Some graft transformations and its applications on Q-spectral radius of a graph*

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Abstract

A generalized θ-graph is composed of at least three internal disjoint paths (at most one of them is with length 1) which have the same initial vertex and the same terminal vertex. If the initial vertex and the terminal vertex are the same in a generalized θ-graph, then the generalized θ-graph is called a degenerated θ-graph or a petal graph. In this paper, two graft transformations that increase or decrease Q-spectral radius of a graph are represented. With them, for the generalized θ-graphs and petal graphs with order n, the extremal graphs with the maximal Q-spectral radius and the extremal graphs with the minimal Q-spectral radius are characterized respectively.

AMS Classification: 05C50
Keywords: Signless Laplacian; Spectral radius; θ-graph; Petal graph

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*Supported by NSFC (Nos. 11271315, 11171290).

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1 Introduction

Throughout this article, all graphs considered are simple, connected and undirected. Let $G = G[V(G), E(G)]$ be a graph with vertex set $V(G) = \{v_1, v_2, \cdots, v_n\}$ and edge set $E(G)$, where $|V(G)| = n$ is the order and $|E(G)| = m$ is the size of G. In a graph G, if vertex v_i is adjacent to v_j, we denote by $v_i \sim v_j$. We denote by $N_G(v)$ or $N(v)$ the neighbor set of vertex v in graph G. The degree of vertex v in G, denoted by $d_G(v)$ or $d(v)$, is equal to $|N_G(v)|$. In this paper, we denote by K_n, P_n, C_n for a complete graph, a path and a cycle with order n respectively.

Let $A = (a_{ij})_{n \times n}$ be the (0,1)-adjacency matrix of G, and let D be the diagonal matrix $\text{diag}(d_1, d_2, \cdots, d_n)$. The matrix $L(G) = D - A$ is the Laplacian of G, while $Q(G) = D + A$ is called the signless Laplacian of G. If M is the $n \times m$ vertex-edge incidence matrix of the (n, m)-graph G, then $Q(G) = MM^T$. Thus, if G is connected, then $Q(G)$ is positive semi-definite, and its eigenvalues can be arranged as: $q = q_1 \geq q_2 \geq \cdots \geq q_n \geq 0$. q is called the signless Laplacian spectral radius or Q-spectral radius of G. By the Perron-Frobenius theorem [7], for a connected graph G, we know that there exists a unit positive vector corresponding to $q(G)$ which is called *Q-Perron eigenvector*.

Within spectral graph theory, studying the properties of a graph using its signless Laplacian became recently the most dynamic area of research. Cvetković and Simić [1]-[4] recently investigated the theory of Q-spectra of graphs, and they gave some reasons for studying graphs by using Q-spectra being more efficient than studying them using their A-spectra (adjacency spectra) or L-spectra (Laplacian spectra). For some recent results on Q-spectra of graphs, the reader can be referred to [1]-[4], [9, 12, 13].

Definition 1.1 *A generalized θ-graph is composed of internal disjoint paths P_{n_1+1}, P_{n_2+1}, $\cdots$, P_{n_s+1} $(s \geq 3)$, where P_{n_1+1}, P_{n_2+1}, $\cdots$, P_{n_s+1} have the same initial vertex and the same terminal vertex, denoted by $\widehat{\theta}(n_1, n_2, \cdots, n_s)$ or $\widehat{\theta}_n(n_1, n_2, \cdots, n_s)$ if $n_1 + n_2 + \cdots + n_s - s + 2 = n$ (at most one of n_1, n_2, $\cdots$, n_s is 1). Each $P_{n_{i2}+1}$ is called a meridian line; the common initial vertex is called the head, denoted by v_0; the common terminal vertex is called the tail, denoted by u_0. If $s = 3$, $\widehat{\theta}_n(n_1, n_2, n_3)$ is the general θ-graph, denoted by $\theta_n(n_1, n_2, n_3)$ usually.*

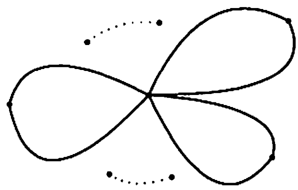

Fig. 1.1 $\mathscr{P}(n_1, n_2, \cdots, n_s)$

A s-petal graph is obtained by attaching s independent cycles C_{n_1}, C_{n_2}, $\cdots$, C_{n_s} to a vertex, denoted by $\mathscr{P}(n_1, n_2, \cdots, n_s)$ (see Fig. 1.1). In fact, a s-petal graph is a degenerated generalized θ-graph, namely, $v_0 = u_0$ in this θ-graph.

The study about θ-graphs are always interesting. For example, in [5], Y. Feng and Q. Huang considered the consecutive edge-coloring of the generalized θ-graphs; in [6], J. Fialaa, J. Kratochvla and Attila Pár considered the the computational complexity of partial covers of θ-graphs; in [10], F. Ramezani, N. Broojerdian and B. Tayfeh-Rezaie considered the spectral characterization of θ-graphs; in [11], for generalized θ-graphs, the relationship between the structure and the spectral radius was discussed. In this paper, two graft transformations that increase or decrease Q-spectral radius of a graph are represented. With them, for generalized θ-graphs and petal graphs with order n, the extremal graphs with the maximal Q-spectral radius and the extremal graphs with the minimal Q-spectral radius are characterized respectively. This paper is organized as follows: Section 1 introduces the basic ideas and their supports; Section 2 gives two graft transformations; Section 3 introduces some applications of the two graft transformations.

2 Graft transformations

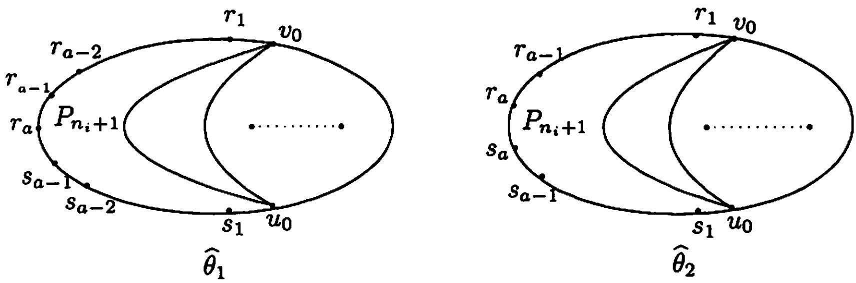

Fig. 2.1. $\widehat{\theta}$

Lemma 2.1 *Let $X = (x_{v_0}, x_{u_0}, \cdots)^T$ denote the Q-Perron vector of a generalized θ-graph $\widehat{\theta}_n(n_1, n_2, \cdots, n_s)$ in which x_v corresponds to vertex v. Denote by q the Q-spectral radius of $\widehat{\theta}_n(n_1, n_2, \cdots, n_s)$. For a meridian line P_{n_i+1},*

(1) if $P_{n_i+1} = v_0 r_1 r_2 \cdots r_{a-1} r_a s_{a-1} \cdots s_1 u_0$ (see Fig. 2.1 $\widehat{\theta}_1$), then

(i) $x_{r_{a-1}} = x_{s_{a-1}}$, $x_{r_{a-2}} = x_{s_{a-2}}$, $\cdots$, $x_{r_1} = x_{s_1}$, $x_{v_0} = x_{u_0}$;

(ii) let $f_1 = \frac{1}{2}(q-2)$ and $f_{i+1} = q - 2 - \frac{1}{f_i}$. Then $x_{r_{a-i}} = f_i x_{r_{a-i+1}}$ for $1 \le i \le a$ and $f_i > 1$ for any $i \ge 1$;

(2) if $P_{n_i+1} = v_0 r_1 r_2 \cdots r_{a-1} r_a s_a s_{a-1} \cdots s_1 u_0$ (see Fig. 2.1 $\widehat{\theta}_2$), then

(i) $x_{r_a} = x_{s_a}$, $x_{r_{a-1}} = x_{s_{a-1}}$, $x_{r_{a-2}} = x_{s_{a-2}}$, $\cdots$, $x_{r_1} = x_{s_1}$, $x_{v_0} = x_{u_0}$;

(ii) let $g_1 = q - 3$ and $g_{i+1} = q - 2 - \frac{1}{g_i}$. Then $x_{r_{a-i}} = g_i x_{r_{a-i+1}}$ for $1 \le i \le a$ and $g_i > 1$ for any $i \ge 1$.

(3) for f_i, g_i, we have $f_i < g_i < f_{i+1}$ for any $i \ge 1$.

Proof. By symmetry, (i) of (1), (2) follows.

For (1), it is easy to check that $x_{r_{a-1}} = f_1 x_{r_a}$, and by induction, $x_{r_{a-i}} = f_i x_{r_{a-i+1}}$ for $1 \le i \le a$. Because $\widehat{\theta}_n(n_1, n_2, \cdots, n_s) \not\cong C_n$, so $q > 4$, and it is easy to check that $f_1 > 1$. Suppose that $f_i > 1$ holds for $i < N$, then $f_N = q - 2 - \frac{1}{f_{N-1}} \ge q - 3 > 1$. By induction, $f_i > 1$ holds for $i \ge 1$. Thus (ii) of (1) is proved. (ii) of (2) can be proved in a same way.

It is easy to see that $f_1 < g_1 < f_2$ because $q \geq 4$ and $f_i > 1$. Suppose that $f_i < g_i < f_{i+1}$ holds for $i < N$, then $q - 2 - \frac{1}{f_{N-1}} < q - 2 - \frac{1}{g_{N-1}} < q - 2 - \frac{1}{f_N}$, namely, $f_N < g_N < f_{N+1}$. By induction, $f_i < g_i < f_{i+1}$ holds for $i \geq 1$. This complete the proof of (3). □

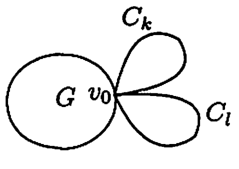

Fig. 2.2. $G_c(k, l)$

Remark 1 Let $G_c(k, l)$ denote the graph obtained by attaching two independent cycles to a vertex of G (see Fig. 2.2). In a same way as Lemma 2.1, for the Perron vector of $G_c(k, l)$, there is the same lemma indicating the relations among the coordinates corresponding to the cycles C_k or C_l.

Theorem 2.2 *In $\widehat{\theta}_n(n_1, n_2, \cdots, n_s)$, if there exist $2 \leq n_i < n_j$, then $q(\widehat{\theta}_n(n_1, n_2, \cdots, n_s)) < q(\widehat{\theta}_n(n_1, n_2, \cdots, n_i - 1, \cdots, n_j + 1, \cdots, n_s))$.*

Proof.

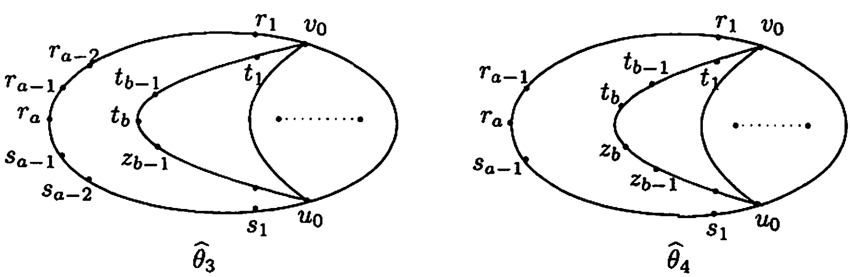

Fig. 2.3. $\widehat{\theta}(n_1, n_2, \cdots, n_s)$

Let $X_1 = (x_{v_0}, x_{u_0}, \cdots)^T$ denote the Q-Perron vector of $\widehat{\theta}_n(n_1, n_2, \cdots, n_s)$ in which x_v corresponds to vertex v.

Case 1 Meridian line $P_{n_i+1} = v_0 r_1 r_2 \cdots r_{a-1} r_a s_{a-1} \cdots s_1 u_0$ in $\widehat{\theta}_n(n_1, n_2, \cdots, n_s)$ (see Fig. 2.3). For meridian line P_{n_j+1}, there are just two subcases as follows.

Subcase 1.1 $P_{n_j+1} = v_0t_1t_2\cdots t_{b-1}t_bz_{b-1}\cdots z_1u_0$ in $\widehat{\theta}_n(n_1, n_2, \cdots, n_s)$.

Subcase 1.2 $P_{n_j+1} = v_0t_1t_2\cdots t_{b-1}t_bz_bz_{b-1}\cdots z_1u_0$ in $\widehat{\theta}_n(n_1, n_2, \cdots, n_s)$.

Whenever for Subcase 1.1 or 1.2, then $x_{r_a} \geq x_{t_b}$ by Lemma 2.1. There must be $x_{r_a} \in [x_{t_\sigma}, x_{t_{\sigma-1}}]$ $(1 \leq \sigma \leq b)$.

Let $\widehat{\theta}_1^1 = \widehat{\theta}_n(n_1, n_2, \cdots, n_s) - s_{a-1}r_a - r_ar_{a-1} - t_\sigma t_{\sigma-1} + s_{a-1}r_{a-1} + t_{\sigma-1}r_a + t_\sigma r_a$. Then

$$\begin{aligned}
&X_1^TQ(\widehat{\theta}_1^1)X_1 - X_1^TQ(\widehat{\theta}_n(n_1, n_2, \cdots, n_s))X_1\\
&= (x_{t_\sigma} + x_{r_a})^2 + (x_{r_{a-1}} + x_{t_{\sigma-1}})^2 + (x_{r_{a-1}} + x_{s_{a-1}})^2 - (x_{t_\sigma} + x_{t_{\sigma-1}})^2\\
&\quad -2(x_{r_{a-1}} + x_{r_a})^2 = 2(x_{t_\sigma}x_{r_a} + x_{r_a}x_{t_{\sigma-1}} + x_{r_{a-1}}^2 - x_{t_\sigma}x_{t_{\sigma-1}} - 2x_{r_{a-1}}x_{r_a})\\
&= 2((x_{r_a} - x_{t_\sigma})x_{t_{\sigma-1}} + x_{r_a}(x_{t_\sigma} - x_{r_{a-1}}) + x_{r_{a-1}}(x_{r_{a-1}} - x_{r_a}))\\
&\quad \geq 2((x_{r_a} - x_{t_\sigma})x_{t_{\sigma-1}} + x_{r_a}(x_{t_\sigma} - x_{r_{a-1}}) + x_{r_a}(x_{r_{a-1}} - x_{r_a}))\\
&\quad = 2((x_{r_a} - x_{t_\sigma})x_{t_{\sigma-1}} + x_{r_a}(x_{t_\sigma} - x_{r_a})) = 2(x_{r_a} - x_{t_\sigma})(x_{t_{\sigma-1}} - x_{r_a})) \geq 0.
\end{aligned}$$

This means that $q(\widehat{\theta}_1^1) \geq q(\widehat{\theta}_n(n_1, n_2, \cdots, n_s))$. If $q(\widehat{\theta}_1^1) = q(\widehat{\theta}_n(n_1, n_2, \cdots, n_s))$, then $q(\widehat{\theta}_1^1) = X_1^TQ(\theta_1^1)X_1$. From linear algebra, we get $Q(\widehat{\theta}_1^1)X_1 = q(\theta_1^1)X_1$, but $Q(\widehat{\theta}_1^1)_{r_{a-1}}X_1 > q(\widehat{\theta}_n(n_1, n_2, \cdots, n_s))x_{r_{a-1}}$, where $Q(\widehat{\theta}_1^1)_{r_{a-1}}$ denote the row in $Q(\widehat{\theta}_1^1)$ corresponding to vertex r_{a-1}, which contradicts $Q(\widehat{\theta}_1^1)X_1 = q(\widehat{\theta}_1^1)X_1$. Hence $q(\widehat{\theta}_1^1) > q(\widehat{\theta}_n(n_1, n_2, \cdots, n_s))$.

Case 2 Meridian line $P_{n_i+1} = v_0r_1r_2\cdots r_{a-1}r_as_as_{a-1}\cdots s_1u_0$ in $\widehat{\theta}_n(n_1, n_2, \cdots, n_s)$. For meridian line P_{n_j+1}, there are just two subcases as follows.

Subcase 1.1 $P_{n_j+1} = v_0t_1t_2\cdots t_{b-1}t_bz_{b-1}\cdots z_1u_0$ in $\widehat{\theta}_n(n_1, n_2, \cdots, n_s)$.

Subcase 1.2 $P_{n_j+1} = v_0t_1t_2\cdots t_{b-1}t_bz_bz_{b-1}\cdots z_1u_0$ in $\widehat{\theta}_n(n_1, n_2, \cdots, n_s)$.

Whenever for Subcase 1.1 or 1.2, then $x_{r_a} \geq x_{t_b}$ by Lemma 2.1. There must be $x_{r_a} \in [x_{t_\sigma}, x_{t_{\sigma-1}}]$ $(1 \leq \sigma \leq b)$.

Note that there must be $x_{t_b} \in [x_{r_\sigma}, x_{r_{\sigma-1}}]$ $(1 \leq \sigma \leq a)$. Let

$$\widehat{\theta}_1^1 = \widehat{\theta}_n(n_1, n_2, \cdots, n_s) - t_{b-1}t_b - z_bt_b - t_\sigma t_{\sigma-1} + z_bt_{b-1} + t_{\sigma-1}t_b + t_\sigma t_b.$$

$$X_1^TQ(\widehat{\theta}_1^1)X_1 - X_1^TQ(\widehat{\theta}_n(n_1, n_2, \cdots, n_s))X_1$$

$$= (x_{r_\sigma} + x_{t_b})^2 + (x_{r_{\sigma-1}} + x_{t_b})^2 - (x_{r_\sigma} + x_{r_{\sigma-1}})^2 - (x_{t_b} + x_{z_b})^2$$
$$= 2(x_{t_b} - x_{r_\sigma})(x_{r_{\sigma-1}} - x_{t_b}) \geq 0.$$

As Case 1, we can prove that $q(\widehat{\theta}_1^1) > q(\widehat{\theta}_n(n_1, n_2, \cdots, n_s))$. □

As Theorem 2.2, we can get

Theorem 2.3 *If $k \leq l$, then $q(G_c(k,l)) < q(G_c(k-1, l+1))$.*

Remark 2

In [2], D. Cvetković and S.K. Simić got that, for graph $G(k,l)$ $(k, l \geq 0)$ obtained from a non-trivial connected graph G by attaching pendant paths of lengths k and l at some vertex v, if $k \geq l \geq 1$, then $q(G(k,l)) > q(G(k+1, l-1))$. An interesting is that, for graph $G_c(k,l)$, namely, the two pendant paths in $G(k,l)$ are replaced by two cycles, but the conclusion for Q-spectral radius is converse.

3 Applications

By Theorem 2.2, we can get the following Theorems 3.1 and 3.2.

Theorem 3.1 *If $n_1 + n_2 + \cdots + n_s - s + 2 = n$, then $q(\widehat{\theta}_n(n_1, n_2, \cdots, n_s)) \leq q(\widehat{\theta}_n(1, 2, 2, \cdots, 2, n-s+1))$, with equality if and only if $\widehat{\theta}_n(n_1, n_2, \cdots, n_s) \cong \widehat{\theta}_n(1, 2, 2, \cdots, 2, n-s+1)$.*

Theorem 3.2 *If $n_1 + n_2 + \cdots + n_s - s + 2 = n$, then $q(\widehat{\theta}_n(n_1, n_2, \cdots, n_s)) \geq q(\widehat{\theta}_n(\kappa_1, \kappa_2, \kappa_3, \cdots, \kappa_{s-1}, \kappa_s))$ ($|\kappa_i - \kappa_j| \leq 1$ for $i, j = 1, 2, \cdots, s$), with equality if and only if $\widehat{\theta}_n(n_1, n_2, \cdots, n_s) \cong \widehat{\theta}_n(\kappa_1, \kappa_2, \kappa_3, \cdots, \kappa_{s-1}, \kappa_s)$.*

Corollary 3.3 *If $j + k + l - 1 = n$, then $q(\theta_n(j,k,l)) \leq q(\theta_n(1, 2, n-2))$, with equality if and only if $\theta_n(j,k,l) \cong \theta_n(1, 2, n-2)$.*

Corollary 3.4 *If $j+k+l-1 = n$, then $q(\theta_n(j,k,l)) \geq q(\theta_n(\kappa_1, \kappa_2, \kappa_3))$ ($|\kappa_i - \kappa_j| \leq 1$ for $i, j = 1, 2, 3$), with equality if and only if $\theta_n(j,k,l) \cong \theta_n(\kappa_1, \kappa_2, \kappa_3)$.*

By Theorem 2.3, we can get the following theorem.

Theorem 3.5 *(1)* $q(\mathscr{P}(n_1, n_2, \cdots, n_s)) \leq q(\mathscr{P}(3, 3, \cdots, 3, n-2s+2))$, *with equality if and only if* $\mathscr{P}(n_1, n_2, \cdots, n_s) \cong \mathscr{P}(3, 3, \cdots, 3, n-2s+2)$;

(2) $q(\mathscr{P}(n_1, n_2, \cdots, n_s)) \geq q(\mathscr{P}(\kappa_1, \kappa_2, \cdots, \kappa_s))$ ($|\kappa_i - \kappa_j| \leq 1$ for $i, j = 1, 2, \cdots, s$), *with equality if and only if* $\mathscr{P}(n_1, n_2, \cdots, n_s) \cong \mathscr{P}(\kappa_1, \kappa_2, \cdots, \kappa_s)$.

Acknowledgment

We offer many thanks to the referees for their kind reviews and helpful suggestions.

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