Proper Nearly Perfect Sets in Graphs

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Abstract

In this paper we present a new combinatorial problem, called the Nearly Perfect Bipartition Problem, which is motivated by a computer networks application. The leads to a new graph parameter $PN_p(G)$ which equals the maximum cardinality of a proper nearly perfect set. We show that the corresponding decision problem is NP-hard, even when restricted to graphs of diameter 3. We present several bounds for $PN_p(G)$ and determine the value of $PN_p(G)$ for several classes of graphs.

1 Introduction

Upgrading computer networks to keep up with rapidly developing software, and optimizing the software to meet customer needs are important tasks. When upgrading a network, dealing with a single system is much easier

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than dealing with the entire network, since upgrading an entire network requires complex decisions about performance and financial issues. Even when one decides not to upgrade the entire network, there is still the difficulty of resolving conflicts between the newly upgraded components and the old components. One solution to this problem is to find parts of the network that are almost isolated from the rest of the network, that is, parts of the network having a very small number of connections to the rest of the network. We reformulate this as the following problem:

Nearly Perfect Bipartition Problem. *Given a graph G, find a bipartition $\pi = \{S, V - S\}$ such that (i) neither S nor $V - S$ is empty, (ii) every vertex in $V - S$ has at most one neighbor in S, and (iii) the size of S is as large as possible.*

In the remainder of this section we review related approaches to this problem in the graph theory literature.

As one of the key concepts in modern graph theory, the concept of domination, has prompted many definitions and parameters of graphs [18], for example perfect dominating sets [22, 16, 13, 6, 3, 4], efficient dominating sets [1, 2] and nearly perfect sets. Nearly perfect sets, along with perfect dominating sets, were defined for the first time in [10], but were not studied there. In a graph G, a set of vertices S is *nearly perfect* if every vertex in $V(G) \setminus S$ is adjacent to at most one vertex in S.

Naturally, and similar to many other topics in graph theory, finding nearly perfect sets with minimum or maximum cardinalities in a graph are the first problems to be considered. However, these two problems are trivial since a minimum cardinality set is the empty set, while a maximum cardinality set is $V(G)$.

To overcome this difficulty, J. E. Dunbar et al. [12] considered 1-maximal and 1-minimal nearly perfect sets and introduced two new parameters, $N_p(G)$ and $n_p(G)$, as the maximum cardinality of a 1-minimal and the minimum cardinality of a 1-maximal nearly perfect set in G, respectively. Following B. Bollobás et al. [5] they called a nearly perfect set 1-minimal if for every vertex $u \in S$, the set $S \setminus \{u\}$ is not nearly perfect and similarly S is called to be 1-maximal if for every vertex u in $V(G) \setminus S$, $S \cup \{u\}$ is not a nearly perfect set. They calculated n_p for some classes of graphs such as paths and cycles. They proved that the decision problem for $n_p(G)$ is NP-complete, while $N_p(G)$ can be calculated in polynomial time for any graph G.

In this paper, we introduce a new parameter related to nearly perfect sets. This parameter, $PN_p(G)$, is defined as the maximum cardinality of a *proper nearly perfect set*, that is a proper subset of vertices which is a nearly perfect set.

If one considers $(S, V(G)\setminus S)$ as a bipartition of the vertices of the graph G, it is perfectly clear that another way for looking at this problem will be in bipartition context.

In fact, we are focusing on bipartitions of vertices (A, B) such that every vertex in A has at most one neighbor in B. This kind of decomposition (bipartition) with a forbidden subgraph ($K_{1,2}$ in our case) is studied in [23] where it is shown that a $K_{1,2}$-free decomposition can be found in polynomial time. Since A and B in every bipartition (A, B) are nonempty, finding a proper nearly perfect set in a graph can be seen as a bipartition problem. Thus the maximum cardinality of Bs in a $K_{1,2}$-free decomposition is another way of defining PN_p.

Obviously, the Nearly Perfect Bipartition Problem (A, B), as stated above, is a special case of a more general problem where any vertex in A can be adjacent to at most i vertices in B. Due to this generalized Nearly Perfect Bipartition Problem one can define generalized PN_p numbers, PN_p^i. Studying these parameters is potentially an attractive subject but here we restricted ourselves to the special case where $i = 1$.

In Section 2 we consider some basic properties of PN_p and determine its value for several classes of graphs. In Section 3 we consider the concept of the nearly perfect closure of a set of vertices. In Section 4 we consider the value of PN_p in terms of several other parameters of graphs, and in Section 5 we show that the decision problem corresponding to $PN_p(G)$ is NP-hard.

2 Definitions and Preliminaries

Throughout this paper we will assume that all graphs are finite, simple, and undirected. We use [11] for terminology and notations not defined here.

Let $G = (V, E)$ be a graph with $|V| = n$. For any nonempty subset $S \subseteq V(G)$, the subgraph of G induced by S is the graph $G[S] = (S, E \cap (S \times S))$. For any vertex $v \in V(G)$, the open neighborhood of v is the set $N(v) = \{u : uv \in E\}$, while the closed neighborhood of v is the set $N[v] = N(v) \cup \{v\}$. For a subset $S \subseteq V(G)$ the open and closed neighborhoods of S is defined by $N(S) = \bigcup_{v \in S} N(v)$ and $N[S] = \bigcup_{v \in S} N[v]$, respectively.

A set $S \subseteq V(G)$ is called *nearly perfect* if for every vertex $v \in V \setminus S$, $| N(v) \cap S | \leq 1$; such a set is abbreviated as an *np*-set. A nearly perfect set $S \subseteq V(G)$ is called *1-maximal* if S is an *np*-set but for every $v \in V \setminus S$, $S \cup \{v\}$ is not an *np*-set. J. E. Dunbar et al. [12] defined $n_p(G)$ to equal the minimum cardinality of a 1-maximal nearly perfect set; such a set is abbreviated as an n_p-set.

Definition 2.1. A proper subset $S \subsetneq V(G)$ is called a *proper nearly perfect set* (or a *pnp-set*, shortly) *in* G , if for each $x \in V(G) \setminus S$, there exists at most one vertex $y \in S$ adjacent to x. If there is no confusion about the underlying graph, we also call S *proper nearly perfect.* For any graph G, the maximum cardinality of *pnp*-sets in G is denoted by $PN_p(G)$; such a set is abbreviated a $PN_p - set$.

Obviously, the empty set and all singleton sets in graphs of order $n \geq 2$ are proper nearly perfect sets, so $PN_p(G)$ is well defined. It is obvious that $0 \leq PN_p(G) \leq n-1$ for any graph G. It is easy to check that if G is nontrivial, then $0 < PN_p(G) \leq n-1$.

Proposition 2.2. *For any graph G of order n,*
a) $PN_p(G) = 0$ if and only if $G = K_1$,
b) $PN_p(G) = n-1$ if and only if $\delta(G) \leq 1$.

Proof. Part (a) is obvious. For part (b), let v be a vertex of G with $deg_G(v) \leq 1$. Then $S = V(G) - \{v\}$ is a maximum *pnp*-set. For the converse, suppose that S is a *pnp*-set of size $n-1$ and $\{v\} = V(G) \setminus S$. By definition, $| N(v) \cap S | \leq 1$. So $\delta(G) \leq deg_G(v) \leq 1$. □

Let $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ be two vertex disjoint graphs and let f be a function $f : V_1 \to V_2$. Then a *function graph*, $G_1 f G_2 = (V, E)$ has the vertex set $V = V(G_1) \cup V(G_2)$ and the edge set

$$E = E(G_1) \cup E(G_2) \cup \{uv : u \in V(G_1), v \in V(G_2), v = f(u)\}$$

Function graphs are closely related to the special classes like permutation graphs, prisms, and generalized Petersen graphs [7, 8, 9, 14, 19]. The following proposition is an immediate consequence of the definitions.

Proposition 2.3. *If $G_1 f G_2$ is a function graph defined by two non-empty, vertex disjoint graphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ and a function $f : V_1 \to V_2$, then V_2 is a proper nearly perfect set in $G_1 f G_2$.*

Theorem 2.4. *For any graph G of order $n \geq 2$ and minimum degree δ, $PN_p(G) \leq n - \delta$, with equality if and only if G is a function graph of the form $K_\delta f H$.*

Proof. Let S be a PN_p-set of G and let $v \in V \setminus S$. Then, by definition, $|N(v) \cap S| \leq 1$. Thus, $| N[v] \cap (V \setminus S) | \geq deg(v)$, and therefore

$$n - PN_p(G) = n - | S | \geq deg(v) \geq \delta.$$

Now suppose that $PN_p(G) = n - \delta$ and S is a PN_p-set. Let $X = V \setminus S = \{x_1, x_2, \ldots, x_\delta\}$. For every $x_i \in X$ we have $deg(x_i) \geq \delta$ and x_i has at most one neighbor in S. Therefore, $deg(x_i) = \delta$ and all of the vertices in $X \setminus \{x_i\}$ are adjacent to x_i. Therefore, $G[X] \simeq K_\delta$ and G is a function graph of the form $G = K_\delta f G[S]$. Therefore by Proposition 2.3, S is a proper nearly perfect set of cardinality $n - \delta$, and since we showed that for any graph G $PN_p(G) \leq n - \delta$, the proof is complete. □

Corollary 2.5. *For any graph G, if $PN_p(G) = n - 2$, then $G = HfK_\delta$, for some graph H.*

Proof. By Theorem 2.4, $PN_p(G) = n - 2$ implies $\delta(G) \leq 2$. If $\delta(G) \leq 1$, then $PN_p(G) = n - 1$ by Proposition 2.2(b). So $\delta(G) = 2$ and using Theorem 2.4 the proof is complete. □

Intuitively one would expect that the value of $PNp(G)$ is generally greater for sparse graphs than it is for non-sparse graphs. This is because every nearly perfect set in G is also a nearly perfect set in $G \setminus e$. The next theorem captures this intuition formally.

Proposition 2.6. *For any graph $G = (V, E)$ and any edge $e \in E$, $PN_p(G - e) \geq PN_p(G)$.*

In the following proposition, $PN_p(G)$ is determined in terms of two standard binary operations on graphs, where $G_1 \cup G_2$ denotes the *disjoint union* of two graphs, and $G_1 + G_2$ denotes the *join* of two graphs.

Proposition 2.7. *Let G_1 and G_2 be two connected vertex disjoint graphs, then*
a) $PN_p(G_1 \cup G_2) = max\{PN_p(G_1) + \mid V(G_2) \mid, PN_p(G_2) + \mid V(G_1) \mid\}$
b) $PN_p(G_1 + G_2) = 1$.

Proof. (a) Let $M = max\{PN_p(G_1) + \mid V(G_2) \mid, PN_p(G_2) + \mid V(G_1) \mid\}$ and S be a *pnp*-set in $G = G_1 \cup G_2$. Then $S_1 = S \cap V_1$ and $S_2 = S \cap V_2$ are both *np*-sets in G_1 and G_2, respectively, at least one of which is a *pnp*-set. Without loss of generality, assume that S_1 is a *pnp*-set. Then $\mid S \cap V(G_1) \mid \leq PN_p(G_1)$ and $\mid S \mid \leq PN_p(G_1) + \mid V(G_2) \mid$. Therefore for any *pnp*-set S in G we have $\mid S \mid \leq M$, so $PN_p(G) \leq M$. On the other hand, suppose that S_1 is a *pnp*-set in G_1 of size $PN_p(G_1)$, then $S = S_1 \cup V(G_2)$ is a *pnp*-set in G, so $PN_p(G) \geq PN_p(G_1) + \mid V(G_2) \mid$. Similarly, $PN_p(G) \geq PN_p(G_2) + \mid V(G_1) \mid$ and the proof is complete.
(b) Let S be a maximum *pnp*-set in G_1+G_2. It is obvious that $\mid S \cap V(G_i) \mid \leq 1$. So $PN_p(G_1+G_2) \leq 2$. Let $\mid S \mid = 2$ and $S \cap V(G_i) = \{x_i\}$. S is a *pnp*-set, therefore at least cardinality of one of $V(G_i)$'s, say $V(G_1)$, is grater than 1. Now suppose $y \in V(G_1)$ be a neighbor of x_1. Then y has two neighbors in S which is a contradiction. □

In the following proposition the values of $PN_p(G)$ are determined for several classes of graphs.

Proposition 2.8. *We have*
a) $PN_p(K_n) = 1$; *for* $n \geq 2$,
b) $PN_p(W_n) = 1$; *for any wheel* W_n *with* $n \geq 3$,
c) $PN_p(K_{m,n}) = 2$; *for* $m, n \geq 2$,
d) $PN_p(C_n) = n - 2$,
e) $PN_p(T) = n - 1$; *for any tree* T.

Proof. (a) and (b) are immediate consequences of Proposition 2.7(b).
(c) Let $K_{m,n}$ be a complete bipartite graph with partite sets X of size m and Y of size n. For any *np*-set S, we have $| S \cap X | \leq 1$ and $| S \cap Y | \leq 1$. Hence $PN_p(K_{m,n}) \leq 2$. Now let $x \in X$ and $y \in Y$. Clearly, $S = \{x, y\}$ is a *pnp*-set and $PN_p(K_{m,n}) = 2$.
(d) Since $C_n = K_2 f P_{n-2}$, by Corollary 2, $PN_p(C_n) = n - 2$.
(e) It is an immediate consequence of part (b) of Proposition 2.2. □

3 *np*-closure

In this section we introduce the notion of *np*-closure of a subset of the vertex set of a graph and use it to prove something more complex about PN_p.
Suppose that S is a subset of $V(G)$. The *np* − *closure* of S, denoted by $cl(S)$, is defined as $\bigcap\{T : S \subseteq T \text{ and } T \text{ is an } np - set \text{ of } G\}$. If S is a nearly perfect set, then $cl(S) = S$. The *np* − *closure* of a subset S can be computed by initially setting $cl(S) = S$, and then repeatedly adding to $cl(S)$ vertices having two or more neighbors in $cl(S)$. This process stops when the resulting set is a nearly perfect set. Notice that $cl(S)$ is not necessarily a *pnp*-set. In order to prove that $PN_p(G) = k$, it is sufficient to show that there exists a proper subset $S \subset V$ such that $| S |= k$, $cl(S) = S$, and $cl(T) = V$ for every $T \subset V$ with $| T |= k + 1$.

As a simple application of these ideas, we compute $PN_p(L(K_n))$, where $L(G)$ is the line graph of G.

Proposition 3.1. $PN_p(L(K_n)) = 1$, *if* $n \geq 3$.

Proof. It is enough to prove that for any subset $T \subseteq V(L(K_n))$ of size two, $cl(T) = V(L(K_n)$. Let $T = \{\{a, b\}, \{c, d\}\}$. Without loss of generality, suppose that $c \notin \{a, b\}$. It is obvious that $\{a, c\}$ and $\{b, c\}$ are in $cl(T)$. So for each vertex x of K_n we have $\{c, x\} \in cl(T)$. Now if $\{x, y\}$ is an arbitrary vertex of $L(K_n)$, then $\{x, y\}$ has at least two neighbors in $cl(T)$ and hence $\{x, y\} \in cl(T)$. Therefore $cl(T) = V(G)$. □

In [21] Kwasnik and Perl proved that if S and S' are nearly perfect sets in G and G', respectively, then $S \times S'$ is a nearly perfect set in the Cartesian product $G\Box G'$. They also proved the following inequality

$$n_p(G\Box G') \leq \min\{n_p(G)\times \mid V(G') \mid, n_p(G')\times \mid V(G) \mid\}$$

The following theorem yields a much stronger assertion about PN_p.

Theorem 3.2. *Let G and G' be two connected graphs. Then*

$$PN_p(G\Box G') = \max\{PN_p(G)\times \mid V(G') \mid, PN_p(G')\times \mid V(G) \mid\}$$

Proof. If one of the graphs G or G' is a trivial graph, then the result holds obviously. Hence suppose that G and G' are nontrivial graphs.
Let S and S' be nearly perfect sets in G and G', respectively. First we show that $S \times S'$ is a nearly perfect set in $G\Box G'$. Suppose that $(v, v') \in V(G\Box G') \setminus S \times S'$. We must show that at most one vertex in $S \times S'$ is adjacent to (v, v'). Without loss of generality assume that $v \notin S$ and $(u, u') \in S \times S'$ be a neighbor of (v, v'). Then $u' = v'$ and u is a neighbor of v in G. Now since S is a nearly perfect set in G, hence v has no other neighbor in S. Therefore (v, v') has no other neighbor in $S \times S'$ and $S \times S'$ is a nearly perfect set in $G\Box G'$.
Now if S and S' are two *pnp*-sets in G and G', respectively, then $S \times V(G')$ and $V(G) \times S'$ are two *pnp*-sets in $G\Box G'$ so

$$PN_p(G\Box G') \geq \max\{PN_p(G)\times \mid V(G') \mid, PN_p(G')\times \mid V(G) \mid\}.$$

In order to show that $PN_p(G) \leq \max\{PN_p(G)\times \mid V(G') \mid, PN_p(G')\times \mid V(G) \mid\}$, we proceed as follows: Let $V(G) = \{v_1, v_2, \dots, v_n\}$, $V(G') = \{v'_1, v'_2, \dots, v'_m\}$, and $S \subsetneq V(G\Box G')$ be a *pnp*-set in $G\Box G'$. Define

$$S_l = \{v_i \mid (v_i, v'_l) \in S\}; \qquad 1 \leq l \leq m,$$
$$S'_k = \{v'_j \mid (v_k, v'_j) \in S\}; \qquad 1 \leq k \leq n.$$

First we show that S_l is an *np*-set in G for any $1 \leq l \leq m$. Otherwise, if for some l, S_l is not a nearly perfect set then there exist $x \in V(G) \setminus S_l$, and two distinct vertices $v_i, v_j \in S_l$ such that $v_i, v_j \in N(x)$. Now $(x, v'_l) \notin S$ and $\{(v_j, v'_l)\} \subseteq S \cap N((x, v'_l))$, which is a contradiction to the assumption that S is a nearly perfect set in $G\Box G'$. Similarly, for any $1 \leq k \leq n$, S'_k is an *np*-set in G'.
To complete the proof, it is enough to show that either all S_l's are *pnp*-sets in G or all S'_k's are *pnp*-sets in G'. It is enough, because if the first condition occurs then for any $1 \leq l \leq m$, $S_l \leq PN_p(G)$, hence $\mid S \mid \leq PN_p(G) \mid V(G') \mid$. Similarly, for the second condition we have $\mid S \mid \leq$

$PN_p(G') \mid V(G) \mid$, and the proof is complete.
Assume now, on the contrary, that for some l and k we have $S_l = V(G)$ and $S'_k = V(G')$. Assume without loss of generality that $l = k = 1$ and we relabeled the vertices of G (and G') such that each v_i ($v_{i'}$, respectively) has at least one neighbor in $\{v_1, v_2, \ldots, v_{i-1}\}$ ($\{v'_1, v'_2, \ldots, v'_{i-1}\}$, respectively). This is possible, because of the assumption of connectivity.
Let $\mathcal{S}_1 = S_1 \times \{v'_1\}$ and $\mathcal{S}'_1 = \{v_1\} \times S'_1$. Now we prove that $cl(\mathcal{S}_1 \cup \mathcal{S}'_1) = V(G) \times V(G')$. Assume that $cl(\mathcal{S}_1 \cup \mathcal{S}'_1) \neq V(G) \times V(G')$ and $(v_i, v'_j) \notin cl(\mathcal{S}_1 \cup \mathcal{S}'_1)$ such that for each $1 \leq k \leq n$ and $1 \leq l \leq m$ with $k + l < i + j$ we have $(v_k, v'_l) \in cl(\mathcal{S}_1 \cup \mathcal{S}'_1)$. Let $v_* \in \{v_1, v_2, \ldots, v_{i-1}\} \cap N(v_i)$ and $v'_* \in \{v'_1, v'_2, \ldots, v'_{j-1}\} \cap N(v'_j)$. Then $\{(v_i, v'_*), (v_*, v'_j)\} \subseteq cl(\mathcal{S}_1 \cup \mathcal{S}'_1) \cap N((v_i, v'_j))$. So we have $(v_i, v'_j) \in cl(\mathcal{S}_1 \cup \mathcal{S}'_1)$ and then $cl(\mathcal{S}_1 \cup \mathcal{S}'_1) = V(G) \times V(G')$. This contradicts the assumption that S is a *pnp*-set. □

As an application of the theorem above, we obtain the following corollary.

Corollary 3.3. *a)* $PN_p(Q_n) = 2^{n-1}$,
b) If $m \leq n$, *then* $PN_p(P_n \Box P_m) = mn - m$.

Proof. Using Theorem 3.2 and the fact $Q_n = K_2 \times Q_{n-1}$, a proof can be constructed using induction. The assertion (b) is a straightforward result from proposition 2.8(e) and Theorem 3.2. □

4 PN_p and Some Other Parameters of Graphs

The purpose of this section is to establish some relationships between $PN_p(G)$ and some other parameters of graphs, including the girth, the diameter, the 2-packing number, and n_p of a graph G.
As mentioned in the introduction, J. E. Dunbar et al. [12] introduced $n_p(G)$ to be the minimum cardinality of 1-maximal nearly perfect sets in graph G. The reader can find the formal definition of n_p in the introduction.
Naturally, one expects $PN_p(G)$ to be greater than or equal to $n_p(G)$, but this is not always the case. For example in $K_{1,n}$ the only 1-maximal nearly perfect set is the whole vertex set of the graph, so $n_p(K_{1,n}) = n + 1$ but by Proposition 2.6(e), $PN_p(K_{1,n}) = n$. Rather surprisingly, $K_{1,n}$ is the only exception to the rule. This fact is a consequence of the following result in [12].

Theorem 4.1. *For any connected graph G with n vertices $n_p(G) = n$ if and only if G is a star.*

Now, we can state the following theorem.

Theorem 4.2. *For any connected graph G, $n_p(G) \leq PN_p(G) + 1$, and equality holds if and only if G is a star.*

Proof. If G is a star, $n_p(G) = n$ and $PN_p(G) = n - 1$. Suppose G is not a star, then by Theorem 4.1 any 1-maximal *np*-set is also a *pnp*-set and so $n_p(G) \leq PN_p(G)$. □

In view of the Theorem 4.2, it is interesting to determine the largest possible difference between PN_p and n_p in graphs. Consider the graph $G = mK_2 + K_1$. A straightforward calculation shows that $n_p(G) = 1$ and $PN_p(G) = 2m - 1$. Therefore $PN_p(G) - n_p(G)$ can be arbitrarily large.

Theorem 4.3. *Let G be a 3-regular connected graph of order n and girth g. Then $PN_p(G) = n - g$.*

Proof. Let S be a PN_p-set of G. For each vertex x of $V(G) \setminus S$ we have $\mid N(x) \cap (V(G) \setminus S) \mid \geq 2$. So $\delta(< G \setminus S >) \geq 2$ and $< G \setminus S >$ has a cycle. Therefore $\mid G \setminus S \mid \geq g$ and hence $PN_p(G) \leq n - g$. On the other hand, let C be a cycle of size g. It is obvious that $S = V(G) \setminus S$ is a *pnp*-set in G, so $PN_p(G) \geq n - g$ and the proof is complete. □

It is worth noting that the Theorem 4.3 restates an elementary result about *defensive alliances* [20]. In a graph G, a nonempty set of vertices S is a *(defensive) alliance* if for every vertex $v \in S, \mid N[v] \cap S \mid \geq \mid N[v] \cap (V(G) \setminus S) \mid$. In fact in 3-regular graphs maximum proper nearly perfect sets are complements of minimum defensive alliances.

As an application of the theorem above we can compute $PN_p(P)$ for the Petersen graph P. Although direct computation of $PN_p(P)$ is not straightforward, using Theorem 4.3 and the well known fact that the girth $g(P) = 5$, it follows that $PN_p(P) = 5$. The following theorem establishes a relationship between PN_p and the diameter of a graph.

Theorem 4.4. *Let G be a connected graph.*
a) If S is a PN_p-set in G, then we have $diam(G \bullet S) \leq 4$, where $G \bullet S$ is the graph obtained by contracting the vertices of S to a single vertex.
b) If $PN_p(G) = 1$, then $diam(G) \leq 2$.

Proof. a) It is enough to prove that for each $v \in V(G) \setminus S$, $d(v, S) \leq 2$. Assume on the contrary that there exists a vertex $v_0 \in V(G) \setminus S$, such that $d(v_0, S) \geq 3$. Then $N(v_0) \cap N(S) = \varnothing$ and there is not any vertex in $V(G) \setminus (S \cup \{v_0\})$ with more than one neighbor in $S \cup \{v_0\}$. Therefore $S \cup \{v_0\}$ is a *pnp*-set and this contradicts the maximality of S.
b) Let $diam(G) > 2$. Then there exist two vertices u and v such that $N(u) \cap N(v) = \varnothing$. Now $S = \{u, v\}$ is a *pnp*-set, which implies $PN_p(G) \geq 2$, a contradiction. □

The following example shows that the bound of Theorem 4.4(a) is tight.

Example 4.5. Let G be the graph shown in Figure 1. It is easy to see that S, the vertex set of K_{100}, is a maximum proper nearly perfect set in G and $diam(G \bullet S) = 4$.

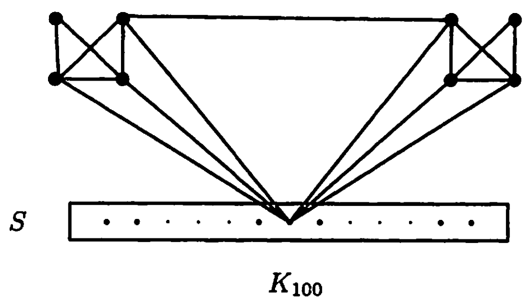

Figure 1: A graph G with $diam(G \bullet S) = 4$.

We conclude this section by comparing the 2-packing number with PN_p. A set $S \subseteq V(G)$ is called a *2-packing* in G if for every pair of vertices $u, v \in S$, $N[u] \cap N[v] = \emptyset$. The *2-packing number*, $\rho_2(G)$, is the maximum cardinality of a 2-packing in G. In order to make a comparison between $PN_p(G)$ and $\rho_2(G)$, it is convenient to restate the definition of $PN_p(G)$. A set $S \subseteq V(G)$ is called an *np-set* in G if for every pair of vertices $u, v \in S$, $N[u] \cap N[v] \cap (V(G) \setminus S) = \emptyset$ and $PN_p(G)$ is the maximum cardinality of an *np*-set in G. Obviously, every 2-packing is a *pnp*-set and the the following theorem holds.

Theorem 4.6. *For every graph G, $PN_p(G) \geq \rho_2(G)$.*

5 Complexity Issues

In this section we investigate the complexity of the following problem:

Proper Nearly Perfect Set (PNPS).
INSTANCE: A graph $G = (V, E)$ and a positive integer k .
QUESTION: Does G have a proper nearly perfect set of cardinality at least

k?

To show that PNPS is NP-hard for arbitrary graphs, we use a well known NP-hardness result, called Exact Three Cover(X3C), which is defined in [17] on page 221, as follows.

Exact cover by 3-sets (X3C).
INSTANCE: A set X with $|X| = 3q$ and a collection C of 3-element subsets of X.
QUESTION: Does C contain an exact cover for X, that is, a sub-collection $C' \subseteq C$ such that every element of X occurs in exactly one member of C'? (Note that if C' exists, then its cardinality is precisely q.)

Theorem 5.1. *PNPS is NP-hard.*

Proof. To show that PNPS is an NP-hard problem, we will establish a polynomial transformation from X3C. Let $X = \{x_1, x_2, \dots, x_{3q}\}$ and $C = \{C_1, C_2, \dots, C_m\}$ be an arbitrary instance of X3C.

We will construct a graph G and a positive integer k such that the instance (X, C) of X3C has an exact cover by 3-sets if and only if G has a *pnp*-set of cardinality at least k. The vertex set V(G) is the union of X and C and the edge set $E(G)$ is the union of the following sets: $E_X = \{x_i x_j : i \neq j\}$ and $E_C = \{C_i x_j : x_j \in C_i\}$. Finally, Let $k = q$. Clearly, the above transformation can be performed in time that is polynomial in m and q. We now show that the instance (X, C) of X3C contains an exact 3-cover if and only if $PN_p(G) \geq q$. Suppose that C contains an exact 3-cover $C' \subseteq C$. It is easy to verify that C' is a *pnp*-set in G and since the cardinality of C' is q, $PN_p(G) \geq q$. Now suppose $PN_p(G) \geq q$ and let $S \subset X \cup C$ be a *pnp*-set with cardinality at least q. To prove that C contains an exact 3-cover we show that $S \cap X = \emptyset$, so $S \subseteq C$ and S must be an exact 3-cover. First notice that if $| S \cap X | \geq 2$ then every $x_i \in X$ has two neighbors in S so $X \subseteq S$. Now every $C_j \in C$ has at least three neighbors in S so $C \subseteq S$ and S is not a proper subset, a contradiction. Hence, $| S \cap X |$ is at most 1. Suppose $x_i \in S$. Now if $C_j \in S$ at least two vertices in X have more than one neighbor in S, so $S \cap C = \emptyset$ and $| S | = 1$. So without loss of generality one can assume that $S \cap X = \emptyset$. Now it is obvious that S is an exact 3-cover. □

From the proof of Theorem 5.1 it turns out that **PNPS** is NP-hard even if restricted to graphs with $diam(G) = 3$. But we don't know whether there is a polynomial-time algorithm for **PNPS** if we restrict ourselves to graphs with diameter 2? Another interesting problem is the complexity of PNPS for bipartite graphs.

6 Summary and Open Problems

In this paper, a new graph parameter, $PN_p(G)$, has been introduced to deal with the Nearly Perfect Bipartition Problem. It is shown that the problem of determining PN_p for a graph is NP-hard even when restricted to chordal graphs with diameter 3. Some bounds and exact values of PN_p for several classes of graphs and Cartesian products $G\Box H$ are given. Many questions remain to be investigated including the ones listed here.

Problem 1. Determine the exact values of $PN_p(G \times H)$, where $G \times H$ is the tensor product of graphs G and H.

Problem 2. PNPS(Diameter 2)
INSTANCE: Graph $G = (V, E)$ of diameter 2, positive integer $k \leq |V|$.
QUESTION: Does G have a *pnp*-set of cardinality at least k?

By the application of PN_p, a graph G would be interesting when $PN_p(G) = 1$. In Theorem 4.4(b), $diam(G) \leq 2$ is stated as a necessary condition for $PN_p(G) = 1$. By a small revision in Theorem 3.2 in [15], it is not difficult to see that $\delta(G) > \frac{n}{2}$ is a sufficient condition for $PN_p(G) = 1$.

Problem 3. Find some necessary or sufficient conditions for graphs G with $PN_p(G) = 1$.

When $PN_p(G) = 1$, G cannot be sparse. Therefore it is natural to question about the minimum number of edges of graph G when $PN_p(G) = 1$. It is obvious that if $G = K_2 + \bar{K}_n$ and $G = T + K_1$, where T is a tree, then $PN_p(G) = 1$ and $E(G) = 2 \mid V(G) \mid -3$. Based on this observation we make the following conjecture.

Conjecture: If $PN_p(G) = 1$, then $\mid E(G) \mid \geq 2 \mid V(G) \mid -3$.

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