + r^{2b-2} \cdot mi(H) \leq (5r^{n-9} + 3) + r^{2b-2} \cdot (r^{n-2b-3} + 1) = 5r^{n-9} + 3 + r^{n-5} + r^{2b-2} = 9r^{n-9} + r^{2b-2} + 3$. Thus we obtain that $r^{2b-2} + 1 \geq r^{n-9}$. It follows that $2b \geq n - 7$. Hence $2b = n - 7 \geq 8$, $|V(H)| = 6$ and $mi(H) \leq t_1(6) = 5$. Since $b \geq 4$, this implies that $T - N_T[x] \neq T_4(n-2)$ and $mi(T - N_T[x]) \leq 5r^{n-9} + 2$. Hence we obtain that $r^{n-9} \cdot mi(H) = mi(T - N_T[y]) = mi(T) - mi(T - N_T[x]) \geq (5r^{n-7} + 2) - (5r^{n-9} + 2) = 5r^{n-9}$, this implies that $mi(H) \geq 5$. Hence $mi(H) = 5$, by Theorem 2.3, $H = T_1(6)$. In conclusion, $T = T_{5_o}(n)$. □

Lemma 3.5. *If F is a forest of even order $n \geq 12$ having $F \neq F_i(n)$, $i = 1, 2, 3, 4$, then $mi(F) \leq 17r^{n-10}$. Furthermore, the equality holds if and only if $F = T_1(10) \cup \frac{n-10}{2} P_2$.*

Proof. It is straightforward to check that $mi(T_1(10) \cup \frac{n-10}{2} P_2) = 17r^{n-10}$. Let F be a forest of even order $n \geq 12$ having $F \neq F_i(n)$, $i = 1, 2, 3, 4$, such that $mi(F)$ is as large as possible. Then $mi(F) \geq 17r^{n-10}$. Suppose that there exist two odd components H_1 and H_2 of F, where $|H_i| = n_i$ for $i = 1, 2$. By Lemma 3.1, Theorems 2.3 and 2.4, we have that $17r^{n-10} \leq mi(F) = mi(H_1) \cdot mi(H_2) \cdot mi(F - (V(H_1) \cup V(H_2))) \leq r^{n_1-1} \cdot r^{n_2-1} \cdot r^{n-n_1-n_2} = r^{n-2} < 17r^{n-10}$, which is a contradiction. Hence F has no component of odd order. Since $F \neq F_1(n)$, there exists an component H of even order $m \geq 4$.

Suppose that $F - V(H) \neq F_1(n-m)$, it follows that $mi(F - V(H)) \leq f_2(n-m) = 3r^{n-m-4}$. Since $F \neq F_i(n)$ for $i = 1, 2, 3, 4$, by Lemma 3.1, Theorems 2.3, 2.6 and 2.8, we have that

$$\begin{aligned}
17r^{n-10} &\le mi(F) = mi(H)\cdot mi(F-V(H)) \\
&\le \begin{cases} \max\{(t_1(4)-1)\cdot f_2(n-4), t_1(4)\cdot f_3(n-4)\} & \text{if } m=4, \\ t_1(m)\cdot f_2(n-m) & \text{if } m\ge 6, \end{cases} \\
&= \begin{cases} 3\cdot 5r^{n-10} & \text{if } m=4, \\ 3r^{n-6}+3r^{n-m-4} & \text{if } m\ge 6, \end{cases} \\
&< 17r^{n-10},
\end{aligned}$$

which is a contradiction.

Now we assume that $F - V(H) = F_1(n-m)$. Since $F \neq F_i(n)$ for $i = 1, 2, 3, 4$, by Lemma 3.1, Theorems 2.3 and 2.4, we have that

$$\begin{aligned}
17r^{n-10} &\le mi(F) = mi(H)\cdot mi(F-V(H)) \\
&\le \begin{cases} (t_1(m)-1)\cdot f_1(n-m) & \text{if } m=4,6,8, \\ t_1(m)\cdot f_1(n-m) & \text{if } m\ge 10, \end{cases} \\
&= \begin{cases} r^{n-2} & \text{if } m=4,6,8, \\ r^{n-2}+r^{n-m} & \text{if } m\ge 10, \end{cases} \\
&\le 17r^{n-10}.
\end{aligned}$$

Furthermore, the equalities holding imply that $m = 10$, $H = T_1(10)$ and $F - V(H) = \frac{n-10}{2}P_2$. In conclusion, $F = T_1(10) \cup \frac{n-10}{2}P_2$. □

Lemma 3.6. *If F is a forest of odd order $n \ge 13$ having $F \neq F_i(n)$, $i = 1, 2, 3, 4$, then $mi(F) \le 49r^{n-13}$. Furthermore, the equality holds if and only if $F = T_2(13) \cup \frac{n-13}{2}P_2$.*

Proof. It is straightforward to check that $mi(T_2(13) \cup \frac{n-13}{2}P_2) = 49r^{n-13}$. Let F be a forest of odd order $n \ge 13$ having $F \neq F_i(n)$, $i = 1, 2, 3, 4$, such that $mi(F)$ is as large as possible. Then $mi(F) \ge 49r^{n-13}$. Suppose that F has three odd components H_1, H_2 and H_3, where $|H_i| = n_i$ for $i = 1, 2, 3$. By Lemma 3.1, Theorems 2.3 and 2.4, we have that $49r^{n-13} \le mi(F) = (\prod_{i=1}^{3} mi(H_i)) \cdot mi(F - \cup_{i=1}^{3} V(H_i)) \le r^{n_1-1} \cdot r^{n_2-1} \cdot r^{n_3-1} \cdot r^{n-(n_1+n_2+n_3)} = r^{n-3} < 49r^{n-13}$, which is a contradiction. Thus we obtain that F has exactly one component H of odd order $m \ge 1$. For the case of $F - V(H) \neq F_1(n-m)$, by Lemma 3.1, Theorems 2.3 and 2.6, then we have that $49r^{n-13} \le mi(F) = mi(H) \cdot mi(F - V(H)) \le t_1(m) \cdot f_2(n-m) = r^{m-1} \cdot 3r^{n-m-4} = 3r^{n-5} < 49r^{n-13}$, which is a contradiction. For the other case of $F - V(H) = F_1(n-m)$, then $m \ge 5$. Since $F \neq F_i(n)$ for $i = 1, 2, 3, 4$, by Lemma 3.1, Theorems 2.4

and 2.5, we have that

$$
\begin{aligned}
49r^{n-13} &\le mi(F) = mi(H) \cdot mi(F - V(H)) \\
&\le \begin{cases} t_2(5) \cdot f_1(n-5) & \text{if } m = 5, \\ (t_2(m) - 1) \cdot f_1(n-m) & \text{if } m = 7, 9, 11, \\ t_2(m) \cdot f_1(n-m) & \text{if } m \ge 13, \end{cases} \\
&= \begin{cases} 3r^{n-5} & \text{if } m = 5, 7, 9, 11, \\ 3r^{n-5} + r^{n-m} & \text{if } m \ge 13, \end{cases} \\
&\le 49r^{n-13}.
\end{aligned}
$$

Furthermore, the equalities holding imply that $m = 13$, $H = T_2(13)$ and $F - V(H) = \frac{n-13}{2}P_2$. In conclusion, $F = T_2(13) \cup \frac{n-13}{2}P_2$. □

The results for the fifth largest numbers of maximal independent sets among all trees and forests, now follow from the above discussion, and they are summarized in the following theorems.

Theorem 3.7. *If T is a tree with $n \ge 14$ vertices having $T \ne T_i(n)$, $i = 1, 2, 3, 4$ then $mi(T) \le t_5(n)$, where*

$$
t_5(n) = \begin{cases} 7r^{n-8}, & \text{if } n \ge 14 \text{ is even;} \\ 5r^{n-7} + 2, & \text{if } n \ge 15 \text{ is odd.} \end{cases}
$$

Furthermore, $mi(T) = t_5(n)$ if and only if $T = T_5(n)$, where

$$
T_5(n) = \begin{cases} T_{5e}(n) \text{ or } B(6; 2, 2), & \text{if } n \ge 14 \text{ is even;} \\ T_{5o}(n), & \text{if } n \ge 15 \text{ is odd.} \end{cases}
$$

Theorem 3.8. *If F is a tree with $n \ge 12$ vertices having $F \ne F_i(n)$, $i = 1, 2, 3, 4$ then $mi(F) \le f_5(n)$, where*

$$
f_5(n) = \begin{cases} 17r^{n-10}, & \text{if } n \ge 12 \text{ is even;} \\ 49r^{n-13}, & \text{if } n \ge 13 \text{ is odd} \end{cases}
$$

Furthermore, $mi(F) = f_5(n)$ if and only if $F = F_5(n)$, where

$$
F_5(n) = \begin{cases} T_1(10) \cup \frac{n-10}{2}P_2, & \text{if } n \ge 12 \text{ is even;} \\ T_2(13) \cup \frac{n-13}{2}P_2, & \text{if } n \ge 13 \text{ is odd.} \end{cases}
$$

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