Enumeration of the degree sequences of 3-connected graphs and cactus graphs

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Abstract

The necessary and sufficient conditions for a given sequence of positive integers $d_1, d_2, ..., d_n$ to be the degree sequence of 3-connected graphs and cactus graphs are proved respectively by S. L. Hakimi[5] and A. R. Rao[6]. In this note, we utilize these results to prove a formula for the function $d_{tc}(2m)$ and $d_{ca}(2m)$, the number of degree sequences with degree sum 2m by 3-connected graphs and cactus graphs respectively. We give generating function proofs and elementary proofs of the formula $d_{tc}(2m)$ and $d_{ca}(2m)$.

Keywords: partition, degree sequence, 3-connected graphs, cactus graphs.

1 Introduction

In this note, we use the standard terminology and notation of graph theory, see [1]. A graph, denoted by $G = (V, E)$ or simply G, con-

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sists of a set $V = \{v_1, v_2, ..., v_n\}$ of vertices and a set $E = \{e_1, \cdots, e_m\}$ of edges. If two edges, say e_k and $e_l \in E$, have the same unordered pair representation, say (v_i, v_j), then they are called parallel edges or multiple edges. A graph may or may not have parallel edges. In this note, the 3-connected graphs under consideration will be finite, undirected, and loopless but may contain multiple edges. And the cactus graphs under consideration will be finite, undirected, and loopless but with no multiple edges.

If G has vertices $v_1, v_2, ..., v_n$, the sequence $(d(v_1), d(v_2), ..., d(v_n))$ is called a degree sequence of G. We say that a sequence $d_1, d_2, ..., d_n$ with $d_1 \leq d_2 \leq \cdots \leq d_n$ is graphic if there is a simple graph G with this degree sequence. And a sequence $d_1, d_2, ..., d_n$ with $d_1 \leq d_2 \leq \cdots \leq d_n$ is multigraphic if there exists a multigraph G with this degree sequence. Such a simple graph or a multigraph is called a realization of the degree sequence. As usual, we say that a nontrivial graph is p-connected if its connectivity $\kappa(G) \geq p$, $\kappa(G) := \min\{p(u, v) : u, v \in V, u \neq v\}$, where $p(u, v)$ is maximum number of pairwise internally disjoint uv-paths in G. A cactus is a connect graph in which each block is an edge or a cycle. Also a graph is said to have property H_k if it is connected and every block is a cycle on k vertices. If $\pi = (d_l, d_2, ..., d_n)$ and P is a property, then we say that π is potentially P if there is a realisation of π (i.e., a graph with degree sequence π) with property P [6]. In this note, we consider the cactus graphs with property H_k and denote them by k-cactus.

Several papers deal with the enumeration of degree sequences of some graphs. Usually, connect graphs, non-separate graphs and line-Hamiltonian multigraphs are well solved. See [3], [4].

In 1974, S. L. Hakimi [5] characterized the degree sequences for which there exists a p-connected graph realization, for $p \geq 3$. And A. R. Rao [6] characterized the degree sequence of cactus graphs with the property H_k. The results are the following:

Theorem 1.1 [5] *Let $\delta = \{d_1, d_2, ..., d_n\}$ be a given set of integers with $d_i \leq d_{i+1}$ for $i = 1, 2, ..., n-1$, and $p \geq 3$. Then δ is realizable* *as a p-connected graph if and only if* $p \le n-1$, $d_1 \ge p$, *and* $\sum_{i=1}^{n} d_i \ge 2d_n + (n-1)(p-1)$.

Theorem 1.2 [6] *Let* $k \ge 3$ *be a fixed integer and let* $n \ge k$. *Then* $d_1, d_2, ..., d_n$ *is potentially* H_k *if and only if the following three conditions are satisfied,*
(1) $d_1 \ge 2$,
(2) d_i *is even for all* i,
(3) $\sum_{i=1}^{n} d_i = \frac{2k}{k-1}(n-1)$.

In this note, our goal is to enumerate all degree sequences of sum 2m for which there exists a realization of a 3-connected graph and those for which there exists a realization of a k-cactus graph. We will denote the number of degree sequences of sum 2m with a 3-connected graph realization by $d_{tc}(2m)$. Similarly, we will let $d_{ca}(2m)$ be the number of degree sequences of sum 2m for which there exists a k-cactus graph. Then our ultimate goal in this note is to prove the following:

Theorem 1.3 *For all* $m \ge 6$,
$d_{tc}(2m) = p(2m) - p(2m-1) - p(2m-2) + p(2m-3) - 1 - p(2m-6,2) - p(2m-9,3) - \sum_{j=4}^{m-1} \big(p(j) - p(j-1) - 1 - p(j-4,2)\big)$,
where $p(k)$ *is the number of unrestricted integer partitions of* k *and* $p(m,k)$ *is the number of partitions of* m *with at most* k *parts.*

Theorem 1.4 *For all* $m \ge 3$,
$d_{ca}(2m) = \sum_{\substack{k \ge 3 \\ m|k}} P\big(m, (k-1)\frac{m}{k} + 1\big)$,
where $P(m,k)$ *is the number of partitions of* m *into exactly* k *parts.*

2 Degree sequences of 3-connected graphs

In [3], the technique utilized in their proofs included generating function and bijections. In this note, the proof techniques also use generating function. First, we will relax the *"evenness"* condition in the statement of Theorem 1.3; namely, we will not concern ourselves at this point with whether the sum of the integers d_i is even. We will invoke this restriction at the end of the proof. Thus, we now consider a function $p(m, n)$, the number of partitions of m into exactly n parts satisfying the inequality in Theorem 1.3.

we can simplify the conditions in Theorem1.3, $n \geq 4$, $d_n \geq d_{n-1} \geq \cdots \geq d_1 \geq 3$, $d_n \leq \sum_{i=1}^{n-1} d_i - 2n + 2$.

The generating function $A_n(q)$ for $p(m, n)$ is,

$$\begin{aligned} A_n(q) &= \sum_{m \geq 0} p(m, n) q^m = \sum_{\substack{d_n \geq d_{n-1} \geq \cdots \geq d_1 \geq 3 \\ d_n \leq d_{n-1} + \cdots + d_1 - 2n + 2}} q^{d_n + d_{n-1} + \cdots + d_1} \\ &= \sum_{d_{n-1} \geq d_{n-2} \geq \cdots \geq d_1 \geq 3} \sum_{d_n = d_{n-1}}^{d_{n-1} + \cdots + d_1 - 2n + 2} q^{d_n + d_{n-1} + \cdots + d_1} \\ &= \sum_{d_{n-1} \geq d_{n-2} \geq \cdots \geq d_1 \geq 3} \frac{1 - q^{d_1 + d_2 + \cdots + d_{n-2} - 2n + 3}}{1 - q} q^{d_1 + d_2 + \cdots + 2d_{n-1}} \\ &= \sum_{d_{n-1} \geq d_{n-2} \geq \cdots \geq d_1 \geq 3} \frac{q^{d_1 + d_2 + \cdots + 2d_{n-1}}}{1 - q} - \sum_{d_{n-1} \geq d_{n-2} \geq \cdots \geq d_1 \geq 3} \frac{q^{2(d_1 + d_2 + \cdots + d_{n-1}) - 2n + 3}}{1 - q}. \end{aligned}$$

There is some wisdom here in considering multivariable generating functions. Thus, for $k \geq 1$, let

$$G_{d,k}(q_1, \cdots, q_k) = \sum_{d_1 \geq d_2 \geq \cdots \geq d_k \geq d} q_1^{d_1} \cdots q_k^{d_k} = \prod_{i=1}^{k} \frac{q_i^d}{1 - q_1 q_2 \cdots q_i}. \quad (1)$$

With this information about $G_{d,k}$ in hand, we have

$$\begin{aligned} A_n(q) &= \frac{1}{1 - q} G_{3,n-1}(q^2, q, \cdots, q) - \frac{q^{-2n+3}}{1 - q} G_{3,n-1}(q^2, q^2, \cdots, q^2) \\ &= q^{3n} \prod_{i=1}^{n} \frac{1}{1 - q^i} - \frac{q^{4n-3}}{1 - q} \prod_{i=1}^{n-1} \frac{1}{1 - q^{2i}}. \end{aligned}$$

So, the generating function $A(q)$ for $p(m)$, the number of integer partitions of m into any number $n \geq 4$ parts which satisfy the inequality in Theorem 1.3, is given by

$$A(q) = \sum_{n\geq 4} q^{3n} \prod_{i=1}^{n} \frac{1}{1-q^i} - \sum_{n\geq 4} \frac{q^{4n-3}}{1-q} \prod_{i=1}^{n-1} \frac{1}{1-q^{2i}}. \tag{2}$$

Now we will consider the two sums in (2) separately and interpret them as generating functions of well-known arithmetic functions. First, we recall a well-known identity of Euler which states that

$$1 + \sum_{n=1}^{\infty} t^n \prod_{i=1}^{n} \frac{1}{1-q^i} = \prod_{n=0}^{\infty} \frac{1}{1-tq^n}. \tag{3}$$

(see Andrews [7, Corollary 2.2].) We will use this identity on the first sum on the right-hand side of (2). By (3) with $t = q^3$, we have,

$$\sum_{n\geq 4} q^{3n} \prod_{i=1}^{n} \frac{1}{1-q^i} = \prod_{n=3}^{\infty} \frac{1}{1-q^n} - 1 - \frac{q^3}{1-q} - \frac{q^6}{(1-q)(1-q^2)} - \frac{q^9}{(1-q)(1-q^2)(1-q^3)}$$

And the generating function for $p(n)$ is

$$\sum_{n=0}^{\infty} p(n)q^n = \prod_{n=1}^{\infty} \frac{1}{1-q^n}. \tag{4}$$

Then let $p(m,k)$ denote the number of partitions of m into at most k parts. (And $p(m,k)$ is also equal to the number of partitions of m into parts no greater than k.) Thus, we have the generating function

$$\sum_{m=0}^{\infty} p(m,k)q^m = \prod_{i=1}^{k} \frac{1}{1-q^i}. \tag{5}$$

By (4)and (5), we have

$$\sum_{n\geq 4} q^{3n} \prod_{i=1}^{n} \frac{1}{1-q^i}$$

$$= (1-q)(1-q^2)\sum_{n=0}^{\infty} p(n)q^n - 1 - \sum_{n=3}^{\infty} q^n - \sum_{n=6}^{\infty} p(n-6,2)q^n$$

$$-\sum_{n=9}^{\infty} p(n-9,3)q^n$$

$$= \sum_{n=9}^{\infty}\Big(p(n) - p(n-1) - p(n-2) + p(n-3) - 1 - p(n-6,2)$$

$$- p(n-9,3)\Big)q^n,$$

where we have used the facts that $p(0) = p(1) = 1,\ p(2) = 2,\ p(3) = 3,\ p(4) = 5,\ p(5) = 7,\ p(6) = 11,\ p(7) = 15,\ p(8) = 22$ *and* $p(0,2) = p(1,2) = 1,\ p(2,2) = 2$.

Now we consider the second sum on the right-hand side of (2). Note that

$$\sum_{n\geq 4} \frac{q^{4n-3}}{1-q} \prod_{i=1}^{n-1} \frac{1}{1-q^{2i}} = \frac{q}{1-q} \sum_{n\geq 3} q^{4n} \prod_{i=1}^{n} \frac{1}{1-q^{2i}}.$$

By (3), first with $t = q^2$, and thereafter replacing q by q^2 throughout. Then, by (4) and (5), we have,

$$\sum_{n\geq 3} q^{4n} \prod_{i=1}^{n} \frac{1}{1-q^{2i}} = (1-q^2)\sum_{n=0}^{\infty} p(n)q^{2n} - 1 - \sum_{n=2}^{\infty} q^{2n} - \sum_{n=4}^{\infty} p(n-4,2)q^2$$

$$= \sum_{n=4}^{\infty}\Big(p(n) - p(n-1) - 1 - p(n-4,2)\Big)q^{2n}.$$

In order to finish the second sum in (2), we must multiply by the factor $\frac{q}{1-q}$. This yields

$$\frac{q}{1-q}\sum_{n\geq 3} q^{4n}\prod_{i=1}^{n}\frac{1}{1-q^{2i}}$$
$$=\sum_{k=1}^{\infty} q^{k}\Big(\sum_{n=4}^{\infty}(p(n)-p(n-1)-1-p(n-4,2))q^{2n}\Big)$$
$$=\sum_{m=4}^{\infty}\sum_{j=4}^{m}\Big(p(j)-p(j-1)-1-p(j-4,2)\Big)q^{2m+1}+\sum_{m=4}^{\infty}\sum_{j=4}^{m}\Big(p(j)$$
$$-p(j-1)-1-p(j-4,2)\Big)q^{2m+2}$$
$$=\sum_{m=5}^{\infty}\sum_{j=4}^{m-1}\Big(p(j)-p(j-1)-1-p(j-4,2)\Big)q^{2m-1}+\sum_{m=5}^{\infty}\sum_{j=4}^{m-1}\Big(p(j)$$
$$-p(j-1)-1-p(j-4,2)\Big)q^{2m}.$$

Since $p(2m) = d_{tc}(2m)$ for all $m \geq 6$, we can obtain that the generating function for $d_{tc}(2m)$ is given by

$$\sum_{m\geq 6} d_{tc}(2m)q^{2m} = \sum_{m\geq 6}\Big(p(2m)-p(2m-1)-p(2m-2)+p(2m-3)$$
$$-1-p(2m-6,2)-p(2m-9,3)\Big)q^{2n}-\sum_{m\geq 6}\sum_{j=4}^{m-1}\Big(p(j)$$
$$-p(j-1)-1-p(j-4,2)\Big)q^{2m}$$

and the proof of Theorem 1.3 is completed.

For example, the number of degree sequences of sum 12 with 3-connected graph realizations is,

$$d_{tc}(12) = p(12)-p(11)-P(10)+p(9)-1-p(6,2)-p(3,3)-(p(4)$$
$$-p(3)-1-p(0,2))-(p(5)-p(4)-1-p(1,2)) = 1.$$

The partition which satisfying the inequality in Theorem 1.3 is only one, $3+3+3+3$, the k_4 complete graph. Besides, when $p \geq 4$, we can also use the generation function like Theorem 1.3. But with p increasing, the result is probably more complex and more difficult to find.

3 Degree sequences of k-cactus

In this section, we will prove the Theorem 1.4. A graph has property H_k if it is connected and every block is a cycle on k vertices. Now we will consider the number of degree sequences of k-cactus, for $k \geq 3$.

We will begin the proof of Theorem 1.4 by considering criterion (2) first. Since d_i is even for all i, we will let $d_i' = \frac{d_i}{2}$, for all i. Then d_i' can relax the evenness condition. And criterion (1) and (2) can change into,

$$\sum_{i=1}^{n} d_i' = \sum_{i=1}^{n} \frac{d_i}{2} = \frac{1}{2}\sum_{i=1}^{n} d_i = \frac{k}{k-1}(n-1) = m \tag{6}$$

$$d_i' = \frac{d_n}{2} \geq 1. \tag{7}$$

Clearly , we have a bijection $\{d_i' : i = 1, 2, ..., n\} \longrightarrow \{d_i : i = 1, 2, ..., n\}$ given by $d_i' = \frac{d_i}{2}$. So we can determine the number of partitions into d_i' of sum m instead of the number of partitions into d_i of sum $2m$, don't need to consider the evenness condition and all $d_i \geq 2$. Next we will consider the restriction on n.

From (6), we have $n = \frac{k-1}{k}m + 1$, so m needs to be divided by k with no remainder. Then the number of the partition of the sum $2m$ satisfying criteria (1),(2) and (3) is equivalent to the number of the partition of the sum m satisfying parts $n = \frac{k-1}{k}m + 1$, and k should be chosen such that m must be a multiple of k. Thus, we have

$$d_{ca}(2m) = \sum_{\substack{k\geq 3\\ m|k}} P\Big(m, (k-1)\frac{m}{k} + 1\Big).$$

The proof of Theorem 1.4 is completed. So, for example, when

• $2m = 6$, the number of degree sequences of sum 6 with property H_k is $d_{ca}(6) = P(3,3) = 1$. The partition is $2 + 2 + 2$, and the corresponding realization is the triangle.

• $2m = 8$, the number of degree sequences of sum 8 with property H_k is $d_{ca}(8) = 0$. None of the corresponding realization is satisfying Theorem 1.2.

• $2m = 12$, the number of degree sequences of sum 6 with property H_k is $d_{ca}(12) = P(6,5) + P(6,6) = 1 + 1 = 2$. The two partitions are $4+2+2+2+2$ and $2+2+2+2+2+2$, and the corresponding realization are the graph constituted by two triangles and the hexagon.

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