On matrices over roots of unity with vanishing permanent *

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Abstract

For any integer $m \geq 2$, let μ_m be the group of mth roots of unity. Let p be a prime and α a positive integer. For $m = p^\alpha$, it is shown that there is no $n \times n$ matrix over μ_m with vanishing permanent if $n < p$.

INTRODUCTION

The permanents of $(0, 1)$-matrices have been studied extensively in the literature for their combinatorial interpretation and significance. Recall that for any $n \times n$ matrix $A = [a_{ij}]$ over a field the permanent of A is defined by $\operatorname{per}(A) := \sum_{\pi \in S_n} a_{1\pi(1)} \cdots a_{n\pi(n)}$. For any integer $m \geq 2$, let $\zeta_m \in \mathbb{C}^\times = \mathbb{C} \setminus \{0\}$ denote a primitive mth root of unity and let $\mu_m = \langle \zeta_m \rangle$ be the cyclic group of all mth roots of unity. For any $n \geq 1$, let $M_n(\mu_m)$ denote the set of all $n \times n$ matrices $A = [a_{ij}]$ with all $a_{ij} \in \mu_m$.

In 1983 Kräuter and Seifter [6], and independently Simion and Schmidt [9], showed that there is no $n \times n$, $(1, -1)$-square matrix with vanishing permanent if $n = 2^\alpha - 1$ for some positive integer α. In [1] we generalized this result to matrices over roots of unity. More precisely, suppose that p is a prime and α is a positive integer. Then we showed that:

If $n = p^\alpha - 1$, then there is no matrix $A \in M_n(\mu_p)$, with $\operatorname{per}(A) = 0$.

In this note, by a similar argument we prove a more general result in this direction and show that:

If $n = p^\alpha - kp^{\alpha-1} - 1$ with $0 \leq k \leq p-2$, then there is no matrix $A \in M_n(\mu_p)$, with $\operatorname{per}(A) = 0$.

Also with the same method we show that:

If $n < p$ and $m = p^\alpha$, then there is no matrix $A \in M_n(\mu_m)$, with $\operatorname{per}(A) = 0$.

Actually this result is known and two different proofs of it are given in [3] (see Theorem 2, Lemma 6 and 7 of [3]), where the authors used this non-vanishing permanent result to give a stronger version of a theorem proved earlier by Alon in [2] concerning a conjecture of Snevily (see [10]) about transversals of additive Latin squares. We used these techniques in [4] to obtain some existence results.

We recall some facts from algebraic number theory. Let $m = p^\alpha$ be a prime power. It is well known that the cyclotomic field $\mathbb{Q}(\zeta_m)$ is of degree $\varphi(m) =$

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$p^{\alpha-1}(p-1)$ over $\mathbb{Q}$ and the ring of integers of $\mathbb{Q}(\zeta_m)$ is $\mathbb{Z}[\zeta_m]$ (see [7]). Also the principal ideal $(1-\zeta_m)\mathbb{Z}[\zeta_m]$ is a prime ideal of $\mathbb{Z}[\zeta_m]$, and the ideal generated by p in $\mathbb{Z}[\zeta_m]$ factorizes as

$$p\,\mathbb{Z}[\zeta_m] = (1-\zeta_m)^{\varphi(m)}\mathbb{Z}[\zeta_m].$$

PROOFS

We give the proofs of our theorems in this section. For the first one, just by a similar argument given in the proof of Theorem 2 (Part (ii)) of [1], one can show that:

Theorem 1. *Let p be a prime and α a positive integer. If $n = p^\alpha - kp^{\alpha-1} - 1$ with $0 \le k \le p-2$, then there is no matrix $A \in M_n(\mu_p)$, with* $\text{per}(A) = 0$.

Actually the proof is almost the same and under the notations of the proof of Theorem 2 (Part (ii)) of [1] which we present below, we have again $S_n > S_j$ under our assumption and this is the key point in the proof. Here we give the techniques used in the above mentioned proof to show our second result:

Theorem 2. *Let $m = p^\alpha$ be a prime power. Then there is no matrix $A \in M_n(\mu_m)$, with* $\text{per}(A) = 0$ *if $n < p$.*

Proof. Assume $A \in M_n(\mu_m)$. As the difference between any two mth roots of unity is divisible by the element $(1-\zeta_m)$, there exists a matrix $B \in M_n(\mathbb{Z}[\zeta_m])$, such that $A = J - (1-\zeta_m)B$, where J is the $n \times n$ matrix with all entries equal to one. By Theorem 1.4 of [8, p.18], we have the following equality

$$\text{per}(A) = \sum_{j=0}^{n}(-1)^j(n-j)!\,(1-\zeta_m)^j\text{per}_j(B)$$

where $\text{per}_j(B)$ denotes the sum of permanents of all submatrices of order j of B (we set $\text{per}_0(B) = 1$). For any $\delta \in \mathbb{C}^\times$, let $\text{ord}_{(1-\zeta_m)}\delta$ be the largest $\beta \ge 0$ such that $(1-\zeta_m)^\beta|\delta$. Now if $\text{per}_j(B) \ne 0$ we find a lower bound for $\text{ord}_{(1-\zeta_m)}P_j(B)$, where we have set $P_j(B) = (-1)^j(n-j)!\,(1-\zeta_m)^j\text{per}_j(B)$ for $1 \le j \le n$. Recall that for $m = p^\alpha$ and $\delta \in \mathbb{C}^\times$ we have $\text{ord}_{(1-\zeta_m)}\delta = \varphi(m)\ \text{ord}_p\delta$. So for $1 \le j \le n$, we have

$$\text{ord}_{(1-\zeta_m)}P_j(B) \ge p^{\alpha-1}(p-1)\,\text{ord}_p(n-j)! + j.$$

It is well known that for any natural number k, $\text{ord}_p(k!) = \dfrac{k-S_k}{p-1}$, where S_k is the sum of the digits of k written to the base p (see [5, p.7]). Hence we obtain

$$\text{ord}_{(1-\zeta_m)}P_j(B) \ge p^{\alpha-1}(n-j-S_{n-j}) + j.$$

On the other hand, we have $\text{ord}_{(1-\zeta_m)}(n!) = p^{\alpha-1}(p-1)\text{ord}_p(n!) = p^{\alpha-1}(n-S_n)$. Now, as $n < p$, it is clear that for any $1 \le j \le n$ we have $S_j = j$. Therefore

$\text{ord}_{(1-\zeta_m)}(n!) = 0$ while for any j, $1 \leq j \leq n$, $\text{ord}_{(1-\zeta_m)} P_j(B) \geq j > 0$. As $\text{per}(A) = n! + \sum_{j=1}^{n} P_j(B)$, we cannot have $\text{per}(A) = 0$. This completes the proof of Theorem 2.

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