Flag-transitive symmetric (v, k, λ) designs admitting primitive automorphism groups with socle PSL(12, 2)

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Abstract

In this paper, we obtained two flag-transitive symmetric (v, k, λ) designs admitting primitive automorphism groups of almost simple type with socle $X = \mathrm{PSL}(12, 2)$.

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1 Introduction

A 2-(v, k, λ) *design* is a finite incidence structure $\mathcal{D}=(P, \mathcal{B})$, where P is a set of v elements called points and $\mathcal{B}$ is a set of k-subsets of P called blocks, such that any two distinct points are incident with exactly λ blocks. And $\mathcal{D}$ is called *symmetric* if $|\mathcal{B}| = v$. The symmetric design $\mathcal{D}$ is *non-trivial* if $\lambda < k < v - 1$. Now we study non-trivial symmetric 2-(v, k, λ) designs which are denoted by symmetric (v, k, λ) designs for simplicity. A symmetric

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design $\mathcal{D}$ is called a *projective plane* if $\lambda = 1$, while a *biplane* if $\lambda = 2$ and a *triplane* if $\lambda = 3$. The *complement* of $\mathcal{D}$, denoted by $\mathcal{D}'$, is a symmetric $(v, v-k, v-2k+\lambda)$ design whose set of points is the same as the set of points of $\mathcal{D}$, and whose blocks are the complements of the blocks of $\mathcal{D}$. A *flag* in a design is an incident point-block pair.

An *automorphism* of $\mathcal{D}$ is a permutation of P that preserves $\mathcal{B}$. A group obtained under composition of automorphisms is an *automorphism group*. The group of all automorphisms of a design is the *full automorphism group* of $\mathcal{D}$, denoted by $\mathrm{Aut}(\mathcal{D})$. For $G \leq \mathrm{Aut}(\mathcal{D})$, the design $\mathcal{D}$ is called *point-primitive* if G is primitive on P, and *flag-transitive* if G is transitive on the set of flags. The socle of a certain group is the product of all its minimal normal subgroups.

In recent years, researchers have tackled many problems related to the designs with an automorphism group which is a linear group acting flag-transitively. In 1986, Delandtsheer [3] classified flag-transitive finite linear spaces where the automorphism group G is one of the simple groups $\mathrm{PSL}(2,q)$ or $\mathrm{PSL}(3,q)$. In [8], Regueiro proved that if a biplane $\mathcal{D}$ admits a flag-transitive automorphism group G of almost simple type with classical socle, then $\mathcal{D}$ is either the unique $(11,5,2)$ or the unique $(7,4,2)$ biplane, and $Soc(G) = \mathrm{PSL}(2,11)$ or $\mathrm{PSL}(2,7)$, respectively. Recently, Zhou et al. proved in [10, 11] that there is only one triplane with flag-transitive linear automorphism group G, namely the $(11,6,3)$ triplane with $G = \mathrm{PSL}(2,11)$ (this is the complement of the (11,5,2) biplane), and there is a unique symmetric $(v,k,4)$ design with a flag-transitive linear automorphism group G, with parameters $(15,8,4)$ and $Soc(G) = \mathrm{PSL}(2,9)$. Now we consider the case in which the automorphism group has $\mathrm{PSL}(12,2)$ as its socle, and obtain the following conclusion.

Theorem 1.1

If $\mathcal{D}$ is a symmetric (v,k,λ) design admitting a point-primitive, flag-transitive automorphism group G of almost simple type with socle $\mathrm{PSL}(12,2)$, *then $\mathcal{D}$ is either the unique projective space* $\mathrm{PG}(11,2)$, *with $v = 4095$, $k = 2047$ and $\lambda = 1023$, or its complement, the unique symmetric* $(4095, 2048, 1024)$ *design.*

2 Some Preliminary Results

The following lemmas give some fundamental information which is essential to the proof of our main theorem.

Lemma 2.1 ([4])

Let $\mathcal{D}$ be a symmetric (v,k,λ) design and $G \leq \mathrm{Aut}(\mathcal{D})$. Suppose that $(k,\lambda)=1$ and G is doubly point transitive, then G is flag-transitive on $\mathcal{D}$.

Lemma 2.2 ([7])

(1) An automorphism group of a symmetric design has as many as orbits on points as on blocks.

(2) A transitive automorphism group of a symmetric design has the same rank whether considered as a permutation group on points or on blocks.

1.0

Lemma 2.3

Let $\mathcal{D}=(P,\mathcal{B})$ be a symmetric (v,k,λ) design admitting a flag-transitive automorphism group G. Suppose that G is 2-transitive on P, then $\mathcal{D}'$, the complement of $\mathcal{D}$, is also flag-transitive.

Proof. Lemma 2.2 (1) implies G_x has the same number of orbits on points and on blocks, similarly for G_B (although here we still don't have that these numbers are equal). Lemma 2.2 (2) implies G_x has as many orbits on points as G_B has on blocks, so now we do know these two numbers are equal. Finally we know this is 2 by the 2-transitivity of G.

The flag-transitivity implies that G_B acts transitively on the points of B (see [9], Lemma 2.3). Then G_B has a orbit Γ_1 of length k on P and the other orbit Γ_2 is of length $v-k$ and $\Gamma_2 = P-\Gamma_1$. Obviously, $\Gamma_2 = B'$ is one of blocks of $\mathcal{D}'$ and $G_B = G_{B'}$. So $G_{B'}$ is transitive on the points of B'. Moreover, by Lemma 2.2, we have that G is block-transitive on $\mathcal{D}'$ since $\mathcal{D}'$ has the same set of points as $\mathcal{D}$ and G is 2-transitive on P. Hence G is flag-transitive on $\mathcal{D}'$. □

3 Proof of Theorem 1.1

Suppose that G is a flag-transitive and point-primitive automorphism group of a symmetric (v,k,λ) design $\mathcal{D}$, and the socle of G is $X = \mathrm{PSL}(12,2)$. It is known that the order of X is

$$644176229278576214187891988140087941529600 0.$$

Since G is primitive, G_x, the stabilizer of G for some $x \in P$, is a maximal subgroup of G. Hence we consider each of the maximal subgroups of G as G_x to search the possible symmetric designs. Note that a subgroup of a classical group must fall into at least one of nine Aschbacher classes C_i, with $1 \leq i \leq 9$. In [2], J. Bray et al. discussed the maximal subgroups of PSL$(12, q)$ with $q = p^e$ be a power of a prime p. Since G_x is a maximal subgroup of G, $G_x \cap X$ is a maximal subgroup of X. Thus Table 1 lists all the potential groups $G_x \cap X$ (combining Table 8.76 and Table 8.77 of [2], and let $q = 2$), such that G_x is maximal in the group G. Because $X \trianglelefteq G$ and G_x is maximal in G, we get $G/X = XG_x/X \cong G_x/(G_x \cap X)$ which implies $|G : G_x| = |X : (G_x \cap X)|$. The index of G_x in G is listed in the last column of Table 1.

Table 1: All the potential maximal subgroups G_x of G with socle X

Case	C_i	$G_x \cap X$	$v = \|G : G_x\| = \|X : (G_x \cap X)\|$
1	C_1	$2^{11}.L(11,2)$	4095
2		$2^{20}.(L(2,2) \times L(10,2))$	2794155
3		$2^{27}.(L(3,2) \times L(9,2))$	408345795
4		$2^{32}.(L(4,2) \times L(8,2))$	13910980083
5		$2^{35}.(L(5,2) \times L(7,2))$	114429029715
6		$2^{36}.(L(6,2) \times L(6,2))$	230674393235
7		$L(11,2)$	8386560
8		$L(2,2) \times L(10,2)$	2929883873280
9		$L(3,2) \times L(9,2)$	54807244843253760
10		$L(4,2) \times L(8,2)$	59747204511792365568
11		$L(5,2) \times L(7,2)$	393175152271149760512 0
12		$2^{21}.L(10,2)$	8382465
13		$2^{36}.(L(2,2)^2 \times L(8,2))$	486884302905
14		$2^{45}.(L(3,2)^2 \times L(6,2))$	321790778562825
15		$2^{48}.(L(4,2)^3)$	2793143957925321
16		$2^{45}.(L(5,2)^2 \times L(2,2))$	305182222249905
17	C_2	S_{12}	1344831059601003867602721970323456 0
18		$L_2(2)^6.S_6$	1917629473875505514914992439164928 00
19		$L_3(2)^4.S_4$	3369429130742311074988598231040 00
20		$L_4(2)^3.S_3$	13103335508442320847177882009 6
21		$L_6(2)^2.S_2$	79259117997517491404 80
22	C_3	$7.L_4(8).3$	8876262199005034444121702400
23		$3.L_6(4).3.2$	9904944486896673751 04
24	C_4	$L_2(2) \times L_6(2)$	5325871851818491699175526236160 0
25		$L_3(2) \times L_4(2)$	1901975355721419755609563929457459200
26	C_8	$S_{12}(2)$	3095295152155210547 2

Note that Out(X) = 2, so $|G_x|$ divides $2|X_x| = 2|G_x \cap X|$. Then $|G_x| = |G_x \cap X|$ or $2|G_x \cap X|$. The cases 17-20, 24, 25 will be ruled out since $|G_x|$ is too small to satisfy the inequality $|G_x|^3 > |G|$ (see [9], Lemma 2.1(iii)).

As $\mathcal{D}$ is a symmetric design, $k(k-1) = \lambda(v-1)$ holds. Since G_x acts transitively on the set of the k blocks which incident with x, we have $k \mid |G_x|$. Now we state the following algorithm, which will be useful to search for designs. The output of the algorithm is a list DESIGNS of parameter sequences (v, k, λ) of potential symmetric designs.

Algorithm 1 (DESIGNS)
INPUT: $|G_x|$, v.
OUTPUT: The list DESIGNS := S.
set S := an empty list;
for each k divides $|G_x|$ and $1 \neq k < v-1$
  $\lambda := k * (k-1)/(v-1)$;
  if λ be an integer
    Add (v, k, λ) to the list S;
return S.

Algorithm 1 checks all possibilities for any given $\{|G_x|, v\}$ pairs coming from the remaining 20 cases. For case 1, We get five parameter sequences (v, k, λ): (4095, 713, 124), (4095, 1335, 435), (4095, 2047, 1023), (4095, 2048, 1024) and (4095, 2760, 1860). For case 7, We get one potential design (8386560, 150144, 2688). For the remaining 18 cases, there is no such 3-tuples (v, k, λ).

We now consider the potential design (8386560, 150144, 2688). In this case, Table 1 shows that $G_x \cap X = \mathrm{PSL}(11, 2)$. For any block $B \in \mathcal{B}$, the flag-transitivity of G implies that G_B is transitive on the k (= 150144) points of B. Thus G_B should have at least one subgroup of index k. Since Out(X) = 2, we have $G = X$ or $X.2$. Let $G = X$, then G has only one conjugacy class of subgroups of index v (= 8386560) which are isomorphic to PSL(11, 2), and PSL(11, 2) has no subgroup of index k(calculated with MAGMA[1]). This is not possible since G_B is a subgroup of index v of G. Then we suppose that $G = X.2$. If $G_B \leq X$, then X should have a subgroup of index k, but PSL(12, 2) has no such subgroup. So $G = XG_B$ holds. The second isomorphism theorem shows that $G_B \cap X \trianglelefteq G_B$ and $G/X \cong G_B/(G_B \cap X)$. Hence $G_B \cap X \cong \mathrm{PSL}(11, 2)$. Let $H \leq G_B$ of index 150144. We have $H(G_B \cap X)/(G_B \cap X) \cong H/(H \cap (G_B \cap X))$. Then $|G_B \cap X : H \cap X| = |H(G_B \cap X) : H|$. Since $|G_B : G_B \cap X| = 2$, we get $|H(G_B \cap X)| = |G_B|$ or $|G_B|/2$. Thus $G_B \cap X$ has a subgroup $H \cap X$ of index k or $k/2$, however, we know that PSL(11, 2) has no such subgroups calculated with MAGMA.

The GAP command `Transitivity(`G, Ω`)` returns the degree k (a non-negative integer) of transitivity of the action implied by the arguments, i.e. the largest integer k such that the action is k-transitive. Thus we know that X acts as a doubly transitive permutation group on the set P of 4095 points by GAP [5, version 4.7.2].

```
gap> G := PSL(12,2);
<permutation group of size 6441762292785762141878919881400879415296000 with 2 generators>
gap> Transitivity(G, [1..4095]);
2
```

The symmetric design $\mathcal{D}$ has the same transitivity as its complement design $\mathcal{D}'$. So we check in [6] that there are exactly two 2-transitive symmetric designs when $v = 4095$, and one is the unique projective space PG$(11,2)$, with $v = 4095$, $k = 2047$ and $\lambda = 1023$, and the other is the unique symmetric (4095, 2048, 1024) design, complement of PG$(11,2)$.

Since $(2047, 1023) = 1$, Lemma 2.1 shows that the symmetric (4095, 2047, 1023) design $\mathcal{D}$ is flag-transitive. By Lemma 2.3, as the complement of $\mathcal{D}$, symmetric (4095, 2048, 1024) design is also flag-transitive. Hence we get two flag-transitive symmetric (v, k, λ) designs admitting a primitive automorphism group of almost simple type with socle PSL$(12,2)$.

This completes the proof of Theorem 1.1.

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