On (Super) Vertex-Graceful Labeling Of Graphs *

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Abstract. In the paper, we discuss properties of the (super) vertex-graceful labeling of cycle C_n , crown graph $C_n \odot K_1$ and generalized crown graph $C_n \odot K_{1,t}$ and prove that C_n , , $C_n \odot K_1$ and $C_n \odot K_{1,t}$ are vertex-graceful if n is odd; C_n is super vertex-graceful if $n \neq 4, 6$; and $C_n \odot K_1$ is super vertex-graceful if n is even. Moreover, we propose two conjectures on (super)vertex-graceful labeling.

Keywords: Crown Graph; Generalized Crown Graph; Vertex-Graceful Graph; Super Vertex-Graceful Graph.

1. Introdution

In 1967, Rosa [15] first introduced the concept of graph labeling and proved some interesting results. In 1980, Graham and Sloane [9] further developed the methods and new notations on graph labeling. Up to now, it has been discovered that theory of labeling graphs can be applied to coding theory, X-ray crystallography, radar, astronomy, circuit design, communication network addressing, data base management, etc [2, 3, 5, 14, 19].

In 2005, Lee, Pan and Tsai [12] called a graph G with p vertices and q edges is vertex-graceful if there is a bijection $f : V(G) \to \{1, 2, \cdots, p\}$

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such that the induced mapping g from E to Z_q defined by $g(uv) = (f(u) + f(v))(mod q)$ is a bijection. In 2006, Lee and Wei [13] defined a graph $G(V, E)$ to be super vertex-graceful if there is a bijection f from V to $\{0, \pm 1, \pm 2, \cdots, \pm \frac{|V|-1}{2}\}$ when $|V|$ is odd and from V to $\{\pm 1, \pm 2, \cdots, \pm \frac{|V|}{2}\}$ when $|V|$ is even such that the induced edge labeling g defined by $g(uv) = f(u)+f(v)$ over all edges uv is a bijection from E to $\{0, \pm 1, \pm 2, \cdots, \pm \frac{|E|-1}{2}\}$ when $|E|$ is odd and from E to $\{\pm 1, \pm 2, \cdots, \pm \frac{|E|}{2}\}$ when $|E|$ is even. They showed that $K_{1,n} \times P_2$ are not super vertex-graceful for n odd; for $n \geq 3$, $P_n^2 \times P_2$ is super vertex-graceful if and only if $n = 3, 4, 5$; $P_{n_1} \times P_{n_2} \times \cdots \times P_{n_m}$ is not super vertex-graceful for each of $m, n_1, n_2, \cdots .n_m$ at least 3; and $C_n \times C_m$ is not super vertex-graceful. They conjecture that $P_n \times P_n$ is super vertex-graceful for $n \geq 3$.

There are two interesting graphs cycle C_n and $C_n \odot K_1$ in the theory of graph labeling, which have been extensively studied. For example, C_n is graceful if and only if $n \equiv 0 or 3 (mod 4)$[15]; C_n is almost graceful[1, 16]; C_n is harmonious if and only if $n \equiv 1$ *or* $3(mod4)$[9]; C_n is odd graceful if and only if n is even[8]; L(2,1)-labeling of C_n[11] and $L(2,1)$-labeling of C_n [6]have been investigated. For crown graph $C_n \odot K_1$, it is known that $C_n \odot K_1$ is graceful[4]; $C_n \odot K_1$ is odd graceful if only if n is even[8]; $C_n \odot K_1$ is harmonious[10]; $C_n \odot K_1$ is integral sum[18]. Moreover, for $C_n \odot K_{1,t}$, it is known that $C_n \odot K_{1,t}$ is super edge-graceful[17] and is odd graceful when n is even [7].

In the paper, we prove that C_n, $C_n \odot K_1$ and $C_n \odot K_{1,t}$ are are vertex-graceful when n is odd; C_n is super vertex-graceful when $n \neq 4, 6$; and $C_n \odot K_1$ is super vertex-graceful when n is even. Two conjectures that C_n , $C_n \odot K_1$ and $C_n \odot K_{1,t}$ are not vertex-graceful when n is even; $C_n \odot K_1$ is not super vertex-graceful when n is odd are proposed.

2. Preliminary

Definition 2.1. Let a graph $G(V, E)$ that has p vertices and q edges, if there is a f bijection from V to $\{1, 2, \cdots, p\}$ such that the induced mapping g from E to Z_q defined by $g(uv) = (f(u) + f(v))(mod q)$ $(u, v \in V,\ uv \in E)$ is a bijection, then graph $G(V, E)$ is called $vertex-graceful\ graph$.

Definition 2.2. Let a graph $G(V, E)$ that has p vertices and q edges, if there is a bijection f from V to $\{0, \pm1, \pm2, \cdots, \pm\frac{|V|-1}{2}\}$ when $|V|$ is odd and from V to $\{\pm1, \pm2, \cdots, \pm\frac{|V|}{2}\}$ when $|V|$ is even such that the induced edge labeling g defined by $g(uv) = f(u) + f(v)$ $(u, v \in V,\ uv \in E)$ over all edges uv is a bijection from E to $\{0, \pm1, \pm2, \cdots, \pm\frac{|E|-1}{2}\}$ when $|E|$ is odd and from E to $\{\pm1, \pm2, \cdots, \pm\frac{|E|}{2}\}$ when $|E|$ is even, then graph $G(V, E)$ is called $super\ vertex-graceful\ graph$.

Definition 2.3. *The crown graph* is obtained by joining a single pendant edge to each vertex of C_n and denoted by $C_n \odot K_1$.

Definition 2.4. *The generalized crown graph* is obtained by joining $t(t > 1)$ pendant edges to each vertex of C_n and denoted by $C_n \odot K_{1,t}$.

In order to prove our results, we introduce the follow operations: let $A = \{a_1, a_2, \cdots, a_m\}$, $B = \{b_1, b_2, \cdots, b_m\}$, be two sets and c is real number. If f is a mapping, then denote $f(A) = \{f(a_1), f(a_2), \cdots, f(a_m)\}$, $A + c = \{a_1 + c, a_2 + c, \cdots, a_m + c\}$.

The Graph C_n is illustrated in Fig.1.

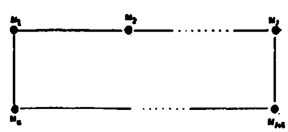

Fig.1. C_n

The Graph $C_n \odot K_1$ is illustrated in Fig.2.

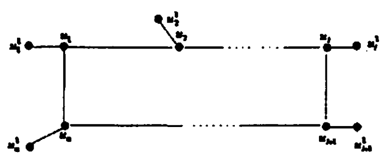

Fig.2. $C_n \odot K_1$

The Graph $C_n \odot K_{1,t}$ is illustrated in Fig.3.

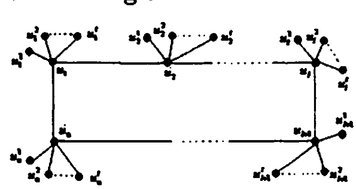

Fig.3. $C_n \odot K_{1,t}$

3. Vertex-graceful Graph

Theorem 3.1. If n is odd, then C_n is vertex-graceful.

Proof For C_n, we suppose $S = \{u_1, u_2, \cdots, u_n\}$, $S_1 = \{u_2, u_3, \cdots, u_n, u_1\}$, let:

$$f(S) = \{1, 3, \cdots, n, 2, 4, \cdots, n-1\},$$

it is easy to see that there is a vertex labelings on C_n in accordance with the definition 2.1. For the edges labelings on C_n,

Case 1. $n \equiv 1(mod4)$, then let
$g(E) = (f(S)+f(S_1))(modn) = \{4, 8, 12, \cdots, n-1, 3, 7, \cdots, n-2, 2, 6, \cdots, n-3, 1, 5, \cdots, n-4, 0\}(modn)$.

Case 2. $n \equiv 3(mod4)$, then let
$g(E) = (f(S)+f(S_1))(modn) = \{4, 8, 12, \cdots, n-3, 1, 5, \cdots, n-2, 2, 6, \cdots, n-1, 3, 7, \cdots, n-4, 0\}(modn)$. No matter what $n \equiv 1(mod4)$ or $n \equiv 3(mod4)$, the edges labelings on C_n are different. Moreover, the maximum value is $(n-1)(modn)$ and the minimum value is $0(modn)$. Hence the mapping g is a bijection from E to Z_n. Therefore C_n is vertex-graceful for n is odd.

Theorem 3.2. If n is odd, then $C_n \odot K_{1,t}$ $(t > 1)$ is vertex-graceful.

Proof Clearly, there are $n(t+1)$ vertices and $n(t+1)$ edges on $C_n \odot K_{1,t}$, Suppose $S = \{u_1, u_2, \cdots, u_n\}$, $S_i = \{u_i^1, u_i^2, \cdots, u_i^t\}$ $(1 \leq i \leq n, t > 1)$. Let

$$f(S) = \{t+1, 2(t+1), \cdots, n(t+1)\},$$

$$f(S_1) = \{n(t+1)-1, n(t+1)-2, \cdots, n(t+1)-t\},$$

$$f(S_n) = \{t+2, t+3, \cdots, 2t+1\},$$

for $2 \leq i \leq \frac{n+1}{2}$, $f(S_i) = f(S_{i-1}) - 2(t+1)$,
for $\frac{n+3}{2} \leq i \leq n-1$, $f(S_i) = f(S_{i+1}) + 2(t+1)$.

By the above rules, we get that the vertex labelings on cycle of $C_n \odot K_{1,t}$ are n numbers that they are $t+1$ number of times, the maximum value is $n(t+1)$, the minimum value is $t+1$. For the vertex u_i^j $(1 \leq i \leq n, 1 \leq j \leq t)$, it is easy to see that their labelings are different from the above rules, and their labelings are difference with the labelings of the vertex labelings on cycle of $C_n \odot K_{1,t}$, the maximum value is $n(t+1)-1$, the minimum value is 1. So f is a bijection from V to $\{1, 2, \cdots, n(t+1)\}$.

For edges, according to definition 2.1., the edge labeling set on cycle is $\{0, t+1, 2(t+1), \cdots, (n-1)(t+1)\}(modn(t+1))$; for edge $u_i u_i^j$ $(1 \leq i \leq n, 1 \leq j \leq t)$, the labeling sets are:

$$\{t, t-1, \cdots, 1\}(modn(t+1)),$$

$$\{n(t+1)-1, n(t+1)-2, \cdots, n(t+1)-t\}(modn(t+1)),$$

$$\{(n-2)(t+1)-1, (n-2)(t+1)-2, \cdots, (n-2)(t+1)-t\}(modn(t+1)),$$

$$\cdots\cdots$$

$$\{2t+1, 2t, \cdots, t+2\}(modn(t+1)).$$

Hence we can get that these edge labelings are difference each other, and differ with the labelings on cycle, where the maximum value is $(n(t+1)-1)(modn(t+1))$, the minimum value is $0(modn(t+1))$, then according to definition 2.1, the mapping g defined by $g(uv) = (f(u)+f(v))(modn(t+1))$ is a bijection from E to $Z_{n(t+1)}$. Hence $C_n \odot K_{1,t}$ is vertex-graceful for n is odd.

Corollary 3.1. If n is odd, then $C_n \odot K_1$ is vertex-graceful.
Proof In the proof of theorem 3.2., let $t = 1$. It is easy to see that the assertion holds.

4. Super Vertex-graceful Graph

Lemma 4.1. C_4, C_6 are not super vertex-graceful.
Proof Suppose C_4 is super vertex-graceful, then the vertex labelings set is $\{\pm 1, \pm 2\}$, but not matter whether labeling of the vertices of C_4, one of the following cases is occur, there is 0 or ± 3 in the edge labelings, contradiction.

Suppose C_6 is super vertex-graceful. Let $f(u_1) = 3$, then $f(u_2) = -1$ or -2.

Case 1. If $f(u_2) = -1$, then $f(u_3) = 2$ or -2. Suppose $f(u_3) = 2$,then $f(u_4) = 1$ or -3. If $f(u_4) = 1$, then $f(u_5), f(u_6)$ are $-3, -2$ or $-2, -3$, hence the edge u_5u_6 labeling is -5, contradiction. If $f(u_4) = -3$, then the edge u_4u_5 labeling is -5 or $g(u_3u_4) = g(u_5u_6)$, contradiction. If $f(u_3) = -2$, then the other vertices labelings are $1, 2, -3$ or $1, -3, 2$, hence $g(u_6u_1)$ is 0 or 5, contrsdiction.
Case 2. If $f(u_2) = -2$, then $f(u_3) = 1$ or -1. Suppose $f(u_3) = 1$, then $f(u_4) = 2$ or -3. If $f(u_4) = 2$, then $g(u_5u_6) = -4$, contradiction; if $f(u_4) = -3$, then $g(u_4u_5) = -1$ or-4, contradiction; if $f(u_3) = -1$, then the other vertices labelings are $2, 1, -3$, or $2, -3, 1$, Thus $g(u_6u_1) = 0$ or 4, contradiction.

Thus, C_4 , C_6 are not super vertex-graceful.

Theorem 4.1. If $n \neq 4, 6$, then C_n is super vertex-graceful.

Proof We consider the following two cases:

Case 1. n is odd. Let $n = 2m + 1, m = 1, 2, \cdots$, we consider the following two subcases:

subcase 1.1. m is odd. Define f as follows:
$f : \{u_2, u_4, \cdots, u_{\frac{n+1}{2}}\} \cup \{u_1, u_3, \cdots, u_{\frac{n-1}{2}}\} \rightarrow \{\frac{n-1}{2}, \frac{n-1}{2} - 1, \cdots, \frac{n-3}{4} + 1\} \cup \{0, -1, -2, \cdots, -\frac{n-3}{4}\}$,
for the other vertices , let $f(u_{n+2-i}) = -f(u_i)$ $2 \leq i \leq \frac{n+1}{2}$. Thus we get that these labelings are different, $\max\limits_{1\leq i\leq n} |f(u_i)| = \frac{n-1}{2}$, $\min\limits_{1\leq i\leq n} |f(u_i)| = 0$. Hence, the mapping f is a bijection from V to $\{0, \pm1, \pm2, \cdots, \pm\frac{n-1}{2}\}$. Next, we consider the edge labelings, according to the above rule, we get the edge labelings as following in proper order: $\frac{n-1}{2}, \frac{n-1}{2} - 1, \cdots, 1, 0, -1, -2, \cdots, -\frac{n-1}{2}$, hence, there is a bijection from E to $\{0, \pm1, \pm2, \cdots, \pm\frac{n-1}{2}\}$.

subcase 1.2. m is even. Define f as follows:
$f : \{u_2, u_4, \cdots, u_{\frac{n-1}{2}}\} \cup \{u_1, u_3, \cdots, u_{\frac{n+1}{2}}\} \rightarrow \{\frac{n-1}{2}, \frac{n-1}{2} - 1, \cdots, \frac{n-1}{4} + 1\} \cup \{0, -1, -2, \cdots, -\frac{n-1}{4}\}$,
for the other vertices , let $f(u_{n+2-i}) = -f(u_i)$ $2 \leq i \leq \frac{n+1}{2}$, we can get the same conclusion by following the proof of the subcase 1.1.

Case 2. n is even, We consider the following four subcases.

subcase 2.1. $n \equiv 0(mod 8)$. For the vertex $u_i (1 \leq i \leq \frac{n}{2})$, define f as follows:
$f : \{u_1, u_3, \cdots, u_{\frac{n-4}{4}}\} \cup \{u_2, u_4, \cdots, u_{\frac{n}{4}}\} \cup \{u_{\frac{n}{4}+1}, u_{\frac{n}{4}+3}, \cdots, u_{\frac{n}{2}-1}\} \cup \{u_{\frac{n}{4}+2},$

$u_{\frac{n}{4}+4}, \cdots, u_{\frac{n}{2}}\} \to \{\frac{n}{2}, \frac{n}{2}-1, \cdots, \frac{3n}{8}+1\} \cup \{-1, -2, \cdots, -\frac{n}{8}\} \cup \{-\frac{3n}{8}, -\frac{3n}{8}+1, \cdots, -\frac{n}{4}-1\} \cup \{\frac{n}{8}, \frac{n}{8}+1, \cdots, \frac{n}{4}\}$,
for the other vertices, let $f(u_{\frac{n}{2}+i}) = -f(u_i) \quad (1 \leq i \leq \frac{n}{2})$, by the above rule, it is easy to see that the vertex labelings are different if their subscripts are different, $\max\limits_{1\leq i\leq n} |f(u_i)| = \frac{n}{2}$, $\min\limits_{1\leq i\leq n} |f(u_i)| = 1$. So the mapping f is a bijection from V to $\{\pm 1, \pm 2, \cdots, \pm\frac{|E|}{2}\}$.

Now, we discuss the edge labelings, following the definition 2.2. we get the mapping g :
$g : \{u_1u_2, u_2u_3, \cdots, u_{\frac{n}{2}}u_{\frac{n}{2}+1}\} \to \{\frac{n}{2}-1, \frac{n}{2}-2, \cdots, \frac{n}{4}+1, -\frac{n}{2}, -\frac{n}{4}+1, -\frac{n}{4}+2, \cdots, -1, -\frac{n}{4}\}$, the other edges labelings are the opposite numbers of the above edges labelings in proper order, from these labelings, we can deduce that these labelings are different and $\max\limits_{1\leq i\leq n} |g(u_iu_{i+1})| = \frac{n}{2}$, $\min\limits_{1\leq i\leq n} |g(u_iu_{i+1})| = 1 \quad (u_{n+1} = u_1)$, so the mapping g is a bijection from E to $\{\pm 1, \pm 2, \cdots, \pm\frac{|E|}{2}\}$.

subcase 2.2. $n \equiv 2(mod8)$. For the vertex $u_i(1 \leq i \leq \frac{n}{2})$, define f as follows:
$f : \{u_1, u_3, \cdots, u_{\frac{n-2}{4}-1}\} \cup \{u_2, u_4, \cdots, u_{\frac{n-2}{4}}\} \cup \{u_{\frac{n-2}{4}+1}, u_{\frac{n-2}{4}+3}, \cdots, u_{\frac{n}{2}}\} \cup \{u_{\frac{n-2}{4}+2}, u_{\frac{n}{4}+4}, \cdots, u_{\frac{n-2}{2}}\} \to \{\frac{n}{2}, \frac{n}{2}-1, \cdots, \frac{3n+2}{8}+1\} \cup \{-1, -2, \cdots, -\frac{n-2}{8}\} \cup \{-\frac{3n+2}{8}, -(\frac{3n+2}{8}+1), \cdots, -\frac{n}{2}\} \cup \{\frac{n-2}{8}, \frac{n-2}{8}-1, \cdots, 1\}$,
$f : \{u_{\frac{n}{2}+1}\} \cup \{u_{\frac{n}{2}+2}, u_{\frac{n}{2}+4}, \cdots, u_{\frac{3n-2}{4}}\} \cup \{u_{\frac{n}{2}+3}, u_{\frac{n}{2}+5}, \cdots, u_{\frac{3n+2}{4}}\} \cup \{u_{\frac{3n+2}{4}+1}, u_{\frac{3n+2}{4}+3}, \cdots, u_{n-1}\} \cup \{u_{\frac{3n+2}{4}+2}, u_{\frac{3n+2}{4}+4}, \cdots, u_n\} \to \{\frac{n+2}{4}\} \cup \{\frac{n-2}{4}, \frac{n-2}{4}-1, \cdots, \frac{n+6}{8}\} \cup \{-\frac{n+2}{4}, -\frac{n+2}{4}-1, \cdots, -\frac{3n-6}{8}\} \cup \{\frac{3n+2}{8}, \frac{3n+2}{8}-1, \cdots, \frac{n+2}{4}+1\} \cup \{-\frac{n+6}{8}, -(\frac{n+6}{8}+1), \cdots, -\frac{n-2}{4}\}$,
by the above rule, we can get that these vertex labelings satisfying the demand from definition 2.2. Hence the mapping f is a bijection from V to $\{\pm 1, \pm 2, \cdots, \pm\frac{|E|}{2}\}$.

For the edges, there is:
$g : \{u_1u_2, u_2u_3, \cdots, u_nu_1\} \to \{\frac{n}{2}-1, \frac{n}{2}-2, \cdots, \frac{n}{4}+1, -\frac{n}{2}, -\frac{n+2}{4}, -(\frac{n}{4+2}+1), \cdots, -\frac{n}{2}+1, -\frac{n+2}{4}+1, \frac{n}{2}, -1, -2, \cdots, -\frac{n+2}{4}+2, 1, \frac{n-2}{4}, \frac{n-2}{4}-1, \cdots, 2, \frac{n+2}{4}\}$,
we can see that these labelings are different and $\max\limits_{1\leq i\leq n} |g(u_iu_{i+1})| = \frac{n}{2}$, $\min\limits_{1\leq i\leq n} |g(u_iu_{i+1})| = 1 \quad (u_{n+1} = u_1)$. So the mapping g is a bijection from E to $\{\pm 1, \pm 2, \cdots, \pm\frac{|E|}{2}\}$.

For $n \equiv 4, 6(mod8)$, by the same argument in subcase 2.2, the assertion holds.

subcase 2.3. $n \equiv 4(mod8)$ $(n \neq 12)$. Let

$f : \{u_1, u_2, u_3\} \cup \{u_5, u_7, \cdots, u_{\frac{n}{2}+3}\} \cup \{u_4, u_6, \cdots, u_{\frac{n}{2}+2}\} \cup \{u_{\frac{n}{2}+4}, u_{\frac{n}{2}+5}, u_{\frac{n}{2}+6}, u_{\frac{n}{2}+7}\} \rightarrow \{\frac{n}{2}, 1, \frac{n}{2}-1\} \cup \{-\frac{n}{2}, -\frac{n}{2}+1, \cdots, -(\frac{n}{4}+1)\} \cup \{1, 2, \cdots, \frac{n}{4}\} \cup \{-\frac{n}{4}+1, \frac{n}{4}+2, -\frac{n}{4}, \frac{n}{4}+1\}$,

$f(u_i) = f(u_{i-4}) + 2$ $(\frac{n}{4}+8 \leq i \leq n, i \neq n-1), f(u_{n-1}) = \frac{n}{2} - 2$.

If $n = 12$, we define the vertex labelings in proper order: $6, -1, 5, 1, -6, 2, -5, 3, -4, -2, 4, -3$.

subcase 2.4. $n \equiv 6(mod8)$ $(n \neq 14, 22)$. Let

$f : \{u_1, u_2, u_3\} \cup \{u_5, u_7, \cdots, u_{\frac{n}{2}+2}\} \cup \{u_4, u_6, \cdots, u_{\frac{n}{2}+3}\} \cup \{u_{\frac{n}{2}+4}, u_{\frac{n}{2}+5}, \cdots, u_{\frac{n}{2}+9}\} \rightarrow \{\frac{n}{2}, 1, \frac{n}{2}-1\} \cup \{-\frac{n}{2}, -\frac{n}{2}+1, \cdots, -(\frac{n}{4}+1)\} \cup \{1, 2, \cdots, \frac{n+2}{4}\} \cup \{-\frac{n+2}{4}+1, -\frac{n+2}{4}, \frac{n+2}{4}+2, -\frac{n+2}{4}+2, \frac{n+2}{4}+1, -\frac{n+2}{4}+2\}$,

$f(u_i) = f(u_{i-4}) + 2$ $(\frac{n}{4}+10 \leq i \leq n, i \neq n-6)$,

$f : \{u_{n-5}, u_{n-4}, \cdots, u_n\} \rightarrow \{\frac{n}{2}-3, -4, \frac{n}{2}-4, -2, \frac{n}{2}-2, -3\}$.

If $n = 14$, we define the vertex labelings in proper order: $7, -1, 6, 1, -7, 2, -6, 3, -5, 4, -2, 5, -4, -3$.

If $n = 22$, we define the vertex labelings in proper order: $11, -1, 10, 1, -11, 2, -10, 3, -9, 4, -8, 5, -7, 6, -5, -6, 8, -4, 7, -2, 9, -3$.

Theorem 4.2. If n is even, then $C_n \odot K_1$ is super vertex-graceful.

Proof Suppose $S_1 = \{u_1, u_2, \cdots, u_n\}, S_2 = \{u_2, u_3, \cdots, u_n, u_1\}, S_3 = \{u_1^1, u_2^1, \cdots, u_n^1\}, E_1 = \{u_1u_2, u_2u_3, \cdots, u_nu_1\}, E_2 = \{u_iu_i^1, 1 \leq i \leq n\}$, there are $2n$ vertices and $2n$ edges.

Case 1. $n \equiv 0(mod4)$, from u_1, we label the vertices u_i $(1 \leq i \leq n)$ on the subscript is added in proper order, we divide them into four parts that the vertices number is same. For the first part, we label in proper order$1, 3, \cdots, \frac{n}{2} - 1$; for the second part, we label in proper order: $\frac{n}{2}, \frac{n}{2} - 2, \cdots, 2$; foe the third part and the forth part, they are the opposite numbers of the first part and the second part respective. For u_i^1 $(1 \leq i \leq n)$, from u_1^1 , we label the vertices them on the subscript is add in proper order, for u_i^1 $(1 \leq i \leq \frac{n}{4} + 1)$ we label in proper order: $n - 1, -n, -n + 2, \cdots, -\frac{n}{2} - 2$, for u_i^1 $(\frac{n}{4} + 2 \leq i \leq \frac{n}{2})$, we label in proper order: $-\frac{n}{2} - 1, -\frac{n}{2} - 3, \cdots, -n + 3$, to u_i^1 $(\frac{n}{2} + 1 \leq i \leq n), f(u_i) = -f(u_{i-\frac{n}{2}}$, thus we get the labelings of all vertices on $C_n \odot K_1$, which satisfy: $f(u_i) \in \{\pm 1, \pm 2, \cdots, \pm \frac{n}{2}\}$, where there are n differ numbers,

$\max\limits_{1\le i\le n}|f(u_i)|=\frac{n}{2}$, $\min\limits_{1\le i\le n}|f(u_i)|=1$, $f(u_i^1)\in\{\pm(\frac{n}{2}+1),\pm(\frac{n}{2}+2),\cdots,\pm n\}$, there are n differ numbers, $\max\limits_{1\le i\le n}|f(u_i^1)|=n$, $\min\limits_{1\le i\le n}|f(u_i)|=\frac{n}{2}+1$. From the above rule, we can get that these vertex labelings satisfying the conditions of definition 2.2. Hence, the mapping f is a bijection from V to $\{\pm1,\pm2,\cdots,\pm n\}$.

Now, we cosider the edge labelings on $C_n\odot K_1$. According to definition 2.2, we get :
$g(E_1)=f(S_1)+f(S_2)=\{4,8,\cdots,n-4,n-1,n-2,n-6,\cdots,10,6,1,-4,-8,\cdots,-n+4,-n+1,-n+2,-n+6,\cdots,-10,-6,-1\}$, expect the edge $u_nu_1, u_{\frac{n}{4}}u_{\frac{n}{4}+1}, u_{\frac{n}{2}}u_{\frac{n}{2}+1}, u_{\frac{3n}{4}}u_{\frac{3n}{4}+1}$ labelings are $-1,n-1,1,-n+1$, the others are even, and the maximum of absolute values is $n-2$ the minimum of absolute value is 4
$g(E_2)=f(S_1)+f(S_3)=\{n,-n+3,-n+7,\cdots,-5,-2,-3,-7,\cdots,-n+5,-n,n-3,n-7,\cdots,5,2,3,7,$
$\cdots,n-5\}$, expect $u_1u_1^1, u_{\frac{n}{4}+1}u^1_{\frac{n}{4}+1}, u_{\frac{n}{2}+1}u^1_{\frac{n}{2}+1}, u_{\frac{3n}{4}+1}u^1_{\frac{3n}{4}+1}$ are $n,-2-n,2$, the others are odd, and the maximum of absolute values is $n-3$, the minimum of absolute value is 3, so these edge labelings satisfy the demand from definition 2.2.

Case 2. $n\equiv 2(mod4)$. We define:
$f:\{u_1,u_2,\cdots,u_{\frac{n+2}{4}}\}\cup\{u_{\frac{n+2}{4}+1},u_{\frac{n+2}{4}+2},\cdots,u_{\frac{n}{2}}\}\to\{1,3,\cdots,\frac{n}{2}\}\cup\{\frac{n}{2}-1,\frac{n}{2}-3,\cdots,2\}$,
for the others vertices, their labelings are opposite numbers of the first $\frac{n}{2}$ vertices in proper order. Similar to case 1, we get the labelings of the vertices on cycle are $\{\pm1,\pm2,\cdots,\pm\frac{n}{2}\}$ there are n differ numbers.
$f:\{u_1^1,u_2^1,\cdots,u_{\frac{n+2}{4}}{}^1\}\cup\{u^1_{\frac{n+2}{4}+1},u^1_{\frac{n+2}{4}+2},\cdots,u^1_{\frac{n}{2}}\}\to\{n-1,-n,-n+2,\cdots,-\frac{n}{2}-1\}\cup\{-\frac{n}{2}-2,-\frac{n}{2}-4,\cdots,-n+3\}$,
for the others vertices, their labelings are opposite numbers of the first $\frac{n}{2}$ vertices in proper order. Thus we get the vertex labelings satisfy the demand from definition 2.2.

Now, we consider the edge labelings.
$g(E_1)=f(S_1)+f(S_2)=\{4,8,\cdots,n-2,n-1,n-4,n-8,\cdots,10,6,1,-4,-8,\cdots,-n+2,-n+1,-n+4,-n+8,\cdots,-10,-6,-1$, expect $u_nu_1, u_{\frac{n+2}{4}}u_{\frac{n+2}{4}+1}, u_{\frac{n}{2}}u_{\frac{n}{2}+1}, u_{\frac{3n+2}{4}}u_{\frac{3n+2}{4}+1}$ labelings are $-1,n-1,1,-n+1$, the others are even, and the maximum of absolute values is $n-2$ the minimum of absolute

value is 4.

$g(E_2) = f(S_1) + f(S_3) = \{n, -n+3, -n+7, \cdots, -3, -2, -5, -9, \cdots, -n+5, -n, n-3, n-7, \cdots, 3, 2, 5, 9, \cdots, n-5\}$, expect $u_1u_1^1, u_{\frac{n+2}{4}+1}u^1_{\frac{n+2}{4}+1}, u_{\frac{n}{2}+1}u^1_{\frac{n}{2}+1}, u_{\frac{3n+2}{4}+1}u^1_{\frac{3n+2}{4}+1}$ labelings are $n, -2-n, 2$, the others are odd, and the maximum of absolute values is $n-3$, the minimum of absolute value is 3, so these edge labelings satisfy the demand from definition 2.2.

Hence, if n is even, $C_n \odot K_1$ is super vertex-graceful.

Conjecture 1. $C_n \odot K_1$ and $C_n \odot K_{1,t}$ are not vertex-graceful for even n

Conjecture 2. $C_n \odot K_1$ is not super vertex-graceful for odd n.

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