Total embedding distributions for two graph families obtained from the dipole D_3 *

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Abstract

The distribution of the set of embeddings of a graph into orientable or non-orientable surfaces is called the total embedding distribution. Chen, Gross and Rieper [Discrete Math. 128(1994) 73-94.] first used the overlap matrix for calculating the total embedding distributions of necklaces, closed-end ladders and cobblestone paths. In this paper, also by using the overlap matrix, closed formulas of the total embedding distributions for two classes of graphs are given.

Key words: Overlap matrix; Total embedding distribution; The dipole D_3

1 Introduction

Let G be a connected graph, allowing self-loops and multiple edges, with vertex set $V(G)$ and edge set $E(G)$. Let $|X|$ denote the cardinality of a set X. A *surface* means a compact connected 2-dimensional manifold without boundary. In topology, surfaces are classified into S_m, the orientable surface with $m(m \geq 0)$ handles and N_n, the non-orientable surface with $n(n \geq 1)$ crosscaps. An *embedding* of G into a closed surface S is a homeomorphism $\varphi : G \longrightarrow S$ of G into S. If every component of $S - \varphi(G)$ is a 2-cell, then φ is said to be a *2-cell embedding.* Throughout this paper, all embeddings of graphs into surfaces are 2-cell embeddings. Two embeddings $\xi : G \to S$ and $\eta : G \to S$ of G into S are said to be equivalent if there is a orientation-preserving homeomorphism $\zeta : S \to S$ such that $\zeta \circ \xi = \eta$. Basic terminologies for graph embedding appear in [6,15,19].

By the total genus polynomial of G, we mean the polynomial

$$I_G(x,y) = \sum_{i=0}^{\infty} g_i x^i + \sum_{i=1}^{\infty} f_i y^i,$$

where g_i is the number of embeddings of G into the orientable surface S_i up to equivalence and f_i is the number of embeddings of G into the non-orientable surface N_i up to equivalence.

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We call the first part of $I_G(x,y)$ the genus polynomial of G and denote it by $g_G(x) = \sum_{i=0}^{\infty} g_i x^i$. Similarly, $f_G(y) = \sum_{i=1}^{\infty} f_i y^i$ is the crosscap number polynomial of G. They are both finite polynomials.

Let $G = \{V, E\}$ be a connected graph. A *rotation* at a vertex v of G is a cyclic permutation of the edge-ends incident on it. A *pure rotation system* of G is a list of rotations, one for each vertex of G. The total number of the pure rotation systems of G equals the product of numbers $(d_v - 1)!$, taken over all vertices $v \in V(G)$. A *general rotation system* of G is a pair (P, λ), where P is a pure rotation system and λ is a mapping $E(G) \longrightarrow \{0,1\}$. The edge e is said to be *twisted*(respectively, *untwisted*) if $\lambda(e) = 1$(respectively, $\lambda(e) = 0$). It is well known that every orientable embedding of G can be described by a general rotation system (P, λ) with $\lambda(e) = 0$, for all $e \in E(G)$. Let T be a spanning tree of G. A *T-rotation system* (P, λ) of G is a general rotation system (P, λ) such that $\lambda(e) = 0$, for all $e \in E(T)$.

Let Φ_G^T be the set of all T-rotation systems of G. Suppose that in these $|\Phi_G^T|$ T-rotation systems of G, there are a_i, $i = 0, 1, \ldots,$ embeddings into the orientable surface S_i and b_i, $i = 0, 1, \ldots,$ embeddings into the non-orientable surface N_i, we call the polynomial

$$I_G^T(x,y) = \sum_{i=0}^{\infty} a_i x^i + \sum_{i=1}^{\infty} b_i y^i$$

the $T-$distribution polynomial of G.

Theorem 1.1.[2] *Let T be a spanning tree of graph G. For any general rotation system (P, λ) of G, there is an equivalent T-rotation system of G. The total genus polynomial $I_G(x,y)$ is equal to the $T-$distribution polynomial $I_G^T(x,y)$. The total number of embeddings of G up to equivalence is*

$$2^{\beta(G)} \prod_{v \in V(G)} (d_v - 1)!.$$

Given a graph $G = \{V, E\}$ such that $|V(G)| = n$ and $|E(G)| = q$. Let T be a spanning tree of G. Let (P, λ) be a T-rotation system of G, and let $E_1 = \{e \mid \lambda(e) = 1, e \in E(G)\}$. Suppose that e_1 and e_2 are two co-tree edges in $E - E(T)$. We say that e_1 and e_2 *overlap* with respect to P, E_1 and T either if $e_1 = e_2$ and $e_1 \in E_1$, or if $e_1 \neq e_2$ and the induced embedding I of $T + e_1 + e_2$ from the pure rotation system P is non-planar (I is obtained by deleting all edges from P except the edges e_1, e_2, and the edges in T). Let $e_1, e_2, \ldots, e_{\beta(G)}$ be the co-tree trees of T, where $\beta(G)$ is the cycle rank number of G, and $\beta(G) = q - n + 1$. The *overlap matrix* of G with respect to P, E_1 and T is a $\beta \times \beta$ matrix M over $GF(2)$ such that the (i,j) element of M is 1 if and only if the edges e_1 and e_2 overlap with respect to P, E_1 and T.

The following proposition is obvious.

Proposition 1.2. *Let T be a spanning tree of graph G, and let e and f be two co-tree edges of T in G. Then e and f overlap only if their fundamental cycles with respect to T have at least one vertex in common.*

Mohar [16] has shown the following interesting result.

Theorem 1.3.[16] *Let (P, λ) be a general rotation system for a graph, and let M be the overlap matrix. Then the rank of M equals twice the genus, if the corresponding embedding surface is orientable, and it equals the crosscap number otherwise. It is independent of the choice of a spanning tree.*

In 1987, Gross and Furst [7] introduced the concept of the genus distribution. Subsequently, many authors have computed the genus distributions for certain classes of graphs. Gross et al. computed the genus distribution for bouquets of circles[8]; Furst et al. computed it for closed-end ladders and cobblestone paths[5]; Tesar computed it for Ringel ladders[21]; Kwak computed it for dipoles[13], and many others, we only list a few, see [1,20,22,23]. However, for the total embedding distributions, only few classes are known. For example, Chen, Gross and Rieper computed the total embedding distributions for necklaces of type$(r, 0)$, closed-end ladders and cobblestone paths[2], by using the overlap matrix; Kwak and Shim computed the total embedding distribution for bouquets of circles [14], by using edge-attaching surgery technique; In [3], Chen, Liu and Wang computed the total embedding distributions of all graphs with maximum genus 1, by using the overlap matrix; Furthermore, in [4], Chen, Ou and Zou obtained a closed formula for the total embedding distribution of Ringel ladders; Yang and Liu computed the total embedding distributions for two classes of 4-regular graphs[24], by using the joint tree model of graph embedding which was established by Y.P. Liu. We use the overlap matrix herein to compute the total embedding distributions of two classes of graphs from the dipole D_3. For complementary work on genus distribution of graph amalgamations, see [9,10,11,12,18].

2 The total embedding distribution of L_n

We introduce the concept of bar-amalgamation, which can also be seen in [7]. Define *the bar-amalgamation* of a single-rooted graph (G, t) and a double-rooted graph (H, u, v) to be the result of joining the root t of G and the root u of H by an edge e. We denote this operation by $\oplus_e$:

$$(G, t) \oplus_e (H, u, v) = (G \oplus_e H, v).$$

In [9], Gross et al. defined *the vertex-amalgamation* of a single-rooted graph (G, t) and a double-rooted graph (H, u, v), which is obtained from their disjoint union by merging the roots t and u. We denote the operation of the vertex-amalgamation by an asterisk:

$$(G,t)*(H,u,v) = (G*H,v).$$

A *dipole graph* D_n is a multi-graph, which consists of two vertices joined by n edges.

Let $\ddot{D}_3$ be the graph obtained by inserting vertices u and v at the midpoints of two different edges of the dipole graph D_3, and by regarding them as the roots. We construct a sequence of graphs recursively: $(L_1,t_1) = (\ddot{D}_3,v)$ (suppressing co-root u); $(L_n,t_n) = (L_{n-1},t_{n-1}) \oplus_e (\ddot{D}_3,u,v)$. (See Figure 2.1)

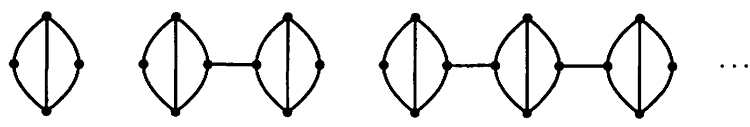

Figure 2.1: L_n is an open chain of n copies of $\ddot{D}_3$.

In [7], Gross and Furst proved the following theorem.

Theorem 2.1. $g_{G\oplus_e H}(x) = d_G(t)d_H(u)g_G(x)g_H(x)$.

In [3], Chen et al. presented the following theorem.

Theorem 2.2. $f_{G\oplus_e H}(y) = d_G(t)d_H(u)(f_G(y)f_H(y) + f_G(y)g_H(y^2) + g_G(y^2)f_H(y))$.

We can obtain a recursion formula about the total genus polynomial of L_n by Theorem 2.1 and Theorem 2.2, but herein expedites the calculation of the total genus polynomial of L_n with the overlap matrix.

Since we are examining topological properties of a graph, we often ignore all 2-valent vertices.

A 3-regular graph at each vertex has two cyclic orderings of its neighbors. One of these two cyclic orderings is denoted as *clockwise* and the other *counterclockwise*. We color the vertex *black*, if that vertex has the clockwise ordering of its neighbors, otherwise, we will color the counterclockwise vertex *white*. This will offer convenience to embed a 3-regular graph into surface, as we can draw an embedding on the plane and only need to color the vertices black and white. We adopt the same notation used by Ringel [17, p.17].

We define an edge is *matched* if it has the same color at both ends, otherwise it is called *unmatched*.

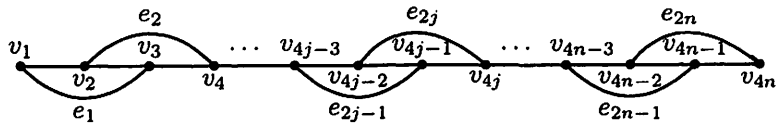

Figure 2.2: L_n.

In Figure 2.2, fix a spanning tree $T = v_1v_2v_3v_4 \cdots v_{4n-3}v_{4n-2}v_{4n-1}v_{4n}$ of L_n, list all its co-tree edges $e_1, e_2, \ldots, e_{2n-1}, e_{2n}$ of T in L_n, and use them as row and column labels of the overlap matrix M_{2n} with respect to T.

By Proposition 1.2, in the case of L_n, it implies that only e_{2i-1} and e_{2i} can overlap, for $i = 1, \ldots, n$. Given a T-rotation system (P, λ) of L_n, the corresponding overlap matrix M_{2n} for (P, λ) be of the following form:

$$M_{2n} = \begin{pmatrix} x_1 & y_1 & 0 & 0 & & 0 & 0 \\ y_1 & x_2 & 0 & 0 & & 0 & 0 \\ 0 & 0 & x_3 & y_2 & & 0 & 0 \\ 0 & 0 & y_2 & x_4 & & 0 & 0 \\ & & & & \ddots & & \\ 0 & 0 & 0 & 0 & & x_{2n-1} & y_n \\ 0 & 0 & 0 & 0 & & y_n & x_{2n} \end{pmatrix},$$

where $X = (x_1, x_2, x_3, x_4, \ldots, x_{2n-1}, x_{2n}) \in (GF(2))^{2n}$, $Y = (y_1, y_2, \ldots, y_n) \in (GF(2))^n$. Note that $x_i = 1$ if and only if the edge e_i is twisted, for all $i = 1, 2, \ldots, 2n$, $y_j = 1$ if and only if e_{2j-1} and e_{2j} overlap, for all $j = 1, 2, \ldots, n$.

Let M_{2n}^0 be a $2n \times 2n$ symmetric matrix over $GF(2)$ that is of the form M_{2n} such that the diagonal elements are all 0.

Let Φ be the set of all matrices over $GF(2)$ that are of the form M_{2n}, and let Φ^0 be the set of all matrices over $GF(2)$ that are of the form M_{2n}^0, we calculate the distributions of ranks of matrices in Φ and Φ^0.

Let $D_\Phi(z) = \Sigma_{i=0}^{2n} c_i z^i$ be the rank distribution polynomial of the set Φ, i.e., for $i = 0, 1, \ldots, 2n$, there are precisely c_i matrices in Φ of rank i, and let $D_{\Phi^0}(z) = \Sigma_{i=0}^{2n} c_i' z^i$ be the rank distribution polynomial of the set Φ^0, for $i = 0, 1, \ldots, 2n$, there are precisely c_i' matrices of rank i in Φ^0.

The following lemma is obvious.

Lemma 2.3. $D_\Phi(z) = (1 + 3z + 4z^2)^n; D_{\Phi^0}(z) = (1 + z^2)^n$.

Proposition 2.4. *Two co-tree edges e_{2j-1} and e_{2j} of T in L_n, for $j = 1, 2, \ldots, n$, overlap if and only if the edge $v_{4j-2}v_{4j-1}$ is unmatched.*

Proposition 2.5. *For a fixed matrix of the form M_{2n}, there are exactly 2^{3n-2} different T-rotation systems corresponding to that matrix.*

Proof. Given a matrix M_{2n}, the values of $y_1, y_2, \ldots, y_n$ are determined. For $j = 1, 2, \ldots, n$, let $V_j = \{v_{4j-2}, v_{4j-1}\}$. Let $V = \cup_{j=1}^{n} V_j$. For $i = 1, 2, \ldots, n-1$, let $U_i = \{v_{4i}, v_{4i+1}\}$. Let $U = \cup_{i=1}^{n-1} U_i$.

- $y_j = 0$. If we color the vertex v_{4j-2} black, by Proposition 2.4, the color of v_{4j-1} is black. Otherwise the vertex v_{4j-2} is colored white, by Proposition 2.4, the color of v_{4j-1} is also white.
- $y_j = 1$. Similar discuss like the case $y_j = 0$.

Therefore, for the matrix M_{2n}, there are two ways for V_j to color and 2^n ways for V to color.

On the other hand, for any fixed matrix M_{2n}, it is independent of the colors of vertices in U. Since there are $2n-2$ vertices in U, and 2 ways to color each vertex in U, there are 2^{2n-2} ways for U to color.

According to Theorem 1.3, Lemma 2.3 and Proposition 2.5, the following theorems are obtained.

Theorem 2.6. *The genus polynomial of the graph L_n is $g_{L_n}(x) = 2^{3n-2}(1+x)^n$.*

Theorem 2.7. *The crosscap number polynomial of the graph L_n is $f_{L_n}(y) = 2^{3n-2}[(1+3y+4y^2)^n - (1+y^2)^n]$.*

Theorem 2.8. *The total genus polynomial of the graph L_n is $I_{L_n}(x,y) = g_{L_n}(x) + f_{L_n}(y) = 2^{3n-2}(1+x)^n + 2^{3n-2}[(1+3y+4y^2)^n - (1+y^2)^n]$.*

The total genus polynomials of the graphs L_n for $n = 1, 2, 3, 4$ are as follows:

$I_{L_1}(x,y) = 2 + 2x + 6y + 6y^2$;

$I_{L_2}(x,y) = 16 + 32x + 16x^2 + 96y + 240y^2 + 384y^3 + 240y^4$;

$I_{L_3}(x,y) = 128 + 384x + 384x^2 + 128x^3 + 1152y + 4608y^2 + 12672y^3 + 19584y^4 + 18432y^5 + 8064y^6$;

$I_{L_4}(x,y) = 1024 + 4096x + 6144x^2 + 4096x^3 + 1024x^4 + 12288y + 67584y^2 + 258048y^3 + 617472y^4 + 1032192y^5 + 1142784y^6 + 786432y^7 + 261120y^8$.

Remark: The above four formulas are consistent with the results which have been obtained by Theorem 2.1 and Theorem 2.2.

3 The total embedding distribution of $L_n + e_0$

Let the graph $L_n + e_0$ be obtained by joining two 2-valent vertices of L_n with an edge e_0. See Figure 3.1. $L_n + e_0$ is a 3-regular graph. When $n = 1$, $L_n + e_0$ is isomorphic to the complete graph K_4.

We choose a path $P = v_1v_2v_3v_4 \cdots v_{4n-3}v_{4n-2}v_{4n-1}v_{4n}$ as the spanning tree T of $L_n + e_0$. List all its co-tree edges $e_0, e_1, e_2, \ldots, e_{2n-1}, e_{2n}$ of T in $L_n + e_0$, and use them as row and column labels of the overlap matrix $\tilde{M}_{2n+1}$ with respect to T.

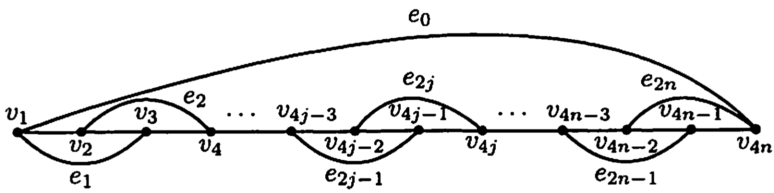

Figure 3.1: $L_n + e_0$.

By Proposition 1.2, it can be seen that the overlap matrix $\tilde{M}_{2n+1}$ over $GF(2)$ with respect to T has the following form:

$$\tilde{M}_{2n+1} = \begin{pmatrix} x_0 & z_1 & z_2 & z_3 & z_4 & \cdots & z_{2n-1} & z_{2n} \\ z_1 & x_1 & y_1 & 0 & 0 & & 0 & 0 \\ z_2 & y_1 & x_2 & 0 & 0 & & 0 & 0 \\ z_3 & 0 & 0 & x_3 & y_2 & & 0 & 0 \\ z_4 & 0 & 0 & y_2 & x_4 & & 0 & 0 \\ \vdots & & & & & \ddots & & \\ z_{2n-1} & 0 & 0 & 0 & 0 & & x_{2n-1} & y_n \\ z_{2n} & 0 & 0 & 0 & 0 & & y_n & x_{2n} \end{pmatrix}.$$

Note that $x_i = 1$ if and only if the edge e_i is twisted, for all $i = 0, 1, 2, \ldots, 2n$, $y_j = 1$ if and only if edges e_{2j-1} and e_{2j} overlap, for all $j = 1, 2, \ldots, n$, $z_k = 1$ if and only if edges e_0 and e_k overlap, for all $k = 1, 2, \ldots, 2n$.

Let $\tilde{M}^0_{2n+1}$ be a $(2n+1) \times (2n+1)$ symmetric matrix over $GF(2)$ that is of the form $\tilde{M}_{2n+1}$ such that the diagonal elements are all 0.

Now we denote $\tilde{\Phi}_{2n+1}$ to be the set of all matrices over $GF(2)$ that are of the form $\tilde{M}_{2n+1}$, and $\tilde{\Phi}^0_{2n+1}$ to be the set of all matrices over $GF(2)$ that are of the form $\tilde{M}^0_{2n+1}$.

Let $D_{\tilde{\Phi}_{2n+1}}(z) = \Sigma_{i=0}^{2n+1} \tilde{c}_i z^i$ be the rank distribution polynomial of the set $\tilde{\Phi}_{2n+1}$, i.e., for $i = 0, 1, \ldots, 2n+1$, there are precisely $\tilde{c}_i$ matrices in $\tilde{\Phi}_{2n+1}$ of rank i. Let $D_{\tilde{\Phi}^0_{2n+1}}(z) = \Sigma_{i=0}^{2n+1} \tilde{c}'_i z^i$ be the rank distribution polynomial of the set $\tilde{\Phi}^0_{2n+1}$, i.e., for $i = 0, 1, \ldots, 2n+1$, there are precisely $\tilde{c}'_i$ matrices in $\tilde{\Phi}^0_{2n+1}$ of rank i.

Lemma 3.1. $D_{\tilde{\Phi}_{2n+1}}(z) = (1 + 6z + 16z^2)^{n-1}(1 + 7z + 20z^2 + 4z^3 - 32z^4) + 2^{2n+1}z^2(1 + 3z + 4z^2)^n$.

Proof. We consider the following different ways to assign the variables x_{2n}, y_n, z_{2n}, x_{2n-1} and z_{2n-1} in the matrix $\tilde{M}_{2n+1}$. Each term from the last column of the following table denote the distribution to $D_{\tilde{\Phi}_{2n+1}}(z)$ under the corresponding case.

x_{2n}	y_n	z_{2n}	x_{2n-1}	z_{2n-1}	terms
0	0	0	0	0	$D_{\tilde{\Phi}_{2n-1}}(z)$
0	0	0	1	1	$zD_{\tilde{\Phi}_{2n-1}}(z)$
0	0	0	1	0	$zD_{\tilde{\Phi}_{2n-1}}(z)$
0	0	0	0	1	$2^{2n-1}z^2(1+3z+4z^2)^{n-1}$
0	0	1	0	0	$2^{2n-1}z^2(1+3z+4z^2)^{n-1}$
0	0	1	1	1	$2^{2n-1}z^3(1+3z+4z^2)^{n-1}$
0	0	1	1	0	$2^{2n-1}z^3(1+3z+4z^2)^{n-1}$
0	0	1	0	1	$2^{2n-1}z^2(1+3z+4z^2)^{n-1}$
1	1	0	0	0	$z^2D_{\tilde{\Phi}_{2n-1}}(z)$
1	1	0	1	1	$2^{2n-1}z^3(1+3z+4z^2)^{n-1}$
1	1	0	1	0	$zD_{\tilde{\Phi}_{2n-1}}(z)$
1	1	0	0	1	$z^2D_{\tilde{\Phi}_{2n-1}}(z)$
1	1	1	0	0	$z^2D_{\tilde{\Phi}_{2n-1}}(z)$
1	1	1	1	1	$zD_{\tilde{\Phi}_{2n-1}}(z)$
1	1	1	1	0	$2^{2n-1}z^3(1+3z+4z^2)^{n-1}$
1	1	1	0	1	$z^2D_{\tilde{\Phi}_{2n-1}}(z)$
1	0		0	0	$2zD_{\tilde{\Phi}_{2n-1}}(z)$
1	0		1	1	$2z^2D_{\tilde{\Phi}_{2n-1}}(z)$
1	0		1	0	$2z^2D_{\tilde{\Phi}_{2n-1}}(z)$
1	0		0	1	$2^{2n-1}2z^3(1+3z+4z^2)^{n-1}$
0	1				$8z^2D_{\tilde{\Phi}_{2n-1}}(z)$

The table implies a recurrence relation:
$D_{\tilde{\Phi}_{2n+1}}(z) = (1+6z+16z^2)\, D_{\tilde{\Phi}_{2n-1}}(z) + 2^{2n-1}(3z^2+6z^3)(1+3z+4z^2)^{n-1}$, the solution of which is:
$D_{\tilde{\Phi}_{2n+1}}(z) = (1+6z+16z^2)^{n-1}D_{\tilde{\Phi}_3}(z) + 2^{2n+1}z^2(1+3z+4z^2)^n - 8z^2(1+3z+4z^2)(1+6z+16z^2)^{n-1} = (1+6z+16z^2)^{n-1}(1+7z+28z^2+28z^3) + 2^{2n+1}z^2(1+3z+4z^2)^n - 8z^2(1+3z+4z^2)(1+6z+16z^2)^{n-1}$.

Lemma 3.2. $D_{\tilde{\Phi}^0_{2n+1}}(z) = (1+4z^2)^{n-1}(1+3z^2-4z^4) + 4^n z^2(1+z^2)^n$.

Proof. We consider the following different ways to assign the variables y_n, z_{2n} and z_{2n-1} in the matrix $\tilde{M}^0_{2n+1}$. Each term from the last column of the following table denote that the distribution to $D_{\tilde{\Phi}^0_{2n+1}}(z)$ under the corresponding case.

y_n	z_{2n}	z_{2n-1}	terms
0	0	0	$D_{\tilde{\Phi}^0_{2n-1}}(z)$
0	1	1	$2^{2n-2}z^2(1+z^2)^{n-1}$
0	1	0	$2^{2n-2}z^2(1+z^2)^{n-1}$
0	0	1	$2^{2n-2}z^2(1+z^2)^{n-1}$
1			$4z^2 D_{\tilde{\Phi}^0_{2n-1}}(z)$

The following recurrence relation is obtained from the table: $D_{\tilde{\Phi}^0_{2n+1}}(z) = (1+4z^2)D_{\tilde{\Phi}^0_{2n-1}}(z) + 3\cdot 2^{2n-2}z^2(1+z^2)^{n-1}$, the solution of which is $D_{\tilde{\Phi}^0_{2n+1}}(z) = (1+4z^2)^{n-1}D_{\tilde{\Phi}^0_3}(z) + 4^n z^2(1+z^2)^n - 4z^2(1+z^2)(1+4z^2)^{n-1} = (1+4z^2)^{n-1}(1+7z^2) + 4^n z^2(1+z^2)^n - 4z^2(1+z^2)(1+4z^2)^{n-1}$.

Proposition 3.3. *For $j = 1, 2, \ldots, n$, two co-tree edges e_{2j-1} and e_{2j} of T in $L_n + e_0$ overlap if and only if the edge $v_{4j-2}v_{4j-1}$ is unmatched.*

Proposition 3.4. *For $k = 1, 2, \ldots, 2n$, two co-tree edges e_0 and e_k of T in $L_n + e_0$ overlap if and only if the edge e_k is unmatched.*

Proposition 3.5. *For a fixed matrix of the form $\tilde{M}_{2n+1}$, there are exactly 2^n different T-rotation systems corresponding to that matrix.*

Proof. Given a matrix $\tilde{M}_{2n+1}$, the values of $y_1, y_2, \ldots, y_n$ and $z_1, z_2, \ldots, z_{2n}$ are determined. Let $V_j = \{v_{4j-3}, v_{4j-2}, v_{4j-1}, v_{4j}\}$, for $j = 1, 2, \ldots, n$. Let $V = \cup_{j=1}^n V_j$.

- *$y_j = 0$. If we color the vertex v_{4j-2} black, by Proposition 3.3, the color of the vertex v_{4j-1} is black. Since the values of z_{2j-1} and z_{2j} are given, by Proposition 3.4, the colors of v_{4j-3} and v_{4j} are determined. Otherwise the vertex v_{4j-2} is colored white, by Proposition 3.3, the color of the vertex v_{4j-1} is white, by the values of z_{2j-1} and z_{2j} and by Proposition 3.4, the colors of v_{4j-3} and v_{4j} are determined.*
- *$y_j = 1$. Similar discuss like the case $y_j = 0$, the details are omitted.*

Therefore, there are two ways for V_j to color and 2^n ways for V to color.

It follows from Theorem 1.3, Lemma 3.1, Lemma 3.2 and Proposition 3.5 that the following theorems are clear.

Theorem 3.6. *The genus polynomial of the graph L_n+e_0 is $g_{L_n+e_0}(x) = 2^n(1+4x)^{n-1}(1+3x-4x^2) + 2^{3n}x(1+x)^n$.*

Theorem 3.7. *The crosscap number polynomial of the graph L_n+e_0 is $f_{L_n+e_0}(y) = 2^n(1+6y+16y^2)^{n-1}(1+7y+20y^2+4y^3-32y^4) + 2^{3n+1}y^2(1+3y+4y^2)^n - 2^n(1+4y^2)^{n-1}(1+3y^2-4y^4) - 2^{3n}y^2(1+y^2)^n$.*

The total genus polynomials of graphs $L_n + e_0$ for $n = 1, 2, 3, 4$ are as follows:

$I_{L_1+e_0}(x,y) = 2 + 14x + 14y + 42y^2 + 56y^3$;

$I_{L_2+e_0}(x,y) = 4 + 92x + 160x^2 + 52y + 348y^2 + 1712y^3 + 3264y^4 + 2560y^5$;

$I_{L_3+e_0}(x,y) = 8 + 600x + 1824x^2 + 1664x^3 + 152y + 1800y^2 + 16512y^3 + 61920y^4 + 145536y^5 + 187776y^6 + 106496y^7$;

$I_{L_4+e_0}(x,y) = 16 + 4336x + 17664x^2 + 27136x^3 + 16384x^4 + 400y + 8688y^2 + 134272y^3 + 734976y^4 + 2671872y^5 + 6312960y^6 + 9945088y^7 + 9486336y^8 + 4194304y^9$.

4 Conclusions

The above calculations are two illustrations of Theorem 1.3 by Mohar that relates topological types of embedding surfaces to ranks of the corresponding overlap matrices. We conclude that, computing the total embedding distribution of a graph with the overlap matrix, one can consider the following three points:

1: Choose a spanning tree T of the objective graph, then derive the corresponding overlap matrix M with respect to T.

2: For each fixed matrix of the form M, analyze how many different T-rotation systems corresponding to that matrix.

3: For each fixed matrix of the form M, calculate the rank of that matrix.

Take another example. We construct a sequence of graphs recursively: $(\dot{L}_1, t_1) = (\ddot{D}_3, v)$ (suppressing co-root u); $(\dot{L}_n, t_n) = (\dot{L}_{n-1}, t_{n-1}) * (\ddot{D}_3, u, v)$. See Figure 4.1.

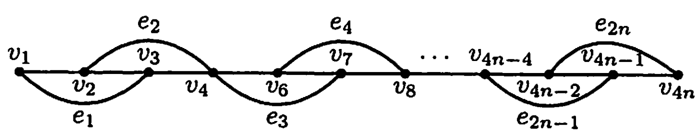

Figure 4.1: Vertex-amalgamations of n copies of $\ddot{D}_3$.

The following theorems can be obtained in a similar way as in Section 2 and Section 3 (It may be helpful to precede the calculation of the total genus polynomial of $\dot{L}_n$ with the reading of [2, p.86-p.93]).

Theorem 4.1. *The genus polynomial of $\dot{L}_n$ is* $g_{\dot{L}_n(x)} = \sum_{i_1,\ldots,i_r>0}^{i_1+\cdots+i_r=2n} 2^{2n-1+c(i_1,i_2,\ldots,i_r)} x^{\sum_{h=1}^{r}\lfloor \frac{i_h}{2} \rfloor}$. *Where $i_1, i_2, \ldots, i_r$ are positive integers, $c(i_1, i_2, \ldots, i_r)$ equals the number of even numbers of the set $\{i_1, i_1 + i_2, \ldots, i_1 + \cdots + i_{r-1}\}$.*

Theorem 4.2. *The crosscap number polynomial of $\dot{L}_n$ is* $f_{\dot{L}_n}(y) = \sum_{i_1,\ldots,i_r>0}^{i_1+\cdots+i_r=2n} 2^{2n-1+c(i_1,i_2,\ldots,i_r)} \prod_{h=1}^{r} (round(\frac{2^{i_h}}{3}) y^{i_h-1} + round(\frac{2^{i_h+1}}{3}) y^{i_h}) - \sum_{i_1,\ldots,i_r>0}^{i_1+\cdots+i_r=2n} 2^{2n-1+c(i_1,i_2,\ldots,i_r)} y^{\sum_{h=1}^{r} 2\lfloor \frac{i_h}{2} \rfloor}$. *Where $i_1, i_2, \ldots, i_r$ are positive integers, $c(i_1, i_2, \ldots, i_r)$ equals the number of even numbers of the set $\{i_1, i_1 + i_2, \ldots, i_1 + \cdots + i_{r-1}\}$ and round(t) equals the closest integer to* *the real number t.*

The total genus polynomials of $\dot{L}_1, \dot{L}_2$ and $\dot{L}_3$ are as follows:

$I_{\dot{L}_1}(x,y) = 2 + 2x + 6y + 8y^2;$

$I_{\dot{L}_2}(x,y) = 16 + 56x + 24x^2 + 104y + 304y^2 + 608y^3 + 424y^4;$

$I_{\dot{L}_3}(x,y) = 128 + 768x + 1120x^2 + 288x^3 + 1280y + 6048y^2 + 21600y^3 + 41760y^4 + 49152y^5 + 25312y^6.$

Remark: The above genus polynomials of $\dot{L}_1, \dot{L}_2$ and $\dot{L}_3$ are consistent with the results which have been obtained by Corollary 3.8 of [9].

Research Problem: Methods for deriving total embedding distribution analogues of the theorems in [9], [10], [12] and [18] would be developed in the future.

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