The maximal (signless Laplacian) spectral radius of connected graphs with given matching number

Xiaodan Chen[1,2], Fuliang Lu[3]

[1] *College of Mathematics and Information Science, Guangxi University, Nanning 530004, Guangxi, P.R. China*

[2] *Department of Mathematics, Hunan Normal University, Changsha 410081, Hunan, P.R. China*
E-mail: x.d.chen@live.cn

[3] *School of Science, Linyi University, Linyi 276000, Shandong, P.R. China*
E-mail: fliang1u@163.com

Abstract: In this paper, the graphs with maximal (signless Laplacian) spectral radius among all connected graphs with given matching number are characterized.

Keywords: connected graph, spectral radius, signless Laplacian, matching number

MSC 2010: 05C50, 05C35

1 Introduction

Let $G = (V, E)$ be a simple graph with vertex set $V = V(G) = \{v_1, v_2, \dots, v_n\}$ and edge set $E = E(G)$. For $i = 1, 2, \dots, n$, let $d(v_i)$ denote the degree of vertex v_i in G. The adjacency matrix of G is $A(G) = (a_{ij})$, where elements $a_{ij} = 1$ if two vertices v_i and v_j are adjacent in G and 0 otherwise. The signless Laplacian matrix of G is defined to be $Q(G) = D(G) + A(G)$, where $D(G) = \text{diag}(d(v_1), d(v_2), \dots, d(v_n))$ is the diagonal matrix of vertex degrees in G. The spectral radius (*resp.*, signless Laplacian spectral radius) of G, denoted by $\rho(G)$ (*resp.*, $\mu(G)$), is the largest eigenvalue of $A(G)$ (*resp.*, $Q(G)$). It is well known that if G is connected, then $A(G)$ (*resp.*, $Q(G)$) is irreducible and nonnegative, and by the Perron-Frobenius theorem, $\rho(G)$ (*resp.*, $\mu(G)$) is simple and has a unique positive unit eigenvector. For more properties and applications about these two graph invariants we refer the readers to [5, 6] and the references cited therein.

A matching in a graph is a set of disjoint edges. The maximum cardinality of a matching over all possible matchings in a graph G is the matching number of G, which is denoted by $\beta = \beta(G)$. A matching is perfect if it contains $n/2$ edges (so n is necessarily even), which is the largest possible value for β.

Let K_n be the complete graph on n vertices. Denote by $\overline{G}$ the complement of the graph G. For two graphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$, the union $G_1 \cup G_2$ of G_1 and G_2 is defined to be $G_1 \cup G_2 = (V_1 \cup V_2, E_1 \cup E_2)$; the join $G_1 \vee G_2$ of G_1 and G_2 is obtained from $G_1 \cup G_2$ by joining edges from each vertex of G_1 to each vertex of G_2.

In [3], Brualdi and Solheid proposed the following classic problem concerning the spectral radius of graphs: *Given a set $\mathcal{G}$ of graphs, find an upper bound for the spectral radius in this set and characterize the graphs in which the maximal spectral radius is attained.* Up to now, Brualdi-Solheid problem has been solved for various sets of graphs, see, for example, [1, 2, 7, 8, 9, 10, 11, 13]. In particular, for the set of graphs with given matching number, Feng, Yu, and Zhang [8] proved the following result.

Theorem 1 *Let G be any graph on n vertices with matching number β.*

(i) If $n = 2\beta$ or $2\beta + 1$, then $\rho(G) \le \rho(K_n)$, with equality if and only if $G \cong K_n$.

(ii) If $2\beta + 2 \le n < 3\beta + 2$, then $\rho(G) \le 2\beta$, with equality if and only if $G \cong K_{2\beta+1} \cup \overline{K_{n-2\beta-1}}$.

(iii) If $n = 3\beta + 2$, then $\rho(G) \le 2\beta$, with equality if and only if $G \cong K_{2\beta+1} \cup \overline{K_{n-2\beta-1}}$ or $G \cong K_\beta \vee \overline{K_{n-\beta}}$.

(iv) If $n > 3\beta + 2$, then $\rho(G) \le \frac{1}{2}\left(\beta - 1 + \sqrt{(\beta-1)^2 + 4\beta(n-\beta)}\right)$, with equality if and only if $G \cong K_\beta \vee \overline{K_{n-\beta}}$.

Recently, Brualdi-Solheid problem has also been extended to the study on signless Laplacian spectral radius [4, 12, 13, 14, 15]. In particular, Yu [14] showed the following signless-Laplacian version of Theorem 1.

Theorem 2 *Let G be any graph on n vertices with matching number β.*

(i) If $n = 2\beta$ or $2\beta + 1$, then $\mu(G) \le \mu(K_n)$, with equality if and only if $G \cong K_n$.

(ii) If $2\beta + 2 \le n < \frac{5\beta+3}{2}$, then $\mu(G) \le 4\beta$, with equality if and only if $G \cong K_{2\beta+1} \cup \overline{K_{n-2\beta-1}}$.

(iii) If $n = \frac{5\beta+3}{2}$, then $\mu(G) \le 4\beta$, with equality if and only if $G \cong K_{2\beta+1} \cup \overline{K_{n-2\beta-1}}$ or $G \cong K_\beta \vee \overline{K_{n-\beta}}$.

(iv) If $n > \frac{5\beta+3}{2}$, then $\mu(G) \le \frac{1}{2}(n-2+2\beta+\sqrt{(n-2+2\beta)^2 - 8\beta^2 + 8\beta})$, with equality if and only if $G \cong K_\beta \vee \overline{K_{n-\beta}}$.

Notice in Theorems 1 and 2 that $K_{2\beta+1} \cup \overline{K_{n-2\beta-1}}$ is not connected. So, in this paper we would like to find upper bounds for the (signless Laplacian) spectral radius of connected graphs with given matching number, and characterize the graphs in which the maximal (signless Laplacian) spectral radius is attained. Observe that if G is a connected graph with matching number $\beta = 1$, then G is nothing but K_3 or $K_1 \vee \overline{K_{n-1}}$. Hence, we would consider only the case of $\beta \geq 2$. Our main results are as follows.

Theorem 3 *Let G be any connected graph on n vertices with matching number $\beta \geq 2$. Then*

(i) If $n = 2\beta$ or $2\beta + 1$, then $\rho(G) \leq n - 1$, with equality if and only if $G \cong K_n$.

(ii) If $2\beta + 2 \leq n \leq 3\beta - 1$ ($\beta \geq 3$), then $\rho(G) \leq \rho^*$, where ρ^* is the maximum root of the following equation

$$x^3 - 2(\beta - 1)x^2 - (n - 1)x + 2(\beta - 1)(n - 2\beta) = 0.$$

The equality holds if and only if $G \cong K_1 \vee \left(K_{2\beta-1} \cup \overline{K_{n-2\beta}}\right)$.

(iii) If $n \geq 3\beta$, then

$$\rho(G) \leq \frac{1}{2}\left(\beta - 1 + \sqrt{(\beta - 1)^2 + 4\beta(n - \beta)}\right),$$

with equality if and only if $G \cong K_\beta \vee \overline{K_{n-\beta}}$.

Theorem 4 *Let G be any connected graph on n vertices with matching number $\beta \geq 2$. Then*

(i) If $n = 2\beta$ or $2\beta + 1$, then $\mu(G) \leq 2n - 2$, with equality if and only if $G \cong K_n$.

(ii) If $2\beta + 2 \leq n \leq 5\beta/2$ ($\beta \geq 4$), then $\mu(G) \leq \mu^*$, where μ^* is the maximum root of the following equation

$$x^3 - (n + 4\beta - 3)x^2 + n(4\beta - 3)x - 4(\beta - 1)(2\beta - 1) = 0.$$

The equality holds if and only if $G \cong K_1 \vee \left(K_{2\beta-1} \cup \overline{K_{n-2\beta}}\right)$.

(iii) If $n \geq (5\beta + 1)/2$, then

$$\mu(G) \leq \frac{1}{2}\left(n - 2 + 2\beta + \sqrt{(n - 2 + 2\beta)^2 - 8\beta^2 + 8\beta}\right),$$

with equality if and only if $G \cong K_\beta \vee \overline{K_{n-\beta}}$.

2 The proofs

In this section, we shall prove Theorems 3 and 4. To this end, we need the following two lemmas, due to Feng *et al.* [8] and Yu [14], respectively.

Let $\mathcal{G}_{n,\beta}$ be the set of graphs on n vertices with matching number β.

Lemma 5 (see [8]) *If G is a graph in $\mathcal{G}_{n,\beta}$ with maximal spectral radius, then there exist nonnegative integers s, q such that*

$$G \cong K_s \vee (K_{n_q} \cup \overline{K_{q-1}}),$$

where $s \leq \beta$, $q = n + s - 2\beta$, and $n_q = 2\beta - 2s + 1$. Moreover, $\rho(G)$ is the maximum root of the equation $f(x) = 0$, where

$$\begin{aligned} f(x) &= x^3 - (2\beta - s - 1)x^2 + (2s\beta + 2s - ns - s^2 - 2\beta)x \\ &\quad + 2s(\beta - s)(n + s - 2\beta - 1). \end{aligned} \tag{1}$$

Lemma 6 (see [14]) *If G is a graph in $\mathcal{G}_{n,\beta}$ with maximal signless Laplacian spectral radius, then there exist nonnegative integers s, q such that*

$$G \cong K_s \vee (K_{n_q} \cup \overline{K_{q-1}}),$$

where $s \leq \beta$, $q = n + s - 2\beta$, and $n_q = 2\beta - 2s + 1$. Moreover, $\mu(G)$ is the maximum root of the equation $g(x) = 0$, where

$$\begin{aligned} g(x) &= x^3 - (n + 4\beta - s - 2)x^2 + (4n\beta + 8s\beta + 4s - 3ns - 4s^2 - 8\beta)x \\ &\quad - 2s(4\beta^2 + s^2 + s - 4s\beta - 2\beta). \end{aligned} \tag{2}$$

Now we are ready to present the proofs of Theorems 3 and 4.

Proof of Theorem 3. Without loss of generality, we first suppose that $G \in \mathcal{G}_{n,\beta}$ is a connected graph with maximal spectral radius. Since the complete graph K_n has the maximum spectral radius uniquely among all the graphs on n vertices (see, e.g., [5]), it is easy to see that if $n = 2\beta$ or $2\beta + 1$, then

$$\rho(G) \leq \rho(K_n) = n - 1,$$

with equality if and only if $G \cong K_n$. Hence, in the following we may assume that $n \geq 2\beta + 2$.

Note that G is connected. Then from Lemma 5, we have $1 \leq s \leq \beta$. Moreover, it follows from (1) that

$$\begin{aligned} &f(-\infty) < 0, \; f(+\infty) > 0, \\ &f(-1) = s(2\beta - 2s + 1)(n + s - 2\beta - 1) > 0, \\ &f(2\beta - s) = -s^2(n + s - 2\beta - 1) < 0, \end{aligned} \tag{3}$$

which imply that the three roots of $f(x) = 0$ lie in three intervals $(-\infty, -1)$, $(-1, 2\beta - s)$, $(2\beta - s, +\infty)$. We now consider the following two cases:

Case 1. $2\beta + 2 \leq n \leq 3\beta - 1$ (obviously, it is required that $\beta \geq 3$). Let $F \cong K_1 \vee (K_{2\beta-1} \cup \overline{K_{n-2\beta}})$. Then we just need to show that $\rho(G) \leq \rho(F)$, with equality if and only if $G \cong F$.

Indeed, by putting $s = 1$ in (1), we know that $\rho(F)$ is the maximum root of the equation $h_1(x) = 0$, where

$$h_1(x) = x^3 - 2(\beta - 1)x^2 - (n - 1)x + 2(\beta - 1)(n - 2\beta).$$

By a direct computation, we have

$$h_1(2\beta - 1) = -(n - 2\beta) < 0,$$

which implies that

$$\rho(F) > 2\beta - 1. \tag{4}$$

On the other hand, after a simple calculation, we obtain

$$f(x) = h_1(x) + (s - 1)r_1(x), \tag{5}$$

where

$$r_1(x) = x^2 - (n + s - 2\beta - 1)x - (4\beta^2 + 2s^2 + 2ns + 2n - 2n\beta - 6s\beta - 4\beta).$$

Taking the derivative of $r_1(x)$ with respect to x, and recalling that $n \leq 3\beta - 1$, we have, for $x \geq 2\beta - 1$,

$$r_1'(x) = 2x - (n + s - 2\beta - 1) \geq 6\beta - n - s - 1 \geq 3\beta - s > 0.$$

This means that the function $r_1(x)$ is strictly increasing with respect to x when $x \geq 2\beta - 1$. Hence, by using (4) and again, recalling that $n \leq 3\beta - 1$ and $s \leq \beta$, we get

$$\begin{aligned} r_1(\rho(F)) &> r_1(2\beta - 1) \\ &= 4\beta^2 + 4s\beta + s - 2s^2 - (2s + 1)n \\ &\geq 4\beta^2 + 4s\beta + s - 2s^2 - (2s + 1)(3\beta - 1) \\ &= (\beta - s)(4\beta + 2s - 3) + 1 > 0, \end{aligned}$$

which, together with (5), would yield that

$$f(\rho(F)) = h_1(\rho(F)) + (s - 1)r_1(\rho(F)) \geq 0, \tag{6}$$

with equality if and only if $s = 1$.

Thus, combining (6), (3) and (4), we may conclude that $\rho(G) \leq \rho(F)$, with equality if and only if $f(\rho(F)) = 0$, that is, $s = 1$, which, together with Lemma 5, implies that $G \cong F$.

Case 2. $n \geq 3\beta$. Let $H \cong K_\beta \vee \overline{K_{n-\beta}}$. Then it suffices to prove that $\rho(G) \leq \rho(H)$, with equality if and only if $G \cong H$.

In fact, by putting $s = \beta$ in (1), we get that $\rho(H)$ is the maximum root of the equation $h_2(x) = 0$, where

$$h_2(x) = x\left[x^2 - (\beta - 1)x - \beta(n - \beta)\right].$$

A direct computation shows that

$$\rho(H) = \frac{1}{2}\left(\beta - 1 + \sqrt{(\beta - 1)^2 + 4\beta(n - \beta)}\right).$$

On the other hand, by some tedious calculation, we may obtain

$$f(x) = h_2(x) + (\beta - s)r_2(x), \tag{7}$$

where

$$r_2(x) = -x^2 + (n + s - \beta - 2)x + 2s(n + s - 2\beta - 1).$$

For notational convenience, let $\theta(n) = \sqrt{(\beta - 1)^2 + 4\beta(n - \beta)}$. Since $n \geq 3\beta$, we have

$$\theta(n) = \sqrt{4n\beta - 3\beta^2 - 2\beta + 1} \geq \sqrt{9\beta^2 - 2\beta + 1} > 3\beta - 1,$$

and hence

$$\rho(H) > 2\beta - 1. \tag{8}$$

Furthermore, we get

$$\begin{aligned}
r_2(\rho(H)) &= \frac{1}{2}\big[(\theta(n) + 4s - \beta - 1)n + (s - 2\beta - 1)\theta(n) \\
&\quad + 4s^2 - 7s\beta - 5s + \beta + 1\big] \\
&\geq \frac{1}{2}\big[(\theta(n) + 4s - \beta - 1)(3\beta) + (s - 2\beta - 1)\theta(n) \\
&\quad + 4s^2 - 7s\beta - 5s + \beta + 1\big] \\
&= \frac{1}{2}\left[(\beta + s - 1)\theta(n) + 4s^2 + 5s\beta - 3\beta^2 - 5s - 2\beta + 1\right] \\
&> \frac{1}{2}\left[(\beta + s - 1)(3\beta - 1) + 4s^2 + 5s\beta - 3\beta^2 - 5s - 2\beta + 1\right] \\
&= 3\beta(s - 1) + s(\beta + 2s - 3) + 1 > 0,
\end{aligned}$$

which, together with (7), would yield that

$$f(\rho(H)) = h_2(\rho(H)) + (\beta - s)r_2(\rho(H)) \geq 0, \tag{9}$$

with equality if and only if $s = \beta$.

Thus, from (9), (3) and (8), we can deduce that $\rho(G) \leq \rho(H)$, with equality if and only if $f(\rho(H)) = 0$, that is, $s = \beta$, which, together with Lemma 5, implies that $G \cong H$.

The proof of Theorem 3 is completed. □

Proof of Theorem 4. As the proof of Theorem 3, we can suppose that $G \in \mathcal{G}_{n,\beta}$ is a connected graph with maximal signless Laplacian spectral radius. Since the complete graph K_n also has the maximum signless Laplacian spectral radius uniquely among all the graphs on n vertices (see, e.g., [6]), it follows easily that if $n = 2\beta$ or $2\beta + 1$, then

$$\mu(G) \leq \mu(K_n) = 2n - 2,$$

with equality if and only if $G \cong K_n$. Hence, in what follows we may assume that $n \geq 2\beta + 2$.

Clearly, $1 \leq s \leq \beta$ since G is connected. Moreover, from (2) we get

$$\begin{aligned} &g(-\infty) < 0, \; g(+\infty) > 0, \\ &g(2\beta - s - 1) = (2\beta - s - 1)(2\beta - 2s + 1)(n + s - 2\beta - 1) > 0, \\ &g(4\beta - 2s) = -2s(2\beta - s)(n + s - 2\beta - 1) < 0, \end{aligned} \tag{10}$$

which imply that the three roots of $g(x) = 0$ lie in three intervals $(-\infty, 2\beta - s - 1)$, $(2\beta - s - 1, 4\beta - 2s)$, $(4\beta - 2s, +\infty)$. We now distinguish the following two cases.

Case 1. $2\beta + 2 \leq n \leq 5\beta/2$ (obviously, it is required that $\beta \geq 4$). Let $F \cong K_1 \vee (K_{2\beta-1} \cup \overline{K_{n-2\beta}})$. Then we just need to prove that $\mu(G) \leq \mu(F)$, with equality if and only if $G \cong F$.

Indeed, by putting $s = 1$ in (2), we know that $\mu(F)$ is the maximum root of the equation $p_1(x) = 0$, where

$$p_1(x) = x^3 - (n + 4\beta - 3)x^2 + n(4\beta - 3)x - 4(\beta - 1)(2\beta - 1).$$

A direct computation shows that,

- If $2\beta + 2 \leq n \leq (5\beta - 1)/2$, then

$$p_1(4\beta - 2) = -2(2\beta - 1)(n - 2\beta) < 0,$$

which implies that

$$\mu(F) > 4\beta - 2. \tag{11}$$

- If $n = 5\beta/2$ (so β is necessarily even), then

$$p_1(4\beta - \frac{5}{3}) = -\frac{2}{27}(3\beta + 4) < 0,$$

which implies that

$$\mu(F) > 4\beta - \frac{5}{3}. \tag{12}$$

On the other hand, after somewhat tedious calculation, we get

$$g(x) = p_1(x) + (s-1)t_1(x), \tag{13}$$

where

$$t_1(x) = x^2 - (3n + 4s - 8\beta)x - (8\beta^2 + 2s^2 + 4s - 8s\beta - 12\beta + 4).$$

Taking the derivative of $t_1(x)$ with respect to x, and recalling that $n \leq 5\beta/2$, we have, for $x \geq 4\beta - 2$,

$$t_1'(x) = 2x - (3n + 4s - 8\beta) \geq 16\beta - 3n - 4s - 4 \geq \frac{17}{2}\beta - 4s - 4 > 0.$$

This means that the function $t_1(x)$ is strictly increasing with respect to x when $x \geq 4\beta - 2$. Hence, it follows that,

- If $2\beta + 2 \leq n \leq (5\beta - 1)/2$, then by using (11) and the fact that

$$s^2 + (4\beta - 2)s \leq 5\beta^2 - 2\beta,$$

we obtain

$$\begin{aligned} t_1(\mu(F)) &> t_1(4\beta - 2) \\ &= 2\left\{20\beta^2 - 10\beta - (6\beta - 3)n - \left[s^2 + (4\beta - 2)s\right]\right\} \\ &\geq 2\left[15\beta^2 - 8\beta - (6\beta - 3)n\right] \\ &\geq 2\left[15\beta^2 - 8\beta - (6\beta - 3)(5\beta - 1)/2\right] \\ &= 5\beta - 3 > 0. \end{aligned}$$

- If $n = 5\beta/2$, then by using (12) and the fact that

$$36s^2 + (144\beta - 48)s \leq 180\beta^2 - 48\beta,$$

we get

$$\begin{aligned} t_1(\mu(F)) &> t_1(4\beta - \frac{5}{3}) \\ &= \frac{1}{18}\left\{180\beta^2 - 39\beta - 22 - \left[36s^2 + (144\beta - 48)s\right]\right\} \\ &\geq \frac{1}{18}(9\beta - 22) > 0 \text{ (as } \beta \geq 4). \end{aligned}$$

Thus, for $2\beta+2 \le n \le 5\beta/2$, by (13) we have

$$g(\mu(F)) = p_1(\mu(F)) + (s-1)t_1(\mu(F)) \ge 0, \tag{14}$$

with equality if and only if $s=1$.

Now, combining (14), (10), (11) and (12), we may deduce that $\mu(G) \le \mu(F)$, with equality if and only if $g(\mu(F)) = 0$, that is, $s=1$, which, together with Lemma 6, implies that $G \cong F$.

Case 2. $n \ge (5\beta+1)/2$. Let $H \cong K_\beta \vee \overline{K_{n-\beta}}$. Then it suffices to show that $\mu(G) \le \mu(H)$, with equality if and only if $G \cong H$.

In fact, by taking $s=\beta$ in (2), we get that $\mu(H)$ is the maximum root of the equation $p_2(x)=0$, where

$$p_2(x) = (x-\beta)\left[x^2-(n+2\beta-2)x+2\beta(\beta-1)\right].$$

A direct computation shows that

$$\mu(H) = \frac{1}{2}\left(n+2\beta-2+\sqrt{(n+2\beta-2)^2-8\beta(\beta-1)}\right).$$

On the other hand, by some calculation, we have

$$g(x) = p_2(x) + (\beta-s)t_2(x), \tag{15}$$

where

$$t_2(x) = -x^2+(3n+4s-4\beta-4)x+(2\beta^2+2s^2+2s-6s\beta-2\beta).$$

For notational convenience, let $\sigma(n) = \sqrt{(n+2\beta-2)^2-8\beta(\beta-1)}$. Since $n \ge (5\beta+1)/2$, we obtain

$$\sigma(n) = \sqrt{n^2+(4\beta-4)n-4\beta^2+4} \ge \frac{1}{2}\sqrt{49\beta^2-22\beta+9} > \frac{7\beta}{2}-1,$$

and hence

$$\mu(H) > 4\beta - \frac{5}{4}. \tag{16}$$

Moreover, noting that $2s^2+2(n+\sigma(n)-\beta-1)s \ge 2n+2\sigma(n)-2\beta$, we get

$$\begin{aligned} t_2(\mu(H)) &= n^2+(\sigma(n)-\beta-3)n-(3\beta+1)\sigma(n)-2\beta^2+2 \\ &\quad +2s^2+2(n+\sigma(n)-\beta-1)s \\ &\ge n^2+(\sigma(n)-\beta-1)n-(3\beta-1)\sigma(n)-2\beta^2-2\beta+2. \end{aligned}$$

Now we consider the following function

$$y(x) = x^2+(\sigma(x)-\beta-1)x-(3\beta-1)\sigma(x)-2\beta^2-2\beta+2,\ x \ge (5\beta+1)/2,$$

where $\sigma(x) = \sqrt{(x+2\beta-2)^2 - 8\beta(\beta-1)}$. For $x \geq (5\beta+1)/2$, since

$$y'(x) = \frac{(x+\sigma(x)-3\beta+1)(x+\sigma(x)+2\beta-2)}{\sigma(x)} > 0,$$

$y(x)$ is increasing monotonously with respect to x. Moreover, observing that

$$5\beta < \sqrt{49\beta^2 - 22\beta + 9} < 7\beta,$$

we obtain

$$\begin{aligned} t_2(\mu(H)) &\geq y(n) \\ &\geq y((5\beta+1)/2) \\ &= \frac{1}{4}\left(7\beta^2 - \beta\sqrt{49\beta^2 - 22\beta + 9} + 3\sqrt{49\beta^2 - 22\beta + 9} - 10\beta + 7\right) \\ &> \frac{1}{4}(5\beta + 7) > 0, \end{aligned}$$

which, together with (15), would yield that

$$g(\mu(H)) = p_2(\mu(H)) + (\beta - s)t_2(\mu(H)) \geq 0, \qquad (17)$$

with equality if and only if $s = \beta$.

Thus, from (17), (10) and (16), we can conclude that $\mu(G) \leq \mu(H)$, with equality if and only if $g(\mu(H)) = 0$, that is, $s = \beta$, which, together with Lemma 6, implies that $G \cong H$.

This completes the proof of Theorem 4. □

Acknowledgements. The authors are grateful to the referees for their helpful comments and suggestions towards improving the original version of this paper. The first author was supported by China Postdoctoral Science Foundation (No. 2015M572252), Guangxi Natural Science Foundation (No. 2014GXNSFBA118008) and Scientific Research Foundation of Guangxi University (No. XBZ130083). The second author was supported by NNSF of China (Nos. 11226288, 11301251) and Promotive Research Fund for Excellent Young and Middle-aged Scientists of Shandong Province (No. BS2013DX026).

References

[1] A. Berman, X. Zhang, On the spectral radius of graphs with cut vertices, J. Combin. Theory Ser. B, 2001, 83: 233–240.

[2] T. Bıyıkoğlu, J. Leydold, Graphs with given degree sequence and maximal spectral radius, Electron. J. Combin., 2008, 15: #R119.

[3] R.A. Brualdi, E.S. Solheid, On the spectral radius of complementary acyclic matrices of zeros and ones, SIAM J. Algebra. Discrete Method, 1986, 7: 265–272.

[4] T. Chang, B. Tam, Graphs with maximal signless Laplacian spectral radius, Linear Algebra Appl., 2010, 432: 1708–1733.

[5] D. Cvetković, P. Rowlinson, The largest eigenvalue of a graph: a survey, Linear Multilinear Algebra, 1990, 28: 3–33.

[6] D. Cvetković, Spectral theory of graphs based on the signless Laplacian, Research Report, 2010. Available at: http://www.mi.sanu.ac.rs/projects /signless_L_reportDec06.pdf.

[7] E.R. van Dam, Graphs with given diameter maximizing the spectral radius, Linear Algebra Appl., 2007, 426: 454–457.

[8] L. Feng, G. Yu, X. Zhang, Spectral radius of graphs with given matching number, Linear Algebra Appl., 2007, 422: 133–138.

[9] Y. Hou, Bounds on the largest eigenvalues of trees with a given size of matching, Linear Algebra Appl., 2002, 342: 203–217.

[10] H. Liu, M. Lu, F. Tian, On the spectral radius of graphs with cut edges, Linear Algebra Appl., 2004, 389: 139–145.

[11] D. Stevanović, M. Aouchiche, P. Hansen, On the spectral radius of graphs with a given domination number, Linear Algebra Appl., 2008, 428: 1854–1864.

[12] J. Wang, Q. Huang, Maximizing the signless Laplacian spectral radius of graphs with given diameter or cut vertices, Linear Multilinear Algebra, 2011, 59: 733–744.

[13] M. Ye, Y. Fan, H. Wang, Maximizing signless Laplacian or adjacency spectral radius of graphs subject to fixed connectivity, Linear Algebra Appl., 2010, 433: 1180–1186.

[14] G. Yu, On the maximal signless Laplacian spectral radius of graphs with given matching number, Proc. Japan Acad. Ser. A, 2008, 84: 163–166.

[15] X. Zhang, The signless Laplacian spectral radius of graphs with given degree sequences, Discrete Appl. Math., 2009, 157: 2928–2937.