Hyperdomination in Groups

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Abstract

Hyperdomination in hypergraphs was defined by J. John Arul Singh and R. Kala in [3]. Let $X = \{x_1, x_2, \ldots, x_n\}$ be a finite set and let $\mathscr{E} = (E_i/1 \leq i \leq m)$ be a family of subsets of X. $H = (X, \mathscr{E})$ is said to be a hypergraph if 1) $E_i \neq \phi$, $1 \leq i \leq m$ and 2) $\bigcup_{i=1}^{m} E_i = X$. The elements $x_1, x_2, \ldots, x_n$ are called the vertices and the sets $E_1, E_2, \ldots, E_m$ are called the edges. A set $D \subset X$ is called a hyperdominating set if for each $v \in X - D$ there exist some edge E containing v with $|E| \geq 2$ such that $E - v \subset D$. The hyperdomination number is the minimum cardinality of all hyperdominating sets.

In this paper, a finite group is viewed as a hypergraph with vertex set as the elements of the group and edge set as the set of all subgroups of the group. We obtain several bounds for hyperdomination number of finite groups and characterise the extremal graphs in some cases.

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1 Introduction

Algebraic Graph theory has developed in various dimensions in the past two decades. In this paper, we relate a group with a hypergraph and study hyperdomination on that hypergraph. Throughout this paper, G denotes

either a group or a hypergraph, which is evident from the context. Terms not defined here are used in the sense of Herstein [2] and Berge [1].
Let $X = \{x_1, x_2, \ldots, x_n\}$ be a finite set and let $\mathscr{E} = (E_i/1 \leq i \leq m)$ be a family of subsets of X. $H = (X, \mathscr{E})$ is said to be a hypergraph if 1) $E_i \neq \phi$, $1 \leq i \leq m$ and 2) $\bigcup_{i=1}^{m} E_i = X$. The elements $x_1, x_2, \ldots, x_n$ are called the vertices and the sets $E_1, E_2, \ldots, E_m$ are called the edges.
A set $S \subset X$ is called stable if it contains no edge E with $|E| > 1$. The stability number $\alpha(H)$ of H is the maximum cardinality of a stable set of H. A strongly stable set is a set $S \subset X$ such that $|S \cap E| \leq 1$ for every edge $E \in \mathscr{E}(H)$. The strong stability number $\overline{\alpha}(H)$ is the maximum cardinality of a strongly stable set of H. A set $T \subset X$ is a transversal of H if it meets all the edges. The transversal number is the smallest cardinality of a transversal. A k-coloring is a partition of the set of vertices X into k classes such that every edge which is not a loop meets at least two classes of the partition. The chromatic number $\chi(H)$ is the smallest integer k for which H admits a k-coloring. A set $D \subset X$ is called a dominating set of H if for every vertex $y \in X - D$ there exists an edge E containing y such that $E \cap D \neq \phi$. The domination number $\gamma(H)$ is the minimum cardinality of a dominating set. The undirected powergraph $\mathscr{G}(G)$ of a group G is an undirected graph whose vertex set is G and two vertices $a, b \in G$ are adjacent if and only if $a \neq b$ and $a^m = b$ or $b^m = a$ for some positive integer m.

2 Main Results

Definition 2.1. *Let $H = (X, \mathscr{E})$ be a hypergraph. A set $D \subset X$ is called a hyperdominating set if for each $v \in X - D$ there exist some edge E containing v with $|E| \geq 2$ such that $E - v \subset D$. The hyperdomination number $\gamma_h(H)$ is the minimum cardinality of all hyperdominating sets.*

A finite group G can be viewed as a hypergraph, considering the vertex set as the elements of the group and edge set as the set of all subgroups of G. We calculate various hypergraph parameters for finite groups.

Remark 2.2. *The following are some immediate observations :*

1. *As the identity element e lies in every subgroup, for any finite group G, $G - e$ is a stable set and so the stability number $\alpha(G) = n - 1$.*

2. *As* $< x, y >$ *is an edge of* G *for every* $x, y \in G$ *, singleton set is the only strongly stable set and so the strong stability number* $\overline{\alpha}(G) = 1$.

3. *As* $(\{e\}, \{G - e\})$ *is a partition of* G *into stable sets, the chromatic number* $\chi(G) = 2$.

4. *As the singleton set containing the identity itself is a dominating set, the domination number* $\gamma(G) = 1$.

We now calculate the hyperdomination number for finite groups.

Theorem 2.3. $\gamma_h(G) = 1$ *if and only if* $G \cong Z_{2^n}$ *for some* $n \geq 1$.

Proof. As the identity element e should lie in any hyperdominating set, it is easy to observe that $\gamma_h(G) = 1$ if and only if $\{e\}$ is a hyperdominating set of G if and only if every element of G lies in a subgroup of order 2. So it is now enough to prove that every element is of order two if and only if $G \cong Z_{2^n}$ for some $n \geq 1$. When $G \cong Z_{2^n}$, that every element is of order two is obvious. For the other case, when each element is of order two, each element is self inverse and so $ab = (ab)^{-1} = b^{-1}a^{-1} = ba$ which says G is abelian. Hence the result follows since the only abelian group in which every element is of order two is Z_{2^n} . □

Theorem 2.4. *Let* p *be the smallest prime divisor of* $|G|$. *If* $p > 2$, $\gamma_h(G) = p - 1$ *if and only if* $G \cong Z_p$.

Proof. For $G \cong Z_p$, it is immediate that $\gamma_h(G) = p - 1$. Assume $\gamma_h(G) = p - 1$. Let D be a hyperdominating set of G . For each $v \in G - D$ there exist a subgroup E of order p such that $E - v = D$. We claim that $G = E$. If not let $w \in G - E$. As $w \in G - D$, w lies in a subgroup F of order p such that $F - w \subset D$. Now as both E and F are of prime order p and $E \neq F$ we have $E \cap F = \{e\}$, which means $F - w \not\subset D$. This contradiction proves the result.

□

Theorem 2.5. *If* G *is any noncyclic finite group with* $o(G) = n$, *then* $\gamma_h(G) \leq n - 3$.

Proof. Let $\frac{n}{t}$ be the maximum order of an element in G. Clearly $t \geq 2$. Choose $a \in G$ such that $o(a) = \frac{n}{t}$. Let b be an element of maximum order in $G - < a >$. Then $o(b) \leq \frac{n}{t}$. Now $|< a > \cup < b >| \leq \frac{2n-1}{t} \leq \frac{2n-1}{2} < n$. So $G - (< a > \cup < b >) \neq \phi$. Choose an element c of maximum order in $G - (< a > \cup < b >)$. By choice of c, $c \notin < a > \cup < b >$. Also $b \notin < a >$. If $b \in < c >$ then $< b > \subseteq < c >$ so that $o(b) \leq o(c)$. Also since $c \in < b >$ we get $o(b) \leq o(c)$, a contradiction to the fact $o(c) \leq o(b)$. Hence $b \notin < c >$ so that $b \notin < a > \cup < c >$. Now if $a \in < b >$ then $< a > \subseteq < b >$ so that $o(a) \leq o(b)$. Also since $b \notin < a >$, $o(a) < o(b)$ a contradiction to the fact that $o(b) \leq o(a)$. Hence $a \notin < b >$. By a similar argument $a \notin < c >$. Hence $a \notin < b > \cup < c >$. Thus $G - \{a, b, c\}$ is a hyperdominating set of G. Hence $\gamma_h(G) \leq n - 3$. □

Theorem 2.6. *For any finite group G of order n, $\gamma_h(G) = n - 1$ if and only if $G \cong Z_{p^\alpha}$.*

Proof. If $G \cong Z_{p^\alpha}$, then the only subgroups of G are $< 1 >, < p >, < p^2 >, \ldots, < p^{\alpha-1} >$ and $< e >$. Also $< p^r > \subseteq < p^k >$ for $k < r$. Hence $\gamma_h(G) = n - 1$. Conversely assume $\gamma_h(G) = n - 1$. Then $|G| = p^\alpha$ for some prime p. If not then G has two prime divisors p and q. By Sylow's theorem, G has an element a of order p and an element b of order q. Clearly $G - \{a, b\}$ is a hyperdominating set, so that $\gamma_h(G) \leq n - 2$ a contradiction. Hence $|G| = p^\alpha$ for some prime p. By theorem 2.5, G is cyclic. The only cyclic subgroup of order p^α is Z_{p^α}. Hence $G \cong Z_{p^\alpha}$. □

Definition 2.7. *Let G be an abelian group. We know that G can be represented as $G \cong Z_{{p_1}^{n_1}} \otimes \ldots \otimes Z_{{p_k}^{n_k}}$ where $\otimes$ represents direct product of groups and p_i^s are primes. We denote by G_D a graph with vertex set $V(G_D) = Z_{(n_1+1)} \times \ldots \times Z_{(n_k+1)}$ where $\times$ represents cartesian product. Two tuples $(x_1, \ldots, x_k)$ and $(y_1, \ldots, y_k)$ in $V(G_D)$ are adjacent if and only if either $x_i \geq y_i$ for all i or $x_i \leq y_i$ for all i.*

Definition 2.8. *For any hypergraph $H = (X, \mathscr{E})$, we call a collection $\mathscr{D} = \{E_i\} \subseteq \mathscr{E}(H)$ as a hyperdominating collection of H if $E_i \not\subseteq \bigcup_{j \neq i} E_j$, $\forall\, i$.*

Proposition 2.9. *For any hypergraph* $H = (X, \mathscr{E})$, $\gamma_h(H) = n - max\{t/$ *there exists a hyperdominating collection* $\mathscr{D}$ *with* $|\mathscr{D}| = t\}$.

Proof. If $\{E_1, E_2, \ldots, E_t\}$ satisfies $E_j \not\subseteq \bigcup_{i\neq j} E_i$, for every j, $1 \leq j \leq t$ then choosing $v_j \in E_j - \bigcup_{i\neq j} E_i$, $1 \leq j \leq t$ we get a hyperdominating set $X - \{v_j, 1 \leq j \leq t\}$ with cardinality n-t. Hence $\gamma_h(H) \leq min\{n - t/$ there exists an edge set $\{E_1, E_2, \ldots, E_t\}$ such that $E_j \not\subseteq \bigcup_{i\neq j} E_i$, for every $j, 1 \leq j \leq t\}$.
On the other hand, if there is a hyperdominating set D with $|D| = \gamma_h(H)$, then by definition, for each $v_i \in X - D$, $1 \leq i \leq n - \gamma_h(H)$, there exist edges E_i such that $E_i - v_i \subseteq D$. $\{E_1, E_2, \ldots, E_t\}$ satisfies $E_j \not\subseteq \bigcup_{i\neq j} E_i$, $1 \leq j \leq n - \gamma_h(H)$. Hence $min\{n - t/$ there exists an edge set $\{E_1, E_2, \ldots, E_t\}$ such that $E_j \not\subseteq \bigcup_{i\neq j} E_i$, for every $j, 1 \leq j \leq t\} \leq \gamma_h(H)$. □

Theorem 2.10. *For any abelian group* G *of order* n, $\gamma_h(G) \leq n - \beta_0(G_D)$. *Moreover* $\gamma_h(G) = n - \beta_0(G_D)$ *whenever* G *is cyclic.*

Proof. Let $G \cong Z_{p_1^{n_1}} \otimes \ldots \otimes Z_{p_k^{n_k}}$, where $p_i's$ are primes. Consider the collection $\mathscr{C}(G) = \{< p_1^{r_1} > \otimes < p_2^{r_2} > \otimes \ldots \otimes < p_k^{r_k} >: 0 \leq r_i \leq n_i\ \forall i\}$. Define $f : \mathscr{C}(G) \longrightarrow V(G)$ by $f(< p_1^{r_1} > \otimes < p_2^{r_2} > \otimes \ldots \otimes < p_k^{r_k} >) = (r_1, r_2, \ldots, r_k)$. Then f is a bijection. Let $\mathscr{D} \subseteq \mathscr{E}(H)$.
Claim $\mathscr{D}$ is a hyperdominating collection of the partial hypergraph of G say H generated by $\mathscr{C}(G)$ if and only if $f(\mathscr{D})$ is an independent set of G_D.
Assume $\mathscr{D}$ is a hyperdominating collection of H. Let $< p_1^{r_1} > \otimes < p_2^{r_2} > \otimes \ldots \otimes < p_k^{r_k} >$ and $< p_1^{s_1} > \otimes < p_2^{s_2} > \otimes \ldots \otimes < p_k^{s_k} >$ be any two elements in $\mathscr{D}$. If $r_i \geq s_i$ $\forall i = 1,\ 2,\ \ldots,\ k$ then $< p_1^{r_1} > \otimes < p_2^{r_2} > \otimes \ldots \otimes < p_k^{r_k} > \subseteq < p_1^{s_1} > \otimes < p_2^{s_2} > \otimes \ldots \otimes < p_k^{s_k} >$, which is not possible in a hyperdominating collection. In the same way, $r_i \leq s_i \forall\ i$ is also not possible. Thus $f(< p_1^{r_1} > \otimes < p_2^{r_2} > \otimes \ldots \otimes < p_k^{r_k} >) = (r_1, r_2, \ldots, r_k)$ and $f(< p_1^{s_1} > \otimes < p_2^{s_2} > \otimes \ldots \otimes < p_k^{s_k} >) = (s_1, s_2, \ldots, s_k)$ are non adjacent. Hence $f(\mathscr{D})$ is independent.
Conversely assume $\mathscr{D} \subseteq \mathscr{C}(G)$ and $f(\mathscr{D})$ is an independent set. Let $(l_1, l_2, \ldots, l_k)$ be any element in $f(\mathscr{D})$. Then as $f(\mathscr{D})$ is independent, $(l_1, l_2, \ldots, l_k)$ is not adjacent to any other vertex in $f(\mathscr{D})$. Then for each $(r_1, r_2, \ldots, r_k) \in f(\mathscr{D})$ there exists $j \in \{1, 2, \ldots, k\}$ such that $l_j < r_j$, so that

$p_j^{l_j} \notin < p_j^{r_j} >$. Now $(p_1^{l_1}, \ldots, p_k^{l_k})$ does not lie in any edge of $\mathscr{D}$, so that $< p_1^{l_1} > \otimes < p_2^{l_2} > \otimes \ldots \otimes < p_k^{l_k} >$ is not contained in union of all other edges. Hence $\mathscr{D}$ is a hyperdominating collection of H. Hence $\beta_0(G_D) = max\,|\mathscr{D}|$ where maximum is taken over all hyperdominating collections of H. By proposition 2.9, we have $\gamma_h(H) = n - \beta_0(G_D)$. Since every hyperdominating collection of H is also a hyperdominating collection of G, $\gamma_h(G) \leq \gamma_h(H) = n - \beta_0(G_D)$. When G is cyclic, every subgroup of G lies in $\mathscr{C}$ and so in this case $G = H$. Thus $\gamma_h(G) = n - \beta_0(G_D)$ when G is cyclic. □

Lemma 2.11. *For any abelian group $G \cong Z_{p_1^{n_1}} \otimes \ldots \otimes Z_{p_k^{n_k}}$, $p_i's$ are primes, the sets $S_l = \{(x_1, x_2, \ldots, x_k) \in Z_{(n_1+1)} \times \ldots \times Z_{(n_k+1)} : \sum x_i = l\}$ for $0 \leq l \leq \sum n_i$ are independent in G_D.*

Proof. Let $x = (x_1, x_2, \ldots, x_k)$ and $y = (y_1, y_2, \ldots, y_k)$ be any two arbitrary points in $S_l \subseteq Z_{(n_1+1)} \times \ldots \times Z_{(n_k+1)}$. As x and y are distinct, there is an integer i such that $x_i \neq y_i$; without loss of generality assume $x_i < y_i$. Then as $\sum x_i = \sum y_i$, there exists some j such that $x_j > y_j$. Hence x and y are non adjacent. This proves the independency of S_l. □

Theorem 2.12. *For any group G of order n with $G \cong Z_{p_1^{n_1}} \otimes \ldots \otimes Z_{p_k^{n_k}}$, $\gamma_h(G) \leq n - |S_{\lfloor \frac{\sum n_i}{2} \rfloor}|$, where $p_i's$ are primes and $S_{\lfloor \frac{\sum n_i}{2} \rfloor} = \{(x_1, x_2, \ldots, x_k) \in G_D : \sum x_i = \lfloor \frac{\sum n_i}{2} \rfloor\}$. Moreover equality holds if G is cyclic.*

Proof. Let $V(G_D) = Z_{(n_1+1)} \times \ldots \times Z_{(n_k+1)}$. We partition G_D into disjoint cliques using the following procedure. In $Z_{(n_1+1)} \times \{0\} \times \{0\} \ldots \times \{0\}$, consider the path $P_0 : ((0,0,0,\ldots,0), (1,0,0,\ldots,0), \ldots, (n_1,0,0,\ldots,0))$. Considering $Z_{(n_1+1)} \times Z_{(n_2+1)} \times \{0\} \times \{0\} \times \ldots \times \{0\}$ we have n_2+1 copies of such paths. We combine them to create new paths as follows. The new paths considered are $P_{0i} : ((0,i,0,0,\ldots,0), (1,i,0,0,\ldots,0), (2,i,0,0,\ldots,0), \ldots, (n_1-i,i,0,0,\ldots,0)) \cup ((n_1-i,i,0,0,\ldots,0), (n_1-i,i+1,0,0,\ldots,0), (n_1-i,i+2,0,0,\ldots,0), \ldots, (n_1-i,n_2,0,0,\ldots,0))$ for $0 \leq i \leq min\{n_1,n_2\}$. In general, let P be the path in $Z_{(n_1+1)} \times Z_{(n_2+1)} \times \ldots \times Z_{(n_{j-1}+1)} \times \{0\} \times$

$\{0\} \times \ldots \times \{0\}$. Suppose that $m = \min_{(x_1,\ldots,x_k)\in P}\{\sum_{i=1}^{k} x_i\}$ and $l =$ length of P. By construction, the vertices in P have distinct co-ordinate sums ranging from m to $m+l$ and $\sum_{i=1}^{j-1} n_i = 2m+l$. In $Z_{(n_1+1)} \times \ldots \times Z_{(n_{j-1}+1)} \times Z_{(n_j+1)} \times \{0\} \times \ldots \times \{0\}$, this path P gives rise to $\min\{n_j, l\}+1$ paths. These paths are given by $P_s = \{(x_1, x_2, \ldots, x_{j-1}, s, 0, 0, \ldots, 0)/(x_1, x_2, \ldots, x_{j-1}, 0, 0 \ldots, 0) \in P$ and $\sum_{i=1}^{j-1} x_i \leq m+l-s\} \bigcup \{(x_1, x_2, \ldots, x_{j-1}, t, 0, 0 \ldots, 0) / \sum_{i=1}^{j-1} x_i = m+l-s, s \leq t \leq n_j\}$ where $0 \leq s \leq \min\{n_j, l\}$. With $n_1 = 6$ and $n_2 = 4$, the construction of paths in $Z_{(n_1+1)} \times Z_{(n_2+1)} \times \{0\} \times \{0\} \times \ldots \times \{0\}$ from paths in $Z_{(n_1+1)} \times \{0\} \times \{0\} \times \ldots \times \{0\}$ is given in Fig. 1.

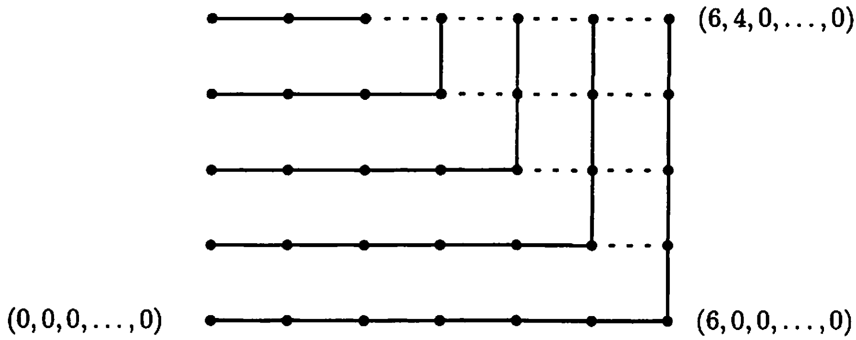

Fig. 1.

Thus we obtain a partition of G_D into paths which are all cliques. Moreover by construction, the coordinate sum of the last point in each path is same as the difference of $\sum_{i=1}^{k} n_i$ and the coordinate sum of the zeroth point in the same path, so that every path in G_D intersects the set $S_{\lfloor \frac{\sum n_i}{2} \rfloor}$. Also we note that the total number of such paths is same as $|S_{\lfloor \frac{\sum n_i}{2} \rfloor}|$ as each point in $S_{\lfloor \frac{\sum n_i}{2} \rfloor}$ should lie in such a path. As an independent set can have at most one vertex from a clique, $\beta_o(G_D) = |S_{\lfloor \frac{\sum n_i}{2} \rfloor}|$. By lemma 2.11, $S_{\lfloor \frac{\sum n_i}{2} \rfloor}$ itself is an independent set ot G_D. Hence $\beta_o(G_D) = |S_{\lfloor \frac{\sum n_i}{2} \rfloor}|$. Theorem now follows from the theorem 2.10. □

Theorem 2.13. *For any group G of order n with $G \cong Z_{p_1^{n_1}} \otimes \ldots \otimes Z_{p_k^{n_k}}$, $(k \geq 2)$, $\gamma_h(G) \leq n - \sum_{i_1=0}^{\min\{n_1,n_2\}} \sum_{i_2=0}^{\min\{n_1+n_2-2i_1,n_3\}} \cdots \sum_{i_{k-2}=0}^{\min\{\sum_{j=1}^{k-2} n_j - 2\sum_{j=1}^{k-3} i_j, n_{k-1}\}}$ $\min\{\sum_{j=1}^{k-1} n_j - 2\sum_{j=1}^{k-2} i_j + 1, n_k + 1\}$. Moreover, equality holds whenever G is cyclic.*

Proof. By theorem 2.12, it is enough to say $|S_{\lfloor \sum \frac{n_i}{2} \rfloor}| = \sum_{i_1=0}^{\min\{n_1,n_2\}} \sum_{i_2=0}^{\min\{n_1+n_2-2i_1,n_3\}} \cdots \sum_{i_{k-2}=0}^{\min\{\sum_{j=1}^{k-2} n_j - 2\sum_{j=1}^{k-3} i_j, n_{k-1}\}} \min\{\sum_{j=1}^{k-1} n_j - 2\sum_{j=1}^{k-2} i_j + 1, n_k + 1\}$. (*) Consider the partition G_D into cliques as in theorem 2.12. We have $|S_{\lfloor \sum \frac{n_i}{2} \rfloor}|$ equals the total number of cliques. So it is sufficient to say that the total number of cliques obtained in the partition of $|G_D|$ equals the expression in the right side of (*).

Claim : The length of the paths obtained in G_D are $\sum_{j=1}^{k} n_j - 2\sum_{j=1}^{k-1} i_j$ (**) where i_1 varies from 0 to $\min\{n_1, n_2\}$, i_2 varies from 0 to $\min\{n_1 + n_2 - 2i_1, n_3\}$, ..., i_{k-1} varies from 0 to $\min\{\sum_{j=1}^{k-1} n_j - 2\sum_{j=1}^{k-2} i_j, n_k\}$.

We prove by induction on k. When $k = 1$ we get a single path of length n_1 and so (**) holds when $k = 1$. Assume the result for $k-1$. That is the length of the obtained paths in $Z_{(n_1+1)} \times \ldots \times Z_{(n_{k-1}+1)}$ are

$$\sum_{j=1}^{k-1} n_j - 2\sum_{j=1}^{k-2} i_j \qquad (***)$$

where i_1 varies from 0 to $\min\{n_1, n_2\}, \ldots, i_{k-2}$ varies from 0 to $\min\{\sum_{j=1}^{k-2} n_j - 2\sum_{j=1}^{k-3} i_j, n_{k-1}\}$. We prove for k. By construction each path of length l in $Z_{(n_1+1)} \times \ldots \times Z_{(n_{k-1}+1)}$ give rise to paths of lengths $l + n_k - 2i_{k-1}$ where i_{k-1} varies from 0 to $\min\{l, n_k\}$. Replacing value of l from (***) we get the obtained paths in $Z_{(n_1+1)} \times \ldots \times Z_{(n_k+1)}$ are of lengths $\sum_{j=1}^{k} n_j - 2\sum_{j=1}^{k-1} i_j$ where i_1 varies from 0 to $\min\{n_1, n_2\}, \ldots, i_{k-2}$ varies from 0 to $\min\{\sum_{j=1}^{k-2} n_j - 2\sum_{j=1}^{k-3} i_j, n_{k-1}\}$, i_{k-1} varies from 0 to

$\min\{\sum_{j=1}^{k-1} n_j - 2\sum_{j=1}^{k-2} i_j, n_k\}$. Hence (**) holds for k. The total number of paths in G_D is

$$\sum_{i_1=0}^{\min\{n_1,n_2\}} \sum_{i_2=0}^{\min\{n_1+n_2-2i_1,n_3\}} \cdots \sum_{i_{k-2}=0}^{\min\{\sum_{j=1}^{k-2} n_j-2\sum_{j=1}^{k-3} i_j, n_{k-1}\}} \sum_{i_{k-1}=1}^{\min\{\sum_{j=1}^{k-1} n_j-2\sum_{j=1}^{k-2} i_j, n_k+\}} 1$$

$$= \sum_{i_1=0}^{\min\{n_1,n_2\}} \cdots \sum_{i_{k-2}=0}^{\min\{\sum_{j=1}^{k-2} n_j-2\sum_{j=1}^{k-3} i_j, n_{k-1}\}} \min\{\sum_{j=1}^{k-1} n_j - 2\sum_{j=1}^{k-2} i_j + 1, n_k + 1\} \quad \square$$

Theorem 2.14. *Let G be a cyclic group isomorphic to $Z_{p_1^{n_1}} \otimes \ldots \otimes Z_{p_k^{n_k}}$ with $n_k \geq n_i \forall i = 1, 2, \ldots, k-1$. Then $n_k \geq \sum_{i=1}^{k-1} n_i$ if and only if $\gamma_h(G) = |G| - \prod_{i=1}^{k-1}(n_i + 1)$.*

Proof. Assume $n_k \geq \sum_{i=1}^{k-1} n_i$. Let $t = \sum_{i=1}^{k-1} n_i$ and $S_t = \{(x_1, x_2, \ldots, x_k) \in V(G_D) : \sum_{i=1}^{k-1} x_i = t\}$. By lemma 2.11, S_t is independent in G_D.
Since $(x_1, x_2, \ldots, x_{k-1}, t - \sum_{i=1}^{k-1} x_i) \in S_t$ for all $(x_1, x_2, \ldots, x_{k-1}) \in Z_{(n_1+1)} \times \ldots \times Z_{(n_{k-1}+1)}$, $\beta_o(G_D) \geq |S_t| \geq \prod_{i=1}^{k-1}(n_i+1)$. And since the sets $T_{(x_1,\ldots,x_k-1)} = \{(x_1, \ldots, x_{k-1}, h) : 0 \leq h \leq n_k\}$ are all cliques in G_D and $\cup T_{(x_1,\ldots,x_{k-1})} = V(G_D)$, $\beta_0 \leq \prod_{i=1}^{k-1}(n_i + 1)$. Hence $\gamma_h(G) = |G| - \prod_{i=1}^{k-1}(n_i + 1)$. Conversely assume $\gamma_h(G) = |G| - \prod_{i=1}^{k-1}(n_i + 1)$.By $\sum n_i$ we mean $\sum_{i=1}^{k} n_i$.
Then $|S_{\lfloor \frac{\sum n_i}{2} \rfloor}| = \prod_{i=1}^{k-1}(n_i+1)$. Since $\{T_{(x_1,\ldots,x_{k-1})}\}$ partitions $V(G_D)$ into cliques and $S_{\lfloor \frac{\sum n_i}{2} \rfloor}$ is a maximum independent set, one element from each $T_{(x_1,x_2,\ldots,x_{k-1})}$ lies in $S_{\lfloor \frac{\sum n_i}{2} \rfloor}$. Let $(0, 0, \ldots, 0, s) \in S_{\lfloor \frac{\sum n_i}{2} \rfloor}$. Then $s = \lfloor \frac{\sum n_i}{2} \rfloor$ and let $(n_1, n_2, \ldots, n_{k-1}, l) \in S_{\lfloor \frac{\sum n_i}{2} \rfloor}$. Then $l = \lfloor \frac{\sum n_i}{2} \rfloor - \sum_{i=1}^{k-1} n_i$, so that $n_k \geq s - t = \sum_{i=1}^{k-1} n_i$. $\square$

Theorem 2.15. *For any finite group G of order n, $\gamma_h(G) = n-2$ if and only if $G \cong Z_{pq^\alpha}$, $\alpha \geq 1$, p, q are distinct primes.*

Proof. If $G \cong Z_{pq^\alpha}$, then $\gamma_h(G) = n - \min\{2, \alpha+1\} = n-2$. For the other part, assume $\gamma_h(G) = n-2$. Then by theorem 2.5, G is cyclic and by theorem 2.4 $|G| \neq p^\alpha$, p prime. So G has at least two prime divisors. If $|G|$ has more than two prime divisors say p, q, l then by Sylow's theorem there is an element b of order 2 and an element c of order l. Now $G - \{a, b, c\}$ is a hyperdominating set, which is a contradiction. Hence $|G| = p^\beta q^\alpha$, p, q primes. If both $\alpha, \beta \leq 2$ then $G - \{p^2, q^2, pq\}$ is a hyperdominating set as $< pq > -pq$, $< q^2 > -q^2$ and $< p^2 > -p^2$ are all subsets of $G - \{p^2, q^2, pq\}$. But now $\gamma_h(G) \leq n-3$ and hence $G \cong Z_{pq^\alpha}$ where p and q are primes. □

Theorem 2.16. *For any finite group G, $\gamma_h(G) = \gamma_h(H)$, where H is the partial hypergraph of G generated by the collection of all cyclic subgroups of G.*

Proof. As every hyperdominating collection of H is again a hyperdominating collection in G, we have $\gamma_h(G) \leq \gamma_h(H)$. Let $\mathscr{D}$ be a hyperdominating collection in G with $|\mathscr{D}| = n - \gamma_h(G)$. Every subgroup E in $\mathscr{D}$ has an element a which does not lie in any other subgroup in $\mathscr{D}$. Replace E by $< a >$. Repeate the process for every noncyclic subgroup in $\mathscr{D}$ until we get a collection $\mathscr{D}'$ containing no noncyclic subgroups. Also $|\mathscr{D}'| = |\mathscr{D}|$. Thus we get a hyperdominating collection $\mathscr{D}'$ of H having cardinality $n - \gamma_h(G)$. Hence $\gamma_h(H) = \gamma_h(G)$. □

Theorem 2.17. *For any finite group G of order n, $\gamma_h(G) \leq n - \sum\limits_{\substack{p|o(G) \\ p \text{ prime}}} \frac{C_p}{p-1}$ where C_p is the number of elements of order p. Moreover equality holds when every element in G is of prime order.*

Proof. Let $\mathscr{D}$ be the collection of all subgroups of G of prime order. Let A, $B \in \mathscr{D}$. If $|A| = p$, $|B| = q$ where $p \neq q$, then as p and q are primes, each element of A other than e is of order p and each element of B other than e has order q, so that $A \cap B = \{e\}$ in this case. Let A and B have the same order say p. If $A \cap B \neq \{e\}$ and $x \in A \cap B$ then as $o(x) = p$, $\{e, x, x^2, \ldots, x^{p-1}\} \subseteq A \cap B$ and so $A = B$. Therefore $A \cap B = \{e\}$ for all $A, B \in \mathscr{D}$ with $A \neq B$. Hence $\mathscr{D}$ is a hyperdominating collection of G. Therefore $\gamma_h(G) \leq n - |\mathscr{D}|$.

$$\text{Also } |\mathscr{D}| = \sum_{\substack{p|o(G) \\ p \text{ prime}}} \text{number of subgroups of order p}$$

$$= \sum_{\substack{p|o(G) \\ p \text{ prime}}} \frac{1}{p-1} \times \text{number of elements of order p}$$

$$= \sum_{\substack{p|o(G) \\ p \text{ prime}}} \frac{C_p}{p-1}, \text{where } C_p \text{ is the number of elements of order p.}$$

When every element of G is of prime order, every cyclic subgroup is of prime order, so that the collection of all cyclic subgroups is same as $\mathscr{D}$. By lemma 2.11 $\gamma_h(G) = \gamma_h(H)$, where H is the partial hypergraph generated by $\mathscr{D}$. We know that $\gamma_h(H) = n - |\mathscr{D}|$ and so $\gamma_h(G) = n - \sum_{\substack{p|o(G) \\ p \text{ prime}}} \frac{C_p}{p-1}$ in this case. □

Corollary 2.18. $\gamma_h(Z_{p^n}) = \frac{p^n-1}{p-1}$, *where* p *is prime.*

Proof. In Z_{p^n} every element except identity has order p and so proof follows from theorem 2.17. □

Theorem 2.19. $\gamma_h(D_{2n}) = \gamma_h(Z_n)$.

Proof. Let D be a hyperdominating set of D_{2n} with $|D| = \gamma_h(D_{2n})$. Then for every $v \in Z_n - (D \cap Z_n)$, by hyperdomination of D, there exist a subgroup $E - v \subseteq D$, so that $< v > -v \subseteq D$. Now since $v \in Z_n$, $< v > \subseteq Z_n$. Therefore $< v > -v \subseteq D \cap Z_n$. Hence $D \cap Z_n$ is a hyperdominating set of Z_n, so that $\gamma_h(D_{2n}) = |D| \geq |D \cap Z_n| \geq \gamma_h(Z_n)$. Conversely let D' be a hyperdominating set of Z_n with $\gamma_h(Z_n) = |D'|$. Then as every element of $D_{2n} - Z_n$ is of order two, D' is again a hyperdominating set of D_{2n}. Thus $\gamma_h(Z_n) = |D'| \geq \gamma_h(D_{2n})$. □

Theorem 2.20. *For any group G of order n, $\gamma_h(G) = n - \beta_o(\mathscr{G}(G))$ where $\mathscr{G}(G)$ is the undirected power graph of G.*

Proof. Let H be a partial hypergraph of G generated by the collection of all cyclic subgroups of G. By theorem 2.14, $\gamma_h(G) = \gamma_h(H)$. We claim that there is a one-one correspondence between independent sets of $\mathscr{G}(G)$ and hyperdominating collections of H. Consider the map $f : G \longrightarrow \mathscr{E}(H)$ defined by $f(a) = < a >$. Let I be an independent set of $\mathscr{G}(G)$. Then $f(I) = \{< a > / a \in I\}$. Fix $a \in I$. Then for any $b \in I$, a and b are nonadjacent, so that $a \neq b^i\ \forall i$. Hence $a \notin < b >\ \forall b \in I$ and $b \neq a$. Then $< a > \not\subseteq \bigcup_{\substack{b \in I \\ b \neq a}} < b >$. Hence $f(I)$ is a hyperdominating collection of H. Conversely let us assume that $\mathscr{D}$ is a hyperdominating collection of H. For each $S_i \in \mathscr{D}$, choose $a_i \in S_i$ such that $< a_i >= S_i$. Let $I = \{a_i\}$. Let $a_i, a_j \in I$ and $i \neq j$. Then by hyperdomination, $< a_i > \not\subseteq < a_j >$ and $< a_j > \not\subseteq < a_i >$ so that $a_i \neq a_j^m$ and $a_j \neq a_i^m$ for all m. Hence a_i and a_j are nonadjacent in $\mathscr{G}(G)$, ie, I is independent with $\mathscr{D} = f(I)$. For any independent set I, $a, b \in I, a \neq b \implies a \neq b, a^i \neq b \forall i \implies < a > \neq < b > \implies f(a) \neq f(b)$. Hence f/I is one to one. Thus $f/I : I \longrightarrow f(I)$ is a bijection. So $|I| = |f(I)|$ for all independent sets of $\mathscr{G}(G)$ and $|\mathscr{D}| = |f^{-1}(\mathscr{D})|$ for all hyperdominating collections of H. Hence $\beta_o(\mathscr{G}(G))$ equals the maximum cardinality of a hyperdominating collection in H, so that $\gamma_h(G) = \gamma_h(H) = |G| - \beta_o(\mathscr{G}(G))$. □

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