On group choosability of graphs, I

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Abstract

We investigate the *group choice number* of a graph G and prove the group list coloring version of Brooks' Theorem, the group list coloring version of Szekeres-Wilf extension of the Brooks' Theorem, and the Nordhaus-Gaddum inequalities for group choice numbers. Furthermore, we characterize all D-group choosable graphs and all 3-group choosable complete bipartite graphs.

Keywords: List coloring; Group coloring; Group choosability.

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1 Introduction

We consider finite and simple graphs. Undefined terms and notations can be found in [1]. Thus for a simple connected graph G, and for any $v \in V(G)$, $d_G(v)$, $\Delta(G)$, $\kappa(G)$, $c(G)$, and $\chi(G)$ denote the degree of vertex v, the maximum degree, the connectivity, the number of components of G and the chromatic number of G, respectively. When the graph G is understood from the context, we also use $d(v)$ for $d_G(v)$. If X is a vertex subset or an edge subset, then $G[X]$ is the subgraph of G induced by X. Throughout this paper, $\mathbf{Z}$ denotes the set of integers, and for $m \in \mathbf{Z}$ with $m > 0$, $\mathbf{Z}_m$ denote the cyclic group of order m.

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Erdős, Rubin and Taylor [3] and Vising [13] introduced graph list colorings. A *list assignment* of a graph G is a function L that assigns to each vertex $v \in V(G)$ a list $L(v)$ of colors. A proper vertex coloring c of G is an *L-coloring* of G if for any $v \in V(G)$, $c(v) \in L(v)$. For an integer k, a *k-list assignment* of G is a list assignment L with $|L(v)| = k$ for each vertex $v \in V(G)$; G is *k-choosable* if G has an L-coloring for every k-list assignment L of G. The *choice number*, $\chi_l(G)$, is the minimum k such that G is k-choosable.

Throughout this paper, A denotes a group with identity 0. We will use addition to denote the binary operation of A even when A is not Abelian. For a graph G, let $F(G, A) = \{f : E(G) \mapsto A\}$. Fix an orientation of G. If for every $f \in F(G, A)$, G has a vertex coloring $c : V(G) \mapsto A$ be a map such that $c(x) - c(y) \neq f(xy)$ for each edge directed from x to y, then G is *A-colorable.* It is known [4] that whether G is A-colorable is independent of the orientation of G. The *group chromatic number* of G, $\chi_g(G)$, is the minimum k such that G is A-colorable for any group A of order at least k.

Král and Nejedlý [5] further introduced the *group choosability* of graphs. Given a digraph G with a list assignment $L : V(G) \mapsto 2^A$, for an $f \in F(G, A)$, an *(A, L, f)-coloring* is an L-coloring $c : V(G) \mapsto A$ such that $c(x) - c(y) \neq f(xy)$ for every edge directed from x to y. If for any $f \in F(G, A)$, G has an (A, L, f)-coloring, then G is *(A, L)-colorable.* It is routine to show that whether G is *(A, L)-colorable* is independent of the orientation.

If G is (A, L)-colorable for each group A of order at least k and for any k-list assignment $L : V(G) \mapsto 2^A$, then G is *k-group choosable.* The minimum k for which G is k-group choosable is the *group choice number* of G and is denoted by $\chi_{gl}(G)$. The following inequalities follow from the definitions.

$$\chi_{gl}(G) \geq \max\{\chi_g(G), \chi_l(G)\} \geq \min\{\chi_g(G), \chi_l(G)\} \geq \chi(G). \qquad (1)$$

A graph G is *D-group choosable* if it is (A, L)-colorable for every group A with $|A| \geq \Delta(G)$, and for every list assignment $L : V(G) \mapsto 2^A$ with $|L(v)| = d(v)$, for any $v \in V(G)$.

The choice number, the group chromatic number and the chromatic number have been intensively studied (see e.g.[3, 8, 11, 12] and the references therein). Erdős, Rubin and Taylor [3] proved a list coloring version of the Brooks' Theorem, while Lai and Zhang [8] obtained its group coloring version. Utilizing D-group choosability, we in this paper prove a group list coloring version of the Brooks' Theorem.

Theorem 1.1 *For any connected simple graph G, we have,*

$$\chi_{gl}(G) \leq \Delta(G) + 1,$$

with equality if and only if G is either a cycle or a complete graph.

In Section 2, we characterize the D-group choosable graphs and present an example G so that $\chi_{gl}(G) > \max\{\chi_g(G), \chi_l(G)\}$. In the other sections, we prove the Szekeres-Wilf extension of Theorem 1.1, a Nordhaus-Gaddum type Theorem for group choice numbers, and a characterization of complete bipartite graphs with group choice number at most 3, respectively.

2 List coloring extension and D-group choosability of graphs

In this section, we shall characterize all D-group choosable graphs. This result is used in the next section to prove a group list coloring version of Brooks' Theorem.

Let $H \subseteq G$, A be a group and $L : V(G) \mapsto 2^A$ be a function. Suppose that $f \in F(G, A)$. If for an $(A, L|_H, f|_H)$-coloring c_0 of H there is an (A, L, f)-coloring c of G such that c is an extension of c_0, then we say that c_0 is *extended to* c. If any $(A, L|_H, f|_H)$-coloring c_0 of H can be extended to an (A, L, f)-coloring c of G, then we say that (G, H) is (A, L, f)*-extensible.* If for any $f \in F(G, A)$, (G, H) is (A, L, f)-extensible then (G, H) is (A, L)*-extensible.* The next lemma follows from the definitions.

Lemma 2.1 *Let G be a graph, A be a group and $L : V(G) \mapsto 2^A$ be a function. Then,*
(i) Suppose that $H \subseteq G$. If (G, H) is (A, L)-extensible and if H is (A, L)-colorable, then G is (A, L)-colorable,
(ii) Suppose that $H_2 \subseteq H_1 \subseteq G$. If (G, H_1) and (H_1, H_2) are (A, L)-extensible, then (G, H_2) is also (A, L)-extensible.

We prepare some lemmas below which are needed in the characterization of D-group choosable graphs, and in other proofs of this paper.

Lemma 2.2 *Suppose that G is a graph and the vertices of G, $v_1, \ldots, v_n$ are so ordered that for $i = 1, \ldots, n$, if $G_i = G[v_1, \ldots, v_i]$, then $d_{G_i}(v_i) \leq k$. For any group A of order at least $k+1$ and for any list assignment $L : V(G) \mapsto$*

2^A *with* $|L(v)| \geq k+1$, *for any* $v \in V(G)$, (G_{i+1}, G_i) *is* (A, L)*-extensible. Consequently,* G *is* (A, L)*-colorable.*

Proof. For any edge $e = v_{j_1}v_{j_2} \in E(G)$ with $j_1 > j_2$ orient e from v_{j_1} to v_{j_2}. Let D denote the resulting orientation. Suppose that $f \in F(G_{i+1}, A)$ and c_1 is an $(A, L|_{G_i}, f|_{G_i})$-coloring of G_i. Let $v_{i_1}, \ldots, v_{i_d}$ denote the neighbors of v_{i+1} in G_{i+1}. As $|L(v_{i+1})| \geq k+1$ and $d_{G_{i+1}}(v_{i+1}) \leq k$, it follows that $B = L(v_{i+1}) - \{f(v_{i+1}v_{i_1}) + c_1(v_{i_1}), \ldots, f(v_{i+1}v_{i_d}) + c_1(v_{i_d})\} \neq \emptyset$. By coloring v_{i+1} with some $t \in B$, we extend c_1 to an $(A, L|_{G_{i+1}}, f|_{G_{i+1}})$-coloring of G_{i+1}. Hence (G_{i+1}, G_i) is (A, L)-extensible for $i = 1, \ldots, n-1$. As G_1 is (A, L)-colorable, G is (A, L)-colorable by Lemma 2.1. ∎

Lemma 2.3 *Let* G *be a graph, then* $\chi_{gl}(G) \leq \max_{H \subseteq G}\{\delta(H)\} + 1$.

Proof. Let $k = \max_{H \subseteq G}\{\delta(H)\}$. Then the vertices of G can be ordered as $v_1, v_2, \ldots, v_n$, satisfying the hypothesis of Lemma 2.2, and so this lemma follows from Lemma 2.2. ∎

Lemma 2.4 *Let* G *be a forest,* $L(v) = \mathbb{Z}_2$ *for each* $v \in V(G)$ *and* $H \subseteq G$. *Then* (G, H) *is* $(\mathbb{Z}_2, L)$*-extensible if and only if any two components of* H *belong to two different components of* G.

Proof. Without loss of generality, we assume that G is a tree, and prove that (G, H) is $(\mathbb{Z}_2, L)$-extensible if and only if H is a connected subgraph of G. First let H be connected and let $e = u_0v_0$ be a directed edge of G such that $u_0 \in V(H)$ and $v_0 \notin V(H)$. For an $f \in F(G, A)$, extend a $(\mathbb{Z}_2, L|_H, f|_H)$-coloring c_1 of H to a $(\mathbb{Z}_2, L|_{H \cup \{u_0v_0\}}, f|_{H \cup \{u_0v_0\}})$-coloring by coloring v_0 with $a \in L(v_0) - \{-f(u_0v_0) + c_1(u_0)\}$. Inductively, a $(\mathbb{Z}_2, L|_H, f|_H)$-coloring c_1 of H can be extended to a $(\mathbb{Z}_2, L, f)$-coloring c of G. This proves the sufficiency.

Conversely, suppose that H is disconnect with H_1 and H_2 being two components of H, and that $v_0v_1 \ldots v_k$ is a directed path of G such that $v_0 \in V(H_1)$, $v_k \in V(H_2)$ and $v_i \notin V(H)$ for $1 \leq i \leq k-1$. Define an $f \in F(G, \mathbb{Z}_2)$ such that $f(e_{k-1}) = 0$ and $f(e) = 1$, for any $e \in E(G) - \{e_{k-1}\}$. Let c_1 be a $(\mathbb{Z}_2, L|_H, f|_H)$-coloring such that $c_1(v) = 1$ for every $v \in V(H)$. It is routine to verify that c_1 can not be extended to a $(\mathbb{Z}_2, L, f)$-coloring for G and so (G, H) is not $(\mathbb{Z}_2, L)$-extensible. ∎

A graph G is *strongly* (A, L)*-colorable* if for every $H \subseteq G$, (G, H) is (A, L)-extensible.

Theorem 2.5 *Let A be a group with $|A| \geq 3$ and $L : V(G) \mapsto 2^A$ be a function with $|L(v)| \geq 3$ for each $v \in V(G)$. If G is a forest, then G is strongly (A, L)-colorable.*

Proof. Let H be a subgraph of G. Without loss of generality, we may assume that G is a tree. Argue by induction on $c(H)$. Argue similarly as in the proof of Lemma 2.4, the theorem holds when $c(H) = 1$. Let $k > 0$ be an integer and assume that the theorem holds when $c(H) \leq k$. Now suppose that H has $k+1$ components. Choose two components H_1 and H_2 of H and a directed path $P = v_0v_1 \dots v_k$ with $v_0 \in H_1$, $v_k \in H_2$ and $v_i \notin V(H)$ $(1 \leq i \leq k-1)$. Assume $f \in F(G, A)$ and $c_1 : V(H) \mapsto A$ is an $(A, L|_H, f|_H)$-coloring of H. Define $c : V(H \cup P) \mapsto A$ as follows. Let $c(v) = c_1(v)$ if $v \in V(H)$, $c(v_i) = a_i \in L(v_i) - \{-f(v_{i-1}v_i) + c(v_{i-1})\}$ $(1 \leq i \leq k-2)$ and $c(v_{k-1}) = a_{k-1} \in L(v_{k-1}) - \{-f(v_{k-2}v_{k-1}) + c(v_{k-2}), f(v_{k-1}v_k) + c(v_k)\}$. Then c is an $(A, L|_{H\cup P}, f|_{H\cup P})$-coloring of $H \cup P$. Since $c(H \cup P) = c(H) - 1 = k$, by induction, $c : V(H \cup P) \mapsto A$ can be extended to an (A, L, f)-coloring c' of G. Hence (G, H) is (A, L)-extensible and so G is strongly (A, L)-colorable. ∎

A θ-graph is a graph obtained by subdividing the edges of the loopless multigraph consisting of two vertices and three parallel edges.

Lemma 2.6 *Each θ-graph is D-group choosable.*

Proof. Orient the edges of a (labelled) θ-graph G as shown in Figure 2. Let

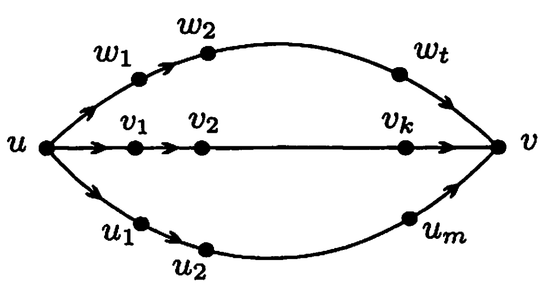

Figure 1: A directed θ-graph.

A be a group of order at least 3, $f \in F(G, A)$ be a function and $L : V(G) \mapsto 2^A$ be a map with $|L(v)| = d(v)$ for each $v \in V(G)$. First color u with $c(u) \in L(u) - \{f(uw_1) + a | a \in L(w_1)\}$. Let $u = u_0 = v_0$, and for each i with

$1 \leq i \leq m$ and j with $1 \leq j \leq k$, color u_i with $c(u_i) \in L(u_i) - \{-f(u_{i-1}u_i) + c(u_{i-1})\}$ and v_j with $c(v_j) \in L(v_j) - \{-f(v_{j-1}v_j) + c(v_{j-1})\}$. Since $d(v) = 3$, v can be colored with $c(v) \in L(v) - \{-f(u_m v) + c(u_m), -f(v_k v) + c(v_k)\}$. Let $v = w_{t+1}$. Then for each $1 \leq l \leq t$, color w_l with $c(w_l) \in L(w_l) - \{f(w_l w_{l+1}) + c(w_{l+1})\}$. Since $c(u) \in L(u) - \{f(uw_1) + a | a \in L(w_1)\}$, c is an (A, L, f)-coloring for G. ∎

Lemma 2.7 *If a connected graph G has a connected induced D-group choosable subgraph H, then G is D-group choosable.*

Proof. We argue by induction on $|V(G) - V(H)|$. If $V(G) = V(H)$, then the lemma holds trivially. Hence we assume that $V(G) - V(H) \neq \emptyset$. Let A be a group with $|A| \geq \Delta(G)$, $f \in F(G, A)$ b a function and $L : V(G) \mapsto 2^A$ be a map with $|L(v)| = d(v)$ for each $v \in V(G)$. Choose $x \in V(G) - V(H)$ to maximize the distance from x to H in G. Then $G - x$ is a connected and contains H. Without loss of generality, suppose that each edge incident at x is directed from x.

Pick any $t \in L(x)$. Define $\overline{L} : V(G - x) \mapsto 2^A$ be a map by

$$\overline{L}(v) = \begin{cases} L(v) - \{-f(xv) + t\} & \text{if } v \text{ is adjacent to } x \text{ in } G \\ L(v) & \text{otherwise.} \end{cases}$$

By induction, $G - x$ is D-group choosable and so it has an $(A, \overline{L}, f|_{G-x})$-coloring c. Extending c by coloring x with t, we obtained an (A, L, f)-coloring for G. ∎

The following theorem plays an important role in the proof of a group list coloring version of the Brook's Theorem.

Theorem 2.8 *Let G be a graph with $\kappa(G) \geq 2$. If G is neither a complete graph nor a cycle, then G has an induced θ-subgraph.*

Proof. Assume first that G contains a 3-cycle. Then G has a maximal clique H with $|V(H)| \geq 3$. Since $G \neq H$ and since $\kappa(G) \geq 2$, G has a path $P = xv_1v_2 \dots v_l y$ with $l \geq 1$ such that $|V(P) \cap V(H)| = 2$ and such that l is minimized. Let $V(P) \cap V(H) = \{x, y\}$ and pick $z \in V(H) - \{x, y\}$. If $zv_i \notin E(G)$ for each $1 \leq i \leq l$, then the induced subgraph on $\{z\} \cup V(P)$ is a θ-graph. Hence for some $1 \leq i \leq l$, $zv_i \in E(G)$. Let $P_1 = xv_1 \dots v_i z$ and $P_2 = zv_i v_{i+1} \dots v_l y$ (as depicted in Figure 2). Since $|V(P)| \leq \min\{|V(P_1)|, |V(P_2)|\}$, we have $l = 1$ and $zv_1 \in E(G)$. Since

H is a maximal clique of G, $G[V(H) \cup \{v_1\}]$ is not a clique of G, and so $tv_1 \notin E(G)$ for some $t \in V(H) - \{x, y\}$. It follows that $G[\{x, v_1, y, t\}]$ is a θ-graph. Hence we may assume that G is triangle free.

Let C be a shortest cycle of G. Since $G \neq C$ and since $\kappa(G) \geq 2$, G has a path $P = xv_1 \dots v_l y$ with $l \geq 1$, such that $|V(C) \cap V(P)| = 2$ and $V(C) \cap V(P) = \{x, y\}$, and such that l is minimized. If $C \cup P$ is not an induced θ-graph, then for some $v_i \in V(P)$ and $z \in V(C) - \{x, y\}$, v_i is adjacent to z. Suppose that $P_1 = xv_1 \dots v_i z$ and $P_2 = zv_i \dots v_l y$. Since $|V(P)| \leq \min\{|V(P_1)|, |V(P_2)|\}$, we have $l = 1$. Let Q' and Q tow internally disjoint (x, y)-paths of C (see Figure 3). Since C is a shortest cycle, both Q' and Q are 2-paths. Hence $v_1 yzv_1$ is a triangle, contrary to the assumption that G is triangle-free. This implies that $C \cup P$ must be an induced θ-graph. ∎

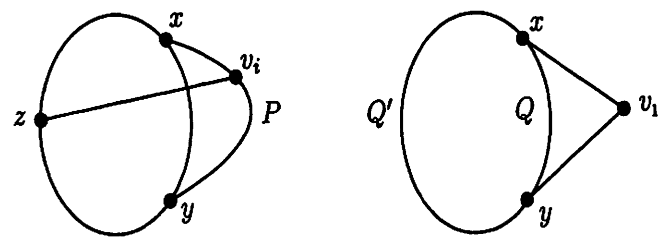

Figure 2: The two graphs in the proof of Theorem 2.8.

The lemma below follows from Lemmas 2.6, 2.7 and Theorem 2.8.

Lemma 2.9 *If G has a block B which is neither a complete graph nor a cycle, then G is D-group choosable.*

Since $\chi_g(K_n) = n$ and $\chi_g(C_n) = 3$ (see [8]), by (1) and Lemma 2.3, both $\chi_{gl}(K_n) = n$ and $\chi_{gl}(C_n) = 3$. Hence both K_n and C_n are not D-group choosable.

Theorem 2.10 *Let G be a connected graph. Then G is D-group choosable if and only if G has a block which is neither a complete graph nor a cycle.*

Proof. If G has a block B which is neither a complete graph nor a cycle, then by Lemma 2.9, it is D-group choosable. Hence it suffices to prove the necessity of the theorem.

Let $b(G)$ the number of blocks of G. We argue by induction on $b(G)$. The theorem holds trivially if $b(G) = 1$, and so we assume that $b(G) > 1$, and the theorem holds for graphs with smaller values of $b(G)$. It remains to show that if every block of G is either a cycle or a complete graph, then G is not D-group choosable.

Let $B_1, \ldots, B_k$ be the blocks of G. It is well known (see Page 121 of [1], for example) if G with $|V(G)| \geq 3$ is connected but not 2-connected, then G has at least two end blocks. Suppose that B_1 is an end block and $V(B_1) \cap (\cup_{i=2}^{k} V(B_i)) = \{v\}$. By induction, $K = G - V(B_1 - v)$ is not D-group choosable. Thus for some group A_1 with $|A_1| \geq \Delta(K)$, an $f_1 \in F(K, A_1)$ and an $L_1 : V(K) \mapsto 2^{A_1}$ with $|L_1(w)| = d_K(w)$ for each $w \in V(K)$, K is not (A_1, L_1, f_1)-colorable. Since K_n and C_m are not D-group choosable, there is a group A' with $|A'| \geq \Delta(B_1)$, an $f_2 \in F(B_1, A')$ and an $L_2 : V(B_1) \mapsto 2^{A'}$ such that B_1 is not (A', L_2, f_2)-colorable. Let $A = A_1 \oplus A'$ be the direct sum of A_1 and A', $L : V(G) \mapsto 2^A$ with $L(v) = L_1(v) \cup L_2(v)$, $L(w) = L_1(w)$ for $w \in V(G - B_1)$ and $L(w) = L_2(w)$ for $w \in V(B_1) - \{v\}$. Define $f \in F(G, A)$ so that $f(e) = f_1(e)$ if $e \in E(K)$, and $f(e) = f_2(e)$ if $e \in E(B_1)$. If G has an (A, L, f)-coloring c, then for $c(v) \in L_1(v)$, K is (A_1, L_1, f_1)-colorable and for $c(v) \in L_2(v)$, B_1 is (A', L_2, f_2)-colorable, contrary to assumptions. Therefore, G is not (A, L, f)-colorable and the proof for the theorem completes. ∎

As suggested by (1), we will investigate the existence of graphs G such that $\chi_{gl}(G) > \max\{\chi_g(G), \chi_l(G)\}$. To do that, the concept of *group connectivity* will be needed. For an Abelian group A, a graph G is A-*connected* if for every $b : V(G) \mapsto A$ with $\sum_{v \in V(G)} b(v) = 0$, there exists a $f \in F(G, A)$ such that for each $e \in E(G)$, $f(e) \neq 0$ and for every $v \in V(G)$, the net out flow at v equals to $b(v)$. A *wheel* of order n, denoted by W_n, is a graph obtained by adjoining a new vertex to the vertices of an n vertex cycle C_n.

Theorem 2.11 *Let A be an Abelian group. Then,*
i) *[2] If $|A| \geq 3$ and $n \in \mathbb{N}$, then W_{2n} is A-connected,*
ii) *[4] If G is a plane graph, then it is A-connected if and only if its dual graph is A-colorable.*

Corollary 2.12 *If A is an Abelian group with $|A| \geq 3$, then W_{2n} is A-colorable.*

Lemma 2.13 *[8] Let G be a connected graph and A be an Abelian group. Then G is A-colorable if and only if each block of G is A-colorable.*

The following example shows that the first inequality in (1) can be strict. Let G denote the graph depicted in Figure 2. We show that $\chi_{gl}(G) = 4$, while $\chi_g(G) = \chi_l(G) = 3$.

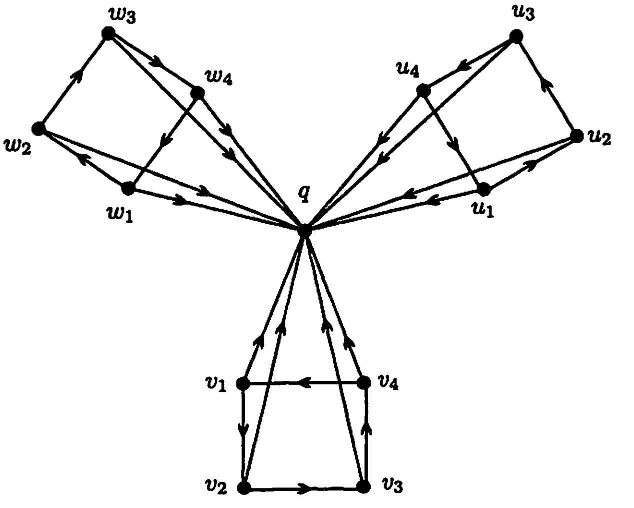

Figure 3: A graph G with $\chi_{gl}(G) > \max\{\chi_g(G), \chi_l(G)\}$.

By Corollary 2.12 and Lemma 2.13, G is $\mathbb{Z}_3$-colorable. On the other hand, by Lemma 2.3, $\chi_g(G) \le \chi_{gl}(G) \le 4$ and so $\chi_g(G) = 3$. Moreover, by an easy argument we get $\chi_l(G) = 3$. Now assume that $A = \mathbb{Z}_8$ is the cyclic group of order 8 and $a \in A$ is the element of order 2 and $x, y, z \in A - \{0, a\}$. Let $L : V(G) \mapsto 2^A$ be a list assignment of G with $L(v_i) = \{x, a, 0\}$, $L(u_i) = \{y, a, 0\}$, $L(w_i) = \{z, a, 0\}$ for $1 \le i \le 4$ and $L(q) = \{x, y, z\}$. Let $f \in F(G, A)$ with $f(e) = a$ for $e \in \{v_2v_3, u_2u_3, w_2w_3\}$ and $f(e) = 0$, otherwise. For each (A, L, f)-coloring $c : V(G) \mapsto A$, there exist v_i, u_j, w_k with $1 \le i, j, k \le 4$ such that $c(v_i) = x$, $c(u_i) = y$ and $c(w_k) = z$. Consequently, the vertex q can not admit any color of $L(q)$ and so $\chi_{gl}(G) \ge 4$. By Lemma 2.3, $\chi_{gl}(G) = 4$.

3 Brooks Type Theorems

We start this section with a proof for a group choice number version of Brooks Coloring Theorem.

Proof of Theorem 1.1: If G is a cycle or a complete graph, then $\chi_{gl}(G) = \Delta(G) + 1$. Now suppose that G is neither a complete graph nor a cycle. If G is not regular, then $\max_{H \subseteq G}\{\delta(H)\} \leq \Delta(G) - 1$ and so by Lemma 2.3, $\chi_{gl}(G) \leq \Delta(G)$. Thus, we assume that G is $\Delta(G)$-regular. If G is a 2-connected graph, then by Theorem 2.10, G is D-group choosable and so $\chi_{gl}(G) \leq \Delta(G)$. So suppose that G has a cut vertex. In this case, regularity of G implies that there is at least a block of G, such as B, which is neither a complete graph nor a cycle. Again by Theorem 2.10, G is D-group choosable and so $\chi_{gl}(G) \leq \Delta(G)$. ∎

Following Szekeres and Wilf [10], define γ to be a real-valued function on graphs satisfying the following two properties:
(P1) If H is an induced subgraph of G, then $\gamma(H) \leq \gamma(G)$.
(P2) If $\delta(G)$ is the minimum degree of G, then $\gamma(G) \geq \delta(G)$ with equality if and only if G is regular.

Szekeres and Wilf [10] presented an extension of the Brooks coloring theorem by replacing $\Delta(G)$ by $\gamma(G)$, as follows.

Theorem 3.1 *(Szekeres and Wilf, [10]) If γ is a real function on graphs with properties (P1) and (P2), then for each graph G, $\chi(G) \leq \gamma(G) + 1$.*

In [7], Lai et al extended Theorem 3.1 to its group coloring version. To determine the structure of graphs satisfying the equality, a concept of χ_g-semi critical graph is introduced in [7]. Following the same idea, we define a graph G to be *k_{gl}-semi critical* if $\chi_{gl}(G-v) < \chi_{gl}(G) = k$ for every vertex $v \in V(G)$ with $d(v) = \delta(G)$. Complete graphs and cycles are examples of semi k_{gl}-critical graphs. By definition, any graph G has a k_{gl}-semi critical subgraph H where $k = \chi_{gl}(G) = \chi_{gl}(H)$.

Lemma 3.2 *Let G be a graph, $v \in V(G)$ and $H = G - v$.*
(i) If $d_G(v) < \chi_{gl}(H)$, then $\chi_{gl}(G) = \chi_{gl}(H)$.
(ii) If G is k_{gl}-semi critical, then $d_G(v) \geq k - 1$ for all $v \in V(G)$.

Proof. We present the proof for (i) only as that for (ii) is similar to that for Lemma 2.3 in [7]. Since $H \subseteq G$, we have $\chi_{gl}(H) \leq \chi_{gl}(G)$. So it is sufficient to show that $\chi_{gl}(G) \leq \chi_{gl}(H)$. Let A be a group of order at least $\chi_{gl}(H)$, $L : V(G) \mapsto 2^A$ be a map with $|L(v)| = \chi_{gl}(H)$ for each $v \in V(G)$ and $f \in F(G, A)$. There is an $(A, L|_H, f|_H)$-coloring c' for H. Assume that $N_G(v) = \{v_1, v_2, \ldots, v_{d(v)}\}$ and G is oriented such that all the edges incident with v are directed from v. Since $d(v) < \chi_{gl}(H)$, by assigning $a \in L(v) - \bigcup_{i=1}^{d(v)}\{f(vv_i) + c'(v_i)\}$ to v we extend c' to an (A, L, f)-coloring for G and so $\chi_{gl}(G) \leq \chi_{gl}(H)$. This proves (i) of the lemma. ∎

Lemma 3.3 *If a connected graph G is k_{gl}-semi critical, then $k = \gamma(G) + 1$ if and only if G is either a cycle or a complete graph.*

Proof. Since a cycle C or a complete graph K_n is a regular graph, by (P2) we have $\gamma(C) = 2$ and $\gamma(K_n) = n - 1$. By definition, cycles C_n and complete graphs K_n are semi critical with $\chi_{gl}(C) = 3$ and $\chi_{gl}(K_n) = n$. Hence the sufficiency follows.

Conversely, suppose that G is a k_{gl}-semi critical graph and $\chi_{gl}(G) = m = \gamma(G) + 1$. By Lemma 3.2(ii) and (P2), $\chi_{gl}(G) - 1 \leq \delta(G) \leq \gamma(G) = \chi_{gl}(G) - 1$. Consequently, $\delta(G) = \gamma(G) = \chi_{gl}(G) - 1$ and so by (P2), G is regular. It follows that $\chi_{gl}(G) = \gamma(G) + 1 = \delta(G) + 1 = \Delta(G) + 1$. By Theorem 1.1, G is either a cycle or a complete graph. ∎

Let $m > 0$ be an integer. Following the same ideas in [7], we define $\mathcal{F}(m)$ to be a family of simple, connected graphs satisfying the following properties.

(F1) $\mathcal{F}(m) = \{K_m\}$ for $m = 1, 2$.

(F2) For $m = 3$, $G \in \mathcal{F}(m)$ if and only if either G is a cycle or $G - v \in \mathcal{F}(m)$ for a vertex v with $d(v) = 1$.

(F3) For $m \geq 4$, $G \in \mathcal{F}(m)$ if and only if either $G = K_m$ or $G - v \in \mathcal{F}(m)$ for a vertex v with $d(v) \leq m - 2$.

By Definition, $\mathcal{F}(3)$ is the set of connected unicyclic graphs. The next theorem extends Theorem 3.1 as well as Theorem 2.4 of [7].

Theorem 3.4 *If G is a connected graph and γ is a real function satisfying (P1) and (P2), then $\chi_{gl}(G) \leq \gamma(G) + 1$. Moreover, if $\chi_{gl}(G) = \gamma(G) + 1$, then $G \in \mathcal{F}(m)$ where $m = \chi_{gl}(G)$.*

Proof. Let $\chi_{gl}(G) = k$ and let $H \subseteq G$ be a k_{gl}-semi critical induced subgraph. By (P1), (P2) and lemma 3.2(ii), we have $k - 1 \leq \delta(H) \leq \gamma(H) \leq \gamma(G)$ and so $\chi_{gl}(G) = k \leq \gamma(G) + 1$.

If G is m_{gl}-semi critical for $m = \chi_{gl}(G)$, by Lemma 3.4, $G \in \mathcal{F}(m)$. Suppose that H_0 is a m_{gl}-semi critical subgraph of G where $m = \chi_{gl}(G)$. We may assume that $\chi_{gl}(G) \geq 3$. Then $\chi_{gl}(H_0) = \chi_{gl}(G) = \gamma(G) + 1 \geq \gamma(H_0) + 1 \geq \chi_{gl}(H_0)$. It follows that $\chi_{gl}(H_0) = \gamma(H_0) + 1$ and $\gamma(H_0) = \gamma(G)$. By Lemma 3.4, H_0 must be a cycle or a complete graph and so $\delta(H_0) = m - 1$. As $\delta(G) \geq m - 1$, we have $m = \chi_{gl}(G) = \gamma(G) + 1 \geq \delta(G) + 1 \geq m$. Hence $\gamma(G) = \delta(G) = m - 1$ and so G is regular. Since G is connected, $G = H_0 \in \mathcal{F}(m)$.

Now assume that $\delta(G) \leq m - 2$. Then G can not be a m_{gl}-semi critical graph, and so G has a vertex v with $d(v) = \delta(G)$ such that $\chi_{gl}(G) = \chi_{gl}(G - v)$. Now by induction on $|V(G)|$, we show that $G \in \mathcal{F}(m)$. By (P1),

$$\chi_{gl}(G - v) = \chi_{gl}(G) = \gamma(G) + 1 \geq \gamma(G - v) + 1 \geq \chi_{gl}(G - v).$$

It follows that $\chi_{gl}(G - v) = \gamma(G - v) + 1$. By induction hypothesis, $G - v \in \mathcal{F}(m)$. By the definition of $\mathcal{F}(m)$, $G \in \mathcal{F}(m)$. ∎

Example 3.5 *The k-degree, $k \geq 1$, of a vertex v of G is the number of walks of length k from v. The maximum k-degree of G is denoted by $\Delta_k(G)$. Let $\lambda(G)$ denote the maximum eigenvalue of G. Then it is routine to verify that both $\Delta_k(G)$ and $\lambda(G)$ satisfy (P1) and (P2). Consequently, $\chi_{gl}(G) \leq \min\{\Delta_k(G), \lambda(G)\} + 1$.*

4 Graphs G with $\chi_{gl}(G) \leq 2$ and Nordhaus-Gaddum Type Theorems

In this section, we apply former results to present a characterization of graphs G with that $\chi_{gl}(G) = 2$, and derive the Nordhaus-Gaddum type theorem for group choice number.

Proposition 4.1 *For any non-trivial graph G, $\chi_{gl}(G) = 2$ if and only if G is a forest.*

Proof. By Corollary 4.2 of [8], if G has a cycle C of length $n \geq 3$, then $\chi_{gl}(G) \geq \chi_{gl}(C) \geq \chi_g(C) \geq 3$. Conversely, if G is a forest, then by Lemma 2.3, $\chi_{gl}(G) \leq 2$. ∎

Assume G^c denotes the complement of a graph G. Nordhaus and Gaddum [9] first investigate the bounds for the sum and product of the chromatic numbers of G and G^c. This has been extended to group chromatic numbers and choice numbers.

Lemma 4.2 *If G is a graph of order n, then*
(i) [8] $2\sqrt{n} \le \chi(G)+\chi(G^c) \le \chi_g(G)+\chi_g(G^c) \le n+1$ and $n \le \chi(G)\chi(G^c) \le \chi_g(G)\chi_g(G^c) \le ((n+1)/2)^2$.
(ii) [3] $2\sqrt{n} \le \chi_l(G)+\chi_l(G^c) \le n+1$ and $n \le \chi_l(G)\chi_l(G^c) \le ((n+1)/2)^2$.

We are ready to present the group choice number version for the Nordhaus-Gaddum Theorem.

Theorem 4.3 *Suppose that G is a graph of order n. Then $2\sqrt{n} \le \chi_{gl}(G)+\chi_{gl}(G^c) \le n+1$ and $n \le \chi_{gl}(G)\chi_{gl}(G^c) \le ((n+1)/2)^2$.*

Proof. Since $\chi_{gl}(G) \ge \chi_g(G)$ and $\chi_{gl}(G^c) \ge \chi_g(G^c)$, by Lemma 4.2, it suffices to prove that $\chi_{gl}(G) + \chi_{gl}(G^c) \le n+1$ and $\chi_{gl}(G)\chi_{gl}(G^c) \le ((n+1)/2)^2$.

We follow a similar argument as in the proof of in [8]. Let $\chi_{gl}(G) = k$ and $\chi_{gl}(G^c) = k'$. Suppose that $d_1 \ge \cdots \ge d_n$ is the degree sequence of G. With a similar argument to Lemma 6.2 in [8], we conclude that G has at least k vertices of degree at least $k-1$. Consequently,

$$\chi_{gl}(G) = \min\{d_k+1, k\} \le \max\{\min\{d_i+1, i\}, 1 \le i \le n\}.$$

Let $d'_1 \ge \cdots \ge d'_n$ be the degree sequence of G^c. Arguing as above, we conclude that there exist integers $p > 0$ and $q > 0$ such that

$$\chi_{gl}(G) \le \min\{d_p+1, p\} \text{ and } \chi_{gl}(G^c) \le \min\{d'_q+1, q\}.$$

If $q \ge n-p+1$, then $n-1 = d_p + d'_{n-p+1} \ge d_p + d'_q \ge (k-1)+(k'-1)$, and so $n+1 \ge k+k' = \chi_{gl}(G)+\chi_{gl}(G^c)$. Since $n-1 \ge d_p + d'_q$,

$$\begin{aligned} kk' &\le (d_p+1)(d'_q+1) = d_pd'_q + d_p + d'_q + 1 \le d_pd'_q + n \\ &\le d_pd'_{n-p+1} + n \le ((n-1)/2)^2 + n = ((n+1)/2)^2. \end{aligned}$$

If $q \le n-p+1$, then $\chi_{gl}(G) \le p$ and $\chi_{gl}(G^c) \le q$ and so $n+1 = p+(n-p+1) \ge p+q \ge \chi_{gl}(G)+\chi_{gl}(G^c)$. Furthermore,

$$kk' \le pq \le p(n-p+1) \le ((n+1)/2)^2.$$

This completes the proof. ∎

5 Group choosability of complete bipartite graphs

Here we study the group choosability of complete bipartite graphs and characterize those with group choice number at most 3.

Proposition 5.1 *If $n \geq m^m$, then $\chi_{gl}(K_{m,n}) = m + 1$.*

Proof. By Lemma 2.3, $\chi_{gl}(K_{m,n}) \leq m+1$. By Theorem 5.1 of [8], when $n \geq m^m$, $\chi_{gl}(K_{m,n}) \geq \chi_g(K_{m,n}) = m+1$. ∎

Proposition 5.2 *Each of the following holds.*
(i) If $n \geq 2$, then $\chi_{gl}(K_{2,n}) = 3$,
(ii) If $n \geq 6$, then $\chi_{gl}(K_{3,n}) = 4$.
(iii) $\chi_{gl}(K_{4,4}) = 4$.
(iv) $\chi_{gl}(K_{3,4}) = 3$.
(v) $\chi_{gl}(K_{3,5}) = \chi_g(K_{3,5}) = 3$.

Proof. By (1) and by Theorems 7.1, 7.2 and Lemma 4.4 of [8], $\chi_{gl}(K_{2,n}) \geq \chi_g(K_{2,n}) = 3$, $\chi_{gl}(K_{3,n}) \geq \chi_g(K_{3,n}) = 4$, and $\chi_{gl}(K_{4,4}) \geq \chi_g(K_{4,4}) = 4$. By Lemma 2.3 or Theorem 1.1, we conclude that $\chi_{gl}(K_{2,n}) \leq 3$, $\chi_{gl}(K_{3,n}) = 4$ and $\chi_{gl}(K_{4,4}) = 4$.

The proofs of (iv) and (v) are similar to the arguments used in the proofs of Lemma 7.3 and 7.4 in [8]. ∎

The corollary below follows immediately from Proposition 5.2.

Corollary 5.3 *Let $K_{m,n}$ be a complete bipartite graph with $m \geq n$. Then $\chi_{gl}(k_{m,n}) = 3$ if and only if either $n = 2$ or $(n, m) \in \{(3,4), (3,5)\}$.*

We conclude this section with the following proposition, which follows by an argument similar to the proofs of Theorem 7.4 in [8].

Proposition 5.4 *For $4 \leq n \leq 10$, $\chi_{gl}(K_{4,n}) = 4$.*

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