TWO-DISTANCE-TRANSITIVE GRAPHS OF VALENCY 7

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ABSTRACT. A graph Γ is said to be $(G, 2)$-distance-transitive if, for $i = 1, 2$ and for any two vertex pairs (u_1, v_1) and (u_2, v_2) with $d_\Gamma(u_1, v_1) = d_\Gamma(u_2, v_2) = i$, there exists $g \in G$ such that $(u_1, v_1)^g = (u_2, v_2)$. This paper classifies the family of $(G, 2)$-distance-transitive graphs of valency 7.

1. INTRODUCTION

In this paper, all graphs are finite, simple, connected and undirected. For a graph Γ, we use $V(\Gamma)$ and $\mathrm{Aut}(\Gamma)$ to denote its *vertex set* and *automorphism group*, respectively. Let $u, v \in V(\Gamma)$. Then the distance between u, v in Γ is denoted by $d_\Gamma(u, v)$. Let $G \leq \mathrm{Aut}(\Gamma)$. A non-complete graph Γ is said to be $(G, 2)$-*distance-transitive*, if for $i = 1, 2$ and for any two vertex pairs (u_1, v_1) and (u_2, v_2) with $d_\Gamma(u_1, v_1) = d_\Gamma(u_2, v_2) = i$, there exists $g \in G$ such that $(u_1, v_1)^g = (u_2, v_2)$. An *arc* is an ordered pair of adjacent vertices. A vertex triple (u, v, w) with v adjacent to both u and w is called a 2-*arc* if $u \neq w$. The graph Γ is said to be $(G, 2)$-*arc-transitive* if G is transitive on both the set of arcs and the set of 2-arcs.

The first remarkable result about $(G, 2)$-arc-transitive graphs comes from Tutte [11, 12], and this family of graphs has been studied extensively, see [6, 9, 10]. By definition, every non-complete $(G, 2)$-arc-transitive graph is $(G, 2)$-distance-transitive. The converse is not necessarily true. If a $(G, 2)$-distance-transitive

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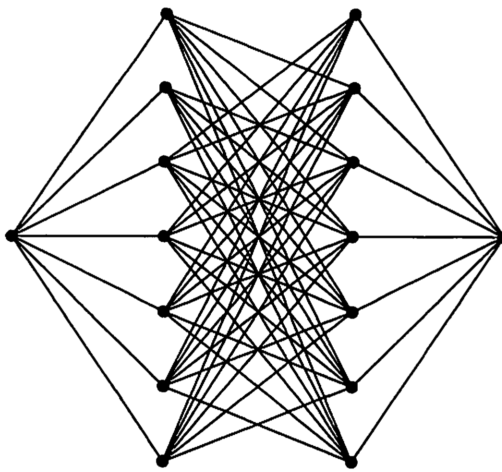

FIGURE 1. $\overline{(2 \times 8)-\text{grid}}$

graph has girth 3 (length of the shortest cycle is 3), then this graph is not $(G, 2)$-arc-transitive. Thus, the family of non-complete $(G, 2)$-arc-transitive graphs is properly contained in the family of $(G, 2)$-distance-transitive graphs. The graph in Figure 1 is $\Gamma = \overline{(2 \times 8) - \text{grid}}$ which is $(G, 2)$-distance-transitive but not $(G, 2)$-arc-transitive of valency 7 for $G = \text{Aut}(\Gamma)$. At the moment, $(G, 2)$-distance-transitive but not $(G, 2)$-arc-transitive graphs of valency at most 6 are classified in [2, 8]. Hence 7 is the next smallest valency for $(G, 2)$-distance-transitive graphs to investigate. Our main theorem classifies such graphs.

The *line graph* $L(\Gamma)$ of a graph Γ has the set of edges of Γ as its vertex set, and two edges are adjacent in $L(\Gamma)$ if and only if they have a common vertex in Γ. The line graph of a complete bipartite graph $\mathrm{K}_{m,n}$ is called an $(m \times n)-\text{grid}$.

Remark 1.1. Let Γ be a connected $(G, 2)$-distance-transitive graph. If Γ has girth at least 5, then for any two vertices u, v with $d_\Gamma(u, v) = 2$, there exists a unique 2-arc between u and v. Hence Γ is $(G, 2)$-distance-transitive implies that it is $(G, 2)$-arc-transitive. If Γ has girth 4, then Γ can be $(G, 2)$-distance-transitive but not $(G, 2)$-arc-transitive. There are infinitely many such graphs. For instance, let Γ be the complement of the $(2 \times p^k)$−grid where p is a prime, and let $M = \mathbb{Z}_p^k : \mathbb{Z}_{p^k-1}$, $G = \mathbb{Z}_2 \times M$. Then Γ is $(G,2)$-distance-transitive but not $(G,2)$-arc-transitive of valency $p^k - 1$ and girth 4. There are also infinitely many $(G,2)$-distance-transitive graphs of girth 4 that are $(G,2)$-arc-transitive, for example the complete bipartite graphs $K_{m,m}$. If Γ has girth 3, then since Γ is non-complete, G_u is not 2-transitive on $\Gamma(u)$, hence it is not $(G,2)$-arc-transitive.

The *complement graph* $\overline{\Gamma}$ of a graph Γ, is the graph with vertex $V(\Gamma)$, and two vertices are adjacent in $\overline{\Gamma}$ if and only if they are not adjacent in Γ.

Theorem 1.2. *Let Γ be a $(G,2)$-distance-transitive graph of valency 7. Let $u, v \in V(\Gamma)$ such that $d_\Gamma(u,v) = 2$. Then either Γ is $(G,2)$-arc-transitive or Γ has girth 4, and one of the following holds:*

(1) $(\Gamma, G) = (\overline{(2\times 8)\text{−grid}}, M : S_2)$ *where M is a 2-transitive but not 3-transitive subgroup of S_8;*

(2) $|\Gamma(u) \cap \Gamma(w)| = 2$, *and $G_u^{\Gamma(u)} \cong ASL(1,7)$ is 2-homogeneous but not 2-transitive on $\Gamma(u)$;*

(3) $|\Gamma(u) \cap \Gamma(w)| = 3$, *and $G_u \cong \mathbb{Z}_2 : (\mathbb{Z}_7 : \mathbb{Z}_2)$ or $\mathbb{Z}_7 : \mathbb{Z}_2$.*

We remark that there exist graphs Γ in Theorem 1.2 (2) and (3), see Examples 2.8 and 2.13.

2. Proof of Theorem 1.2

A graph Γ is said to be *G-distance-transitive* if G is transitive on the ordered pairs of vertices at any given distance. The study of finite G-distance-transitive graphs goes back to Higman's paper [5] in which "groups of maximal diameter" were introduced. By definition, every non-complete G-distance-transitive graph is $(G,2)$-distance-transitive. The *diameter* $\mathrm{diam}(\Gamma)$ of a graph Γ is the maximum distance occurring over all pairs of vertices.

Remark 2.1. Let Γ be a $(G,2)$-distance-transitive graph. Let u, w be two vertices such that $d_\Gamma(u,w) = 2$. Suppose that $|\Gamma_3(u) \cap \Gamma(w)| = 0$. Then since Γ is $(G,2)$-distance-transitive, Γ has diameter 2 and so it is G-distance-transitive.

Suppose that $|\Gamma_3(u) \cap \Gamma(w)| = 1$. Let $(u_0, \dots, u_i)$ be a path with $d_\Gamma(u_0, u_i) = i$ where $i = \mathrm{diam}(\Gamma)$. Then for each $j \leq \mathrm{diam}(\Gamma) - 2$, $|\Gamma_3(u_j) \cap \Gamma(u_{j+2})| = 1$. Note that, $\Gamma_{j+3}(u_0) \cap \Gamma(u_{j+2}) \subseteq \Gamma_3(u_j) \cap \Gamma(u_{j+2})$, and so $|\Gamma_{j+3}(u_0) \cap \Gamma(u_{j+2})| = 1$, hence Γ is also G-distance-transitive.

When $U \subseteq V(\Gamma)$, $[U]$ denotes the subgraph of Γ induced by U.

Lemma 2.2. *Let Γ be a $(G, 2)$-distance-transitive but not $(G, 2)$-arc-transitive graph of valency 7. Then Γ has girth 4.*

Proof. By Remark 1.1, Γ has girth 3 or 4. Assume Γ has girth 3. Let (u, v, w) be a 2-arc such that $d_\Gamma(u, w) = 2$. Since Γ is $(G, 2)$-distance-transitive, G_u is transitive on $\Gamma(u)$, so $[\Gamma(u)]$ is a vertex-transitive graph. Let k be the valency of $[\Gamma(u)]$. Since $[\Gamma(u)]$ is connected and $|\Gamma(u)| = 7$, it follows that $k = 2, 3, 4, 5, 6$. On the other hand, $[\Gamma(u)]$ has $\frac{7k}{2}$ edges, and so k is even, $k = 2, 4, 6$. Set $\Gamma(u) = \{v_1 = v, v_2, v_3, v_4, v_5, v_6, v_7\}$. If $k = 6$, then $[\Gamma(u)] \cong \mathrm{K}_7$, and so $\Gamma \cong \mathrm{K}_8$, contradicting the fact that Γ is non-complete.

Suppose that $k = 4$. Then the complement graph of $[\Gamma(u)]$ is a vertex-transitive graph with valency 2 and order 7, so this complement graph is C_7. Thus $[\Gamma(u)] \cong \overline{C_7}$. Again since $k = 4$, it follows that $|\Gamma_2(u) \cap \Gamma(v_1)| = 2$. Hence there are 14 edges between $\Gamma(u)$ and $\Gamma_2(u)$. Suppose $\Gamma(u) \cap \Gamma(v_1) = \{v_2, v_3, v_4, v_5\}$ and set $\Gamma_2(u) \cap \Gamma(v_1) = \{w_1, w_2\}$. Then $\Gamma(v_1) = \{u, v_2, v_3, v_4, w_1, w_2\}$. Since $[\Gamma(v_1)] \cong \overline{C_7}$, it follows that $|\Gamma(u) \cap \Gamma(w_2)| \geq 4$ and w_1, w_2 are adjacent. As there are 14 edges between $\Gamma(u)$ and $\Gamma_2(u)$, $|\Gamma(u) \cap \Gamma(w_2)|$ divides 14, $|\Gamma(u) \cap \Gamma(w_2)| = 7$, a contradiction.

Thus $k = 2$ and $[\Gamma(u)] \cong C_7$. Let $(v_1, v_2, v_3, v_4, v_5, v_6, v_7)$ be a 7-cycle. Then $|\Gamma_2(u) \cap \Gamma(v_1)| = 4$, and set $\Gamma_2(u) \cap \Gamma(v_1) = \{w_1, w_2, w_3, w_4\}$. Then $\Gamma(v_1) = \{u, v_2, v_7, w_1, w_2, w_3, w_4\}$. Since $[\Gamma(v_1)] \cong C_7$ and (v_2, u, v_7) is a 2-arc, it follows that v_2 is adjacent to one of $\{w_1, w_2, w_3, w_4\}$, say w_1; v_7 is adjacent to one of $\{w_2, w_3, w_4\}$, say w_4. Further, w_1 is adjacent to one of $\{w_2, w_3\}$, say w_2. Hence w_3 is adjacent to both w_2 and w_4. In particular, v_2 is not adjacent to any vertex of $\{w_2, w_3, w_4\}$, and v_7 is not adjacent to any vertex of $\{w_1, w_2, w_3\}$.

Since $|\Gamma_2(u) \cap \Gamma(v_2)| = 4$, there exist w_5, w_6, w_7 in $\Gamma_2(u)$ that are adjacent to v_2, and so $\Gamma(v_2) = \{u, v_1, v_3, w_1, w_5, w_6, w_7\}$. Noting that $[\Gamma(v_2)] \cong C_7$ and (w_1, v_1, u, v_3) is a 3-arc, so v_3 is adjacent to one of $\{w_5, w_6, w_7\}$, say w_7; w_1 is adjacent to one of $\{w_5, w_6\}$, say w_5; and w_6 is adjacent to both w_5 and w_7. Thus, $\{v_1, v_2, w_2, w_5\} \subseteq (\Gamma(u) \cup \Gamma_2(u)) \cap \Gamma(w_1)$. In particular, $2 \leq |\Gamma(u) \cap \Gamma(w_1)| \leq 5$ and $|\Gamma_2(u) \cap \Gamma(w_1)| \geq 2$. Since Γ is $(G, 2)$-distance-transitive and $|\Gamma_2(u) \cap \Gamma(v_1)| = 4$, there are 28 edges between $\Gamma(u)$ and $\Gamma_2(u)$. Since $|\Gamma(u) \cap \Gamma(w_1)|$ divides 28, $|\Gamma(u) \cap \Gamma(w_1)| = 2$ or 4.

Suppose that $|\Gamma(u) \cap \Gamma(w_1)| = 4$. Since there are 28 edges between $\Gamma(u)$ and $\Gamma_2(u)$, it follows that $|\Gamma_2(u)| \cdot |\Gamma(u) \cap \Gamma(w_1)| = 28$, so $|\Gamma_2(u)| = 7$. As $|\Gamma_2(u) \cap \Gamma(w_1)| \geq 2$, $|\Gamma_3(u) \cap \Gamma(w_1)| \leq 1$. Thus by Remark 2.1, Γ is G-distance-transitive. Inspecting the graphs in [1, p. 222-223], such a Γ does not exist.

Thus $|\Gamma(u) \cap \Gamma(w_1)| = 2$. Then $|\Gamma_2(u)| = 14$. Since $|\Gamma_2(u) \cap \Gamma(w_1)| \geq 2$, it follows that $|\Gamma_3(u) \cap \Gamma(w_1)| \leq 3$. If $|\Gamma_3(u) \cap \Gamma(w_1)| \leq 1$, then by Remark 2.1, Γ is G-distance-transitive. Inspecting the graphs in [1, p. 222-223], such a Γ does not exist. Hence $|\Gamma_3(u) \cap \Gamma(w_1)| = 2$ or 3.

Recall that $\Gamma(w_1) = \{v_1, v_2\} \cup (\Gamma_2(u) \cap \Gamma(w_1)) \cup (\Gamma_3(u) \cap \Gamma(w_1))$, $\Gamma(u) \cap \Gamma(w_1) = \{v_1, v_2\}$ and v_1, v_2 are adjacent. Since Γ is $(G, 2)$-distance-transitive and $d_\Gamma(w_1, w_4) = 2$, it follows that $\Gamma(w_1) \cap \Gamma(w_4) = \{v_1, x\}$ and v_1, x are adjacent for some vertex x, that is, $x \in \Gamma(v_1) \cap \Gamma(w_1) \cap \Gamma(w_4)$. Noting that $\Gamma(v_1) = \{u, v_2, v_7, w_1, w_2, w_3, w_4\}$ and y, y' are nonadjacent when $\{y, y'\} \in \{\{u, w_1\}, \{v_7, w_1\}, \{v_2, w_4\}, \{w_2, w_4\}, \{w_1, w_3\}\}$. It follows that $\Gamma(v_1) \cap \Gamma(w_1) \cap \Gamma(w_4) = \emptyset$, so such a vertex x does not exist, which is a contradiction. Therefore Γ has girth 4. □

Lemma 2.3. ([2]) *Let $\Gamma \cong K_{m,m}$ with $m \geq 2$. Then Γ is $(G, 2)$-distance-transitive if and only if it is $(G, 2)$-arc-transitive.*

Lemma 2.4. *Let Γ be a $(G, 2)$-distance-transitive but not $(G, 2)$-arc-transitive graph of valency 7 and girth 4. Let $d_\Gamma(u, w) = 2$. Then 7 divides $|\Gamma_2(u)|$, and either $\Gamma \cong \overline{(2 \times 8)-\text{grid}}$, or $|\Gamma(u) \cap \Gamma(w)| = 2$ or 3.*

Proof. Since Γ is $(G, 2)$-distance-transitive of prime valency 7 and girth 4, there are $7 \times 6 = 42$ edges between $\Gamma(u)$ and $\Gamma_2(u)$. Let $|\Gamma(u) \cap \Gamma(w)| = r$ and $|\Gamma_2(u)| = n$. Then $42 = nr$ and $r \leq 7$.

Suppose that $r = 7$. Then $n = 6$ and $\Gamma(u) = \Gamma(w)$. Since Γ is $(G, 2)$-distance-transitive, it follows that G_u is transitive on $\Gamma_2(u)$, and so $\Gamma(u) = \Gamma(z)$ for any $z \in \Gamma_2(u)$. Hence Γ has diameter 2 with $1 + 7 + n$ vertices. Let $\Delta = \{u\} \cup \Gamma_2(u)$. Then any two vertices $x, y \in \Delta$ are nonadjacent, and x is adjacent to all vertices of $V(\Gamma) \setminus \Delta$. Thus Δ is a block of $\mathrm{Aut}(\Gamma)$ of cardinality $1 + n$, and so $\Gamma(u)$ is another block of cardinality 7. Thus $\Gamma \cong \mathrm{K}_{7,7}$. However, by Lemma 2.3, $\mathrm{K}_{7,7}$ is $(G, 2)$-arc-transitive, which is a contradiction. Thus, $r < 7$. Since $42 = nr$, it follows that $7|n$ and $r|6$.

Suppose that $r = 6$. Then $n = 7$. Set $\Gamma_2(u) = \{w_1 = w, w_2, w_3, w_4, w_5, w_6, w_7\}$, $\Gamma_2(u) \cap \Gamma(v_1) = \{w_1 = w, w_2, w_3, w_4, w_5, w_6\}$ and $\Gamma(u) \cap \Gamma(w_1) = \{v_1, v_2, v_3, v_4, v_5, v_6\}$. If $\Gamma_2(u) \cap \Gamma(x) = \Gamma_2(u) \cap \Gamma(y)$ for any $x, y \in \Gamma(u) \cap \Gamma(w)$, then $\Gamma_2(u) \cap \Gamma(v_7) = \{w_7\}$ contradicting that $|\Gamma_2(u) \cap \Gamma(v_7)| = 6$. Thus there exist two vertices of $\Gamma(u) \cap \Gamma(w_1)$, say v_1, v_2, such that $\Gamma(v_1) \neq \Gamma(v_2)$. Then $(\Gamma(v_1) \cup \Gamma(v_2)) \cap \Gamma_2(u) = \Gamma_2(u)$. Thus $\Gamma_2(u) \cap \Gamma(w_1) = \emptyset$, as Γ has girth 4. Hence $|\Gamma_3(u) \cap \Gamma(w_1)| = 1$, say $\Gamma_3(u) \cap \Gamma(w_1) = \{e\}$. Then $|\Gamma(v_1) \cap \Gamma(e)| = |\Gamma(v_i) \cap \Gamma(e)| = 6$. Since $|\Gamma(v_1) \cap \Gamma_2(u)| = |\Gamma(v_i) \cap \Gamma_2(u)| = 6$, it follows that $\Gamma_2(u) \cap \Gamma(v_1) = \Gamma(v_1) \cap \Gamma(e)$ and $\Gamma_2(u) \cap \Gamma(v_i) = \Gamma(v_i) \cap \Gamma(e)$. Hence $\Gamma_2(u) = \Gamma(e)$, and $\Gamma_3(u) = \{e\}$. Thus, $\Gamma \cong \overline{(2 \times 8)\text{−grid}}$.

Finally, suppose $r < 6$. Since $r|6$, $r \leq 3$. On the other hand, since Γ has girth 4, it follows that $r \geq 2$, so $2 \leq r \leq 3$. □

Lemma 2.5. *Let* $\Gamma = \overline{(2 \times 8)\text{−grid}}$ *be* $(G, 2)$*-distance-transitive but not* $(G, 2)$*-arc-transitive. Then* $G = M : S_2$ *where* M *is a* 2*-transitive but not* 3*-transitive subgroup of* S_8.

Proof. Let (u, v, w) be a 2-arc of $\Gamma = \overline{(2 \times 8)\text{−grid}}$. Then $d_\Gamma(u, w) = 2$, $|\Gamma_2(u) \cap \Gamma(v)| = 6$ and $|\Gamma(u) \cap \Gamma(w)| = 6$. Further there are 42 edges between $\Gamma(u)$ and $\Gamma_2(u)$.

Noting that Γ is $(\mathrm{Aut}(\Gamma), 2)$-arc-transitive. Thus $S_2 < G < \mathrm{Aut}(\Gamma) \cong S_8 \times S_2$. Let the two biparts of Γ be U and W. Let

$u \in U$. Then $\Gamma_2(u) = U \setminus \{1\}$. Since Γ is $(G,2)$-distance-transitive, G_u is transitive on $\Gamma_2(u) = U \setminus \{1\}$. Thus G_U^U is 2-transitive on U. Similarly, G_W^W is 2-transitive on W. If G_u is 2-transitive on $\Gamma_2(u) = U \setminus \{1\}$, then G_u is 2-transitive on $\Gamma(u)$, so Γ is $(G,2)$-arc-transitive, which is a contradiction. Thus G_U^U is a 2-transitive but not 3-transitive subgroup of S_8. Since G is transitive on $V(\Gamma)$, $G = M : S_2$ where $M \cong G_U^U$. □

Lemma 2.6. ([4, Theorem 9.4B]) *Let G be a 2-homogeneous permutation group which is not 2-transitive of degree n. Then $n = p^e \equiv 3 \pmod 4$ where p is a prime and e is odd, and further,*

(1) *$|G|$ is odd and divisible by $\frac{p^e(p^e-1)}{2}$ and*
(2) *$ASL(1,p^e) \le G \le A\Gamma L(1,p^e)$.*

Lemma 2.7. *Let Γ be a $(G,2)$-distance-transitive but not $(G,2)$-arc-transitive graph of valency 7. Let $u, w \in V(\Gamma)$ be such that $d_\Gamma(u,w) = 2$. If Γ has girth 4 and $|\Gamma(u) \cap \Gamma(w)| = 2$, then $G_u^{\Gamma(u)} \cong ASL(1,7)$ is 2-homogeneous but not 2-transitive on $\Gamma(u)$.*

Proof. Suppose that Γ has girth 4 and $|\Gamma(u) \cap \Gamma(w)| = 2$. Then each 2-arc of Γ lies in a unique 4-cycle. Thus, there is a 1-1 mapping between the unordered vertex pairs in $\Gamma(u)$ and vertices in $\Gamma_2(u)$. Since G_u is transitive on $\Gamma_2(u)$, it follows that G_u is transitive on the set of unordered vertex pairs in $\Gamma(u)$. Hence $G_u^{\Gamma(u)}$ is 2-homogeneous on $\Gamma(u)$. Further, since Γ is not $(G,2)$-arc-transitive, $G_u^{\Gamma(u)}$ is not 2-transitive on $\Gamma(u)$. By Lemma 2.6, $G_u^{\Gamma(u)} \cong ASL(1,7)$. □

The Hamming graph H(7, 2) has vertex set $\mathbb{Z}_2^7$, and two vertices are adjacent if and only if they have exactly one different coordinate. This graph is $(G,2)$-distance-transitive for $G = \mathrm{Aut}(\Gamma)$, see [1, 7]. We give an example of graph in Lemma 2.7.

Example 2.8. Let $\Gamma = \mathrm{H}(7,2)$. Then Γ has valency 7 and girth 4, and for each 2-arc (u,v,w) with $d_\Gamma(u,w) = 2$, we have $|\Gamma(u) \cap \Gamma(w)| = 2$. Let $G = S_2 \wr H < S_2 \wr S_7$ where $H = ASL(1,7)$. Then Γ is $(G,2)$-distance-transitive but not $(G,2)$-arc-transitive.

We use $G_u^{[1]}$ to denote the kernel of the G_u action on $\Gamma(u)$, and for an arc (u, v), $G_{uv}^{[1]} = G_u^{[1]} \cap G_v^{[1]}$.

Lemma 2.9. ([13]) *Let Γ be a G-locally primitive arc-transitive graph. Let (u, v) be an arc. Then $G_{uv}^{[1]}$ is a p-group for some prime p. Further, if $G_u^{\Gamma(u)}$ is affine and $|\Gamma(u)| \geq 5$, then $G_{uv}^{[1]} = 1$.*

The following lemma gives a well-known result of Burnside.

Lemma 2.10. ([4, Theorem 3.5B]) *A primitive permutation group G of prime degree p is either 2-transitive, or solvable and $G \leq AGL(1,p)$.*

Lemma 2.11. *Let Γ be a connected G-arc-transitive but not $(G, 2)$-arc-transitive graph of prime valency p. Let (u, v) be an arc. Then $G_u \cong \mathbb{Z}_t.(\mathbb{Z}_p : \mathbb{Z}_r)$, where $t|r|p-1$.*

Proof. Since Γ is G-arc-transitive of prime valency p, G_u is primitive on $\Gamma(u)$. Then by Lemma 2.9, $G_{uv}^{[1]} = 1$. Since Γ is not $(G, 2)$-arc-transitive, by Lemma 2.10, $G_u^{\Gamma(u)} \cong \mathbb{Z}_p : \mathbb{Z}_r$ where $r|p-1$, and $G_{u,v}^{\Gamma(u)} \cong \mathbb{Z}_r$.

Again, since Γ is G-arc-transitive, $G_u^{[1]} \cong G_v^{[1]}$. Note $G_v^{[1]} \cong G_v^{[1]}/(G_v^{[1]} \cap G_u^{[1]}) \cong (G_u^{[1]} G_v^{[1]})/G_u^{[1]} \cong (G_v^{[1]})^{\Gamma(u)}$. Since $G_v^{[1]} \lhd G_{u,v}$, it follows that $(G_v^{[1]})^{\Gamma(u)} \lhd G_{u,v}^{\Gamma(u)} \cong Z_r$, say $(G_v^{[1]})^{\Gamma(u)} \cong \mathbb{Z}_t$, where $t|r$, $t \neq 1$. Hence $G_v^{[1]} \cong (G_v^{[1]})^{\Gamma(u)} \cong \mathbb{Z}_t$.

Since $G_u^{\Gamma(u)} \cong G_u/G_u^{[1]}$, it follows that $G_u \cong \mathbb{Z}_t.(\mathbb{Z}_p : \mathbb{Z}_r)$. □

Lemma 2.12. *Let Γ be a $(G, 2)$-distance-transitive but not $(G, 2)$-arc-transitive graph of valency 7. Let $u, w \in V(\Gamma)$ be such that $d_\Gamma(u, w) = 2$. If Γ has girth 4 and $|\Gamma(u) \cap \Gamma(w)| = 3$, then $G_u \cong \mathbb{Z}_2.(\mathbb{Z}_7 : \mathbb{Z}_2)$ or $\mathbb{Z}_7 : \mathbb{Z}_2$, and $|\Gamma_3(u) \cap \Gamma(w)| = 3$ or 4.*

Proof. Suppose that Γ has girth 4 and $|\Gamma(u) \cap \Gamma(w)| = 3$. Let (u, v, w) be a 2-arc. Then $d_\Gamma(u, w) = 2$ and $|\Gamma_2(u) \cap \Gamma(v)| = 6$. Since Γ is $(G, 2)$-distance-transitive, there are 42 edges between $\Gamma(u)$ and $\Gamma_2(u)$. Since $|\Gamma(u) \cap \Gamma(w)| = 3$ and $|\Gamma(u) \cap \Gamma(w)| \cdot |\Gamma_2(u)| = 42$, it follows that $|\Gamma_2(u)| = 14$. Again since Γ is $(G, 2)$-distance-transitive, G_u is transitive on both $\Gamma(u)$ and $\Gamma_2(u)$, so both $|\Gamma(u)|$ and $|\Gamma_2(u)|$ divide $|G_u|$, hence 14 divides $|G_u|$.

If $|\Gamma_3(u)\cap\Gamma(w)| = 1$, then Γ is distance-transitive, by inspecting the graphs in [1, 223], such a graph does not exist. Hence $|\Gamma_3(u) \cap \Gamma(w)| = 2, 3$ or 4.

Set $\Gamma(u) = \{v_1 = v, v_2, \dots, v_7\}$. Let $\Gamma_2(u) \cap \Gamma(v_1) = \{w_1 = w, w_2, w_3, w_4, w_5, w_6\}$. Noting that for any two vertices x, y if $d_\Gamma(x, y) = 2$, then $|\Gamma(x) \cap \Gamma(y)| = 3$. Thus $|\Gamma(v_i) \cap \Gamma(v_j)| = 3$ whenever $i \neq j$. Suppose $\Gamma(u)\cap\Gamma(w_1) = \{v_1, v_2, v_3\}$ and $\Gamma_2(u)\cap \Gamma(v_1)\cap\Gamma(v_2)=\{w_1, w_2\}$. Let $\Gamma_2(u)\cap\Gamma(v_2)=\{w_1, w_2, w_7, w_8, w_9, w_{10}\}$.

Noting that $|(\Gamma_2(u) \cap \Gamma(v_1)) \cup (\Gamma_2(u) \cap \Gamma(v_2)) \cap \Gamma(v_3)| \leq 3$. Hence $|(\Gamma_2(u) \cap \Gamma(v_3)) \setminus (\Gamma(v_1) \cup \Gamma(v_2))| \geq 3$. Thus $|\Gamma_2(u) \cap (\Gamma(v_1) \cup \Gamma(v_2) \cup \Gamma(v_3))| = 13$ or 14.

Suppose $|\Gamma_3(u)\cap\Gamma(w_1)| = 2$. Then $|\Gamma_2(u)\cap\Gamma(w_1)| = 2$. Since $|\Gamma_2(u)| = 14$, it follows that w_1 is adjacent to one vertex of $\Gamma(v_1) \cup \Gamma(v_2) \cup \Gamma(v_3)$, say x. Thus (w_1, x, y) is a triangle where $y \in \{v_1, v_2, v_3\}$, which is a contradiction. Thus $|\Gamma_3(u) \cap \Gamma(w)| \neq 2$, so $|\Gamma_3(u) \cap \Gamma(w)| = 3$ or 4.

Finally, by Lemma 2.11, $G_u \cong \mathbb{Z}_t.(\mathbb{Z}_7 : \mathbb{Z}_r)$, where $t|r$ and $r = 2$ or 3. If $r = 3$, then $G_u \cong \mathbb{Z}_3.(\mathbb{Z}_7 : \mathbb{Z}_3)$ or $\mathbb{Z}_7 : \mathbb{Z}_3$, contradicting the fact that 14 divides $|G_u|$. Thus $r = 2$, so $G_u \cong \mathbb{Z}_2.(\mathbb{Z}_7 : \mathbb{Z}_2)$ or $\mathbb{Z}_7 : \mathbb{Z}_2$. □

We give an example of graph in Lemma 2.12.

Example 2.13. Let $V = \mathbb{F}_2^4$ be a 4-dimensional vector space over field $\mathbb{F}_2$. Let U and W consist of 1-subspaces and 3-subspaces of $\mathbb{F}_2^4$, respectively. Let Γ be a bipartite graph with biparts U and W such that $u \in U$ and $w \in W$ are adjacent if and only if $u + w = \mathbb{F}_2^4$. This is the point-plane incidence graph of the projective plane $PG(3, 2)$. Further, Γ is 2-distance-transitive satisfying conditions of Lemma 2.12, see [1, p.223], for example.

Now we can prove our main theorem.

Proof of Theorem 1.2. By Lemma 2.2, Γ has girth 4. Let u, w be two vertices such that the distance between them is 2. It follows from Lemma 2.4 that either $\Gamma \cong \overline{(2 \times 8)-\text{grid}}$, or $|\Gamma(u) \cap \Gamma(w)| = 2$ or 3. If $\Gamma \cong \overline{(2 \times 8)-\text{grid}}$, then by Lemma 2.5, $G = S_2 \times M$ where M is a 2-transitive but not 3-transitive subgroup of S_8, so that (1) holds. If $|\Gamma(u)\cap\Gamma(w)|=2$, then by Lemma 2.7, $G_u^{\Gamma(u)}\cong ASL(1,7)$ is 2-homogeneous but not 2-transitive on $\Gamma(u)$, (2) holds. Finally, if $|\Gamma(u)\cap\Gamma(w)|=3$, then by Lemma 2.12, $G_u\cong\mathbb{Z}_2.(\mathbb{Z}_7:\mathbb{Z}_2)$ or $\mathbb{Z}_7:\mathbb{Z}_2$, (3) holds. □

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