# Differential in Cartesian Product Graphs

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**Abstract**

Let $\Gamma(V,E)$ be a graph of order n, $S \subset V$ and let $B(S)$ be the set of vertices in $V \setminus S$ that have a neighbor in a set S. The differential of a set $S$ is defined as $\partial(S) = |B(S)| - |S|$ and the differential of the graph $\Gamma$ is defined as $\partial(\Gamma) = max\{\partial(S) : S \subset V\}$. In this paper we obtain several tight bounds for the differential in Cartesian product graphs. In particular, we relate the differential in Cartesian product graphs with some known parameters of $\Gamma_1 \times \Gamma_2$, namely, its domination number, its maximum and minimum degree and its order. Furthermore, we compute explicitly the differential of some class of product graphs.

*Keywords:* differential; domination number; cartesian product graphs.
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## 1 Introduction

The mathematical properties of the differential in graphs are studied in [illegible]. In particular, several bounds on the differential of a graph were given. The differential in certain classes of graphs is studied in [illegible] and some bounds on he differential in graphs are shown in [illegible]. The parameter $\partial(S)$ is also considered in [illegible] and the differential of an independent set has been considered in [illegible]. The case of the $B$-differential of a graph, defined as $\psi(G) = max\{|B(S)| : S \subseteq V\}$ was investigated in [illegible].

We begin by stating some notation and terminology. $\Gamma = (V,E)$ denotes a simple graph of order $n = |V|$ and size $m = |E|$. The degree of a vertex $v \in V$ will be denoted by $\delta(v)$. We denote by $\delta$ and $\Delta$ the minimum and maximum degree of the graph, respectively. The subgraph induced by a set $S \subseteq V$ will be denoted by $\langle S \rangle$. For a non-empty subset $S \subseteq V$, and any vertex $v \in V$, we denote by $N_S(v)$ the set of neighbors that $v$ has in $S : N_S(v) := \{u \in S : u \sim v\}$ and $\delta_S(v) = |N_S(v)|$. The complement of the set $S$ in $V$ is denoted by $\overline{S}$, so that $N_{\overline{S}}(v)$ is the set of neighbors $v$ has in $\overline{S} = V \setminus S$. Let $\Gamma(V,E)$ be a graph of order $n$ and let $B(S)$ be the set of vertices in $\overline{S}$ that have a neighbor in a set $S$. The differential of a set $S$ is defined as $\partial(S) = |B(S)| - |S|$ and the differential of the graph $\Gamma$ is

defined as $\partial(\Gamma) = max\{\partial(S) : S \subset V\}$.
In this paper we obtain several tight bounds for the differential in Cartesian product graphs. In particular, we relate the differential in Cartesian product graphs with some known parameters of $\Gamma_1 \times \Gamma_2$, namely, its domination number, its maximum and minimum degree and its order.

# 2 Relations between the Differential and the Domination Number of $\Gamma_1 \times \Gamma_2$

Notice that if $\Gamma$ is a connected graph, then $0 \leq \partial(\Gamma) \leq n-2$. It is not difficult to calculate the exact values of the differential in the following families of graphs.

$$\partial(P_2 \times P_n) = \begin{cases} 2\lfloor\frac{n-1}{2}\rfloor & \text{if } n \text{ odd} \\ 2\lfloor\frac{n-1}{2}\rfloor + 1 & \text{if } n \text{ even} \end{cases}$$

$$\partial(P_3 \times P_n) = \begin{cases} 3k & \text{if } n = 3k \\ 3k+1 & \text{if } n = 3k+1 \\ 3(k+1) & \text{if } n = 3k+2 \end{cases}$$

A set $S \subset V$ of a graph $\Gamma$ is a $\partial - set$ of $\Gamma$ if $\partial(S) = \partial(\Gamma)$, and $S$ is a *minimum* $\partial - set$ if

$$|S| = min\{|X| : X \subset V \text{ and } \partial(X) = \partial(\Gamma)\}$$

In [1] appears the following result. For completeness we include a proof of this result.

**Lemma 1.** *If $S$ is a minimum $\partial - set$ of $\Gamma$, then the set $\{S, B(S), C(S)\}$ with $C(S) = V \setminus (B(S) \cup S)$ is a partition of $V$ such that:*

**(a)** *for all $v \in S, \delta_{B(S)}(v) \geq 2$,*

**(b)** *for all $v \in B(S), \delta_{C(S)}(v) \leq 2$,*

**(c)** *for all $v \in C(S), \delta_{C(S)}(v) \leq 1$.*

*Proof.* (a) Assume, to the contrary, that there is a vertex $v \in S$ such that $\delta_{B(S)}(v) \leq 1$. Since $\delta_{B(S)}(v) \leq 1$, then for $S' = S \setminus \{v\}$ we have $\partial(S') \geq \partial(S)$ with $|S'| < |S|$, which is a contradiction.

(b) Assume, to the contrary, that there is a vertex $v \in B(S)$ such that $\delta_{C(S)}(v) \geq 3$. Taking $S' = S \cup \{v\}$ we obtain that $\partial(S') > \partial(S)$, which is a contradiction.

(c) Using the same arguments as in Case (b), we conclude that for all $v \in C(S), \delta_{C(S)}(v) \leq 1$.

□

**Proposition 2.** *Let $\Gamma_i$ be a graph with minimum degree $\delta_i$ for $i = 1, 2$. If $S_i$ is a minimum $\partial -$ set of $\Gamma_i$ with partition $\{S_i, B(S_i), C(S_i)\}$ such that $C(S_i) \neq 0$ for some $i \in \{1, 2\}$, then*

$$\partial(\Gamma_1 \times \Gamma_2) \geq \partial(\Gamma_1) + \partial(\Gamma_2) + \delta_1 |S_2| + \delta_2 |S_1|.$$

*Proof.* Suppose that $u \in C(S_1)$ and $v \notin S_2$. If $A = \{u\} \times S_2$, then $A$ dominates all vertices of $(\{u\} \times B(S_2)) \cup (N(u) \times S_2)$. If $B = S_1 \times \{v\}$, then $B$ dominates all vertices of $(B(S_1) \times \{v\}) \cup (S_1 \times N(v))$. Notice that the sets $\{u\} \times B(S_2)$, $N(u) \times S_2$, $B(S_1) \times \{v\}$ and $S_1 \times N(v)$ are pairwise disjoint. Therefore, we have

$$\begin{aligned}
&\partial(\Gamma_1 \times \Gamma_2) \geq |\{u\} \times B(S_2)| + |N(u) \times S_2| + |B(S_1) \times \{v\}| + |S_1 \times N(v)| \\
&-|A| - |B| = |B(S_2)| + |N(u)||S_2| + |B(S_1)| + |S_1||N(v)| - |S_2| - |S_1| \\
&\qquad = \partial(\Gamma_1) + \partial(\Gamma_2) + |N(u)||S_2| + |N(v)||S_1| \\
&\qquad \geq \partial(\Gamma_1) + \partial(\Gamma_2) + \delta_1 |S_2| + \delta_2 |S_1|
\end{aligned}$$

□

The above bound is attained, for instance, if $\Gamma_1 = C_4$ and $\Gamma_2 = C_3$ we have $\partial(\Gamma_1) = 1$, $\partial(\Gamma_2) = 1$, $|S_1| = 1$, $|S_2| = 1$, $\delta_1 = 2$, $\delta_2 = 2$ and $\partial(\Gamma_1 \times \Gamma_2) = 6$.

A set $S \subset V$ of a graph $\Gamma$ is a dominating set if every vertex not in $S$ is adjacent to a vertex in $S$. The domination number of $\Gamma$, denoted by $\gamma(\Gamma)$, is the minimum cardinality of a dominating set. The reader is referred to [[illegible], [illegible], [illegible]] for more details on domination in graphs.

In [illegible] the following result is proved, it relates the domination number and the differential of a graph.

**Theorem 3.** *For any graph $\Gamma$ of order $n$ without isolated vertices,*

$$n - 2\gamma(\Gamma) \leq \partial(\Gamma) \leq n - \gamma(\Gamma) - 1$$

**Corollary 4.** *If $\Gamma_i$ is a graph of order $n_i$ for $i = 1, 2$, then*

$$n_1 n_2 - 2\gamma(\Gamma_1 \times \Gamma_2) \leq \partial(\Gamma_1 \times \Gamma_2) \leq n_1 n_2 - \gamma(\Gamma_1 \times \Gamma_2) - 1$$

**Theorem 5.** *If $\Gamma_i$ is a graph of order $n_i$ and maximum degree $\Delta_i$ for $i = 1, 2$, then*

$$\begin{aligned}
n_1\gamma(\Gamma_2) + n_2\gamma(\Gamma_1) - 3\gamma(\Gamma_1)\gamma(\Gamma_2) \leq \partial(\Gamma_1 \times \Gamma_2) \leq \\
min\{\gamma(\Gamma_1)n_2, \gamma(\Gamma_2)n_1\}(\Delta_1 + \Delta_2 - 1)
\end{aligned}$$

*Proof.* If $S$ is a minimum $\partial$-set of $\Gamma_1 \times \Gamma_2$, then $\partial(\Gamma_1 \times \Gamma_2) = |B(S)| - |S|$. Note that the maximum degree of $\Gamma_1 \times \Gamma_2$ is $\Delta_1 + \Delta_2$. Moreover,

$$|B(S)| \leq \sum_{v \in S} \delta_{B(S)}(v) \leq (\Delta_1 + \Delta_2)|S|$$

Using now that $\partial(\Gamma_1 \times \Gamma_2) = |B(S)| - |S|$, we can conclude that

$$\partial(\Gamma_1 \times \Gamma_2) \leq |S|(\Delta_1 + \Delta_2 - 1).$$

If $S$ is a minimum $\partial$-set of a graph $\Gamma$, then $|S| \leq \gamma(\Gamma)$. Using now the inequality $\gamma(\Gamma_1 \times \Gamma_2) \leq min\{\gamma(\Gamma_1)n_2, \gamma(\Gamma_2)n_1\}$, see [illegible], the upper bound follows.

For all $S_1 \subseteq V_1$ and $S_2 \subseteq V_2$, we have

$$\begin{aligned}
\partial(S_1 \times S_2) &= |B(S_1 \times S_2)| - |S_1 \times S_2| \\
&= |S_2||B(S_1)| + |S_1||B(S_2)| - |S_1||S_2| \\
&= |S_2|(|B(S_1)| - |S_1|) + |S_1||B(S_2)| \\
&= |S_2|(|B(S_1)| - |S_1|) + |S_1|(|B(S_2)| - |S_2|) + |S_1||S_2|
\end{aligned}$$

By taking $S_i$ as the minimum dominating set of $\Gamma_i$ we obtain

$$\begin{aligned}
\partial(\Gamma_1 \times \Gamma_2) &\geq \partial(S_1 \times S_2) = |S_2|(|B(S_1)| - |S_1|) + |S_1|(|B(S_2)| - |S_2|) \\
+|S_1||S_2| &= \gamma(\Gamma_2)(n_1 - 2\gamma(\Gamma_1)) + \gamma(\Gamma_1)(n_2 - 2\gamma(\Gamma_2)) + \gamma(\Gamma_1)\gamma(\Gamma_2) \\
&= n_1\gamma(\Gamma_2) + n_2\gamma(\Gamma_1) - 3\gamma(\Gamma_1)\gamma(\Gamma_2)
\end{aligned}$$

□

In [illegible] appears the following result. For completeness we include a proof of this result.

**Lemma 6.** *If $S$ is a minimum $\partial$-set of $\Gamma$, then*

**(a)** $2|S| \leq |B(S)|$,

**(b)** $|S| \leq \lfloor \frac{n}{3} \rfloor$,

**(c)** $|B(S)| \leq 2\partial(\Gamma)$.

*Proof.* (a) Let $v_1$ be a vertex in $S$. By Lemma 1, we have that if $v_1 \in S$, then $\delta_{B(S)}(v_1) \geq 2$. Let $u_1, u_2, \dots, u_j$ denote the set of vertices of $B(S)$ adjacent to $v_1$. If $u_2, \dots, u_j$, are adjacent to other vertices of $S$, then $S' = S \setminus \{v_1\}$ is a $\partial$-set of $\Gamma$ such that $|S'| < |S|$, which is a contradiction. Hence there are at least two vertices in $B(S)$ adjacent to $v_1$ that are not adjacent to any vertex of $S$. Thus, we have that $|B(S)| \geq 2|S|$.

(b) If $\{S, B(S), C(S)\}$ with $C(S) = V \setminus (B(S) \cup S)$ is a partition of $V$, then $n = |B(S)| + |S| + |C(S)|$. Since $2|S| \leq |B(S)|$ and $|C(S)| \geq 0$, the result follows.

(c) Since $\partial(\Gamma) = |B(S)| - |S|$ we have that $\partial(\Gamma) + |S| = |B(S)|$. Using that $2|S| \leq |B(S)|$, the result follows.

□

A graph $\Gamma$ is said to be a dominant differential graph if it contains a minimum $\partial$-set which is also a dominating set.

**Theorem 7.** *Let $\Gamma_i$ be a graph of order $n_i$ for $i = 1, 2$.*

**(a)** *If $\Gamma_1 \times \Gamma_2$ is a dominant differential graph, then $\partial(\Gamma_1 \times \Gamma_2) \geq \frac{n_1 n_2}{3}$.*

**(b)** *If $\Gamma_1$ is not a dominant differential graph, then $\partial(\Gamma_1 \times \Gamma_2) \geq n_2 \partial(\Gamma_1) + \partial(\Gamma_2)$.*

*Proof.* (a) If $\Gamma_1 \times \Gamma_2$ is a dominant differential graph, then there exists $S \subseteq V$ such that $n_1 n_2 = |S| + |B(S)|$. By Lemma 6 we can assume that $|S| \leq \partial(\Gamma_1 \times \Gamma_2)$ and $|B(S)| \leq 2\partial(\Gamma_1 \times \Gamma_2)$. Thus $n_1 n_2 \leq 3\partial(\Gamma_1 \times \Gamma_2)$.
(b) Let $S_1$ be a $\partial$-set of $\Gamma_1$ whose associated partition is $\{S_1, B(S_1), C(S_1)\}$. If $\Gamma_1$ is not a dominant differential graph, then $|C(S_1)| \geq 1$. We also notice that if $S_2$ is a minimum $\partial$-set of $\Gamma_2$ and $S = S_1 \times V(\Gamma_2) \cup C(S_1) \times S_2$, then

$$\partial(S) = n_2 \partial(\Gamma_1) + |C(S_1)| \partial(\Gamma_2) \geq n_2 \partial(\Gamma_1) + \partial(\Gamma_2)$$

□

The bound $(a)$ is achieved, for instance, in the graph $P_2 \times P_3$. The bound $(b)$ is achieved in the graph $P_3 \times P_4$.

The $k$-domination number of a graph $\Gamma$, $\gamma_k(\Gamma)$, is the minimum cardinality of a set $X \subset V$ such that any vertex in $V \setminus X$ is adjacent to at least $k$ vertices of $X$.

**Theorem 8.** *If $\Gamma_1 \times \Gamma_2$ is a graph with minimum degree $\delta \geq k$, then*

$$\partial(\Gamma_1 \times \Gamma_2) \geq \frac{\gamma_{k-1}(\Gamma_1 \times \Gamma_2)}{3}$$

*Proof.* Let $S$ be a minimum $\partial$-set of $(\Gamma_1 \times \Gamma_2)$ and set $\{S, B(S), C(S)\}$ be a partition of $V(\Gamma_1 \times \Gamma_2)$; by Lemma 1, we have that for all $v \in C(S)$, $\delta_{C(S)}(v) \leq 1$. Thus, for all $v \in C(S)$, $\delta_{B(S) \cup S}(v) \geq \delta(v) - 1$. If $\Gamma_1 \times \Gamma_2$ is a graph with minimum degree $\delta \geq k$, then for all $v \in C(S)$, we have $\delta_{B(S) \cup S}(v) \geq k - 1$. Hence, the set $B(S) \cup S$ is a $(k-1)$-dominating set of $\Gamma_1 \times \Gamma_2$ and $|B(S) \cup S| \geq \gamma_{k-1}(\Gamma_1 \times \Gamma_2)$.

Since $\partial(\Gamma_1 \times \Gamma_2) = |B(S)| - |S| = |B(S)| + |S| - 2|S|$ we have that $\partial(\Gamma_1 \times \Gamma_2) + 2|S| = |B(S)| - |S| = |B(S)| + |S|$.

By Lemma 6, we have that if $S$ is a minimum $\partial$-set of $\Gamma_1 \times \Gamma_2$; then $2|S| \leq |B(S)|$ and $|B(S)| \leq 2\partial(\Gamma_1 \times \Gamma_2)$. We have $\partial(\Gamma_1 \times \Gamma_2) \geq |S|$. Therefore,

$$3\partial(\Gamma_1 \times \Gamma_2) \geq |B(S)| + |S| \geq \gamma_{k-1}(\Gamma_1 \times \Gamma_2).$$

□

The bound is reached, for instance, in the graph $\Gamma = C_3 \times P_2$ for $k = 3$.

# 3 Relations between the Differential and the Order of $\Gamma_1 \times \Gamma_2$

**Theorem 9.** *If $\Gamma_i$ is a graph of order $n_i$ for $i = 1, 2$, then*

$$\partial(\Gamma_1 \times \Gamma_2) \geq max\{n_2\partial(\Gamma_1), n_1\partial(\Gamma_2)\}.$$

*Proof.* On the one hand, if $S_1$ is a $\partial$-set of $V(\Gamma_1)$, then

$$\begin{aligned}\partial(\Gamma_1 \times \Gamma_2) &\geq \partial(S_1 \times V(\Gamma_2)) = |B(S_1 \times V(\Gamma_2))| - | S_1 \times V(\Gamma_2)| \\ &= |B(S_1) \times V(\Gamma_2)| - n_2|S_1| = n_2|B(S_1)| - n_2|S_1| = n_2\partial(\Gamma_1).\end{aligned}$$

Following this line of reasoning on the other hand, if $S_2$ is a $\partial$-set of $V(\Gamma_2)$, we have that $\partial(\Gamma_1 \times \Gamma_2) \geq n_1\partial(\Gamma_2)$ . Therefore, the result follows.

□

The above bound is sharp as we can see in the following example: $\Gamma = P_3 \times P_n$ with $n = 3k$ or $n = 3k + 1$.

**Corollary 10.** *If $\Gamma_i$ is a graph of order $n_i$ for $i = 1, \dots, k$, then*

$$\partial(\Gamma_1 \times \cdots \times \Gamma_k) \geq max\{(\prod_{j=1, j\neq i}^{k} n_j)\partial(\Gamma_i) : i = 1, \dots, k\}$$

**Theorem 11.** *If $\Gamma_i$ is a graph of order $n_i$ for $i = 1, 2$, then*

$$max\{\tfrac{n_1}{2}\lfloor\tfrac{n_2}{2}\rfloor, \tfrac{n_2}{2}\lfloor\tfrac{n_1}{2}\rfloor\} \leq \partial(\Gamma_1 \times \Gamma_2) \leq \begin{cases} n_1(n_2 - 2) & if \;\; n_1 < n_2 \\ (n_1 - 1)^2 & if \;\; n_1 = n_2 \end{cases}$$

*Proof.* Recall that we have seen in the proof of Theorem 5 that for all $S_1 \subseteq V_1$ and $S_2 \subseteq V_2$,

$$\partial(S_1 \times S_2) = |S_2|(|B(S_1)| - | S_1|) + |S_1|(|B(S_2)| - | S_2|) + |S_1||S_2|.$$

If $S$ is a dominating set of $\Gamma$, then $\partial(S) = n - 2|S|$. By taking $S_1$ as a dominating set of $\Gamma_1$ such that $|S_1| = \gamma(\Gamma_1)$, we obtain

$$\begin{aligned}\partial(\Gamma_1 \times \Gamma_2) \geq \partial(S_1 \times S_2) &= |S_2|(n_1 - 2\gamma(\Gamma_1)) + \gamma(\Gamma_1)(n_2 - 2|S_2|) + \gamma(\Gamma_1)|S_1| \\ &= |S_2|(n_1 - \gamma(\Gamma_1)) + \gamma(\Gamma_1)(n_2 - 2|S_2|)\end{aligned}$$

It is known that if $\Gamma$ is a graph of order $n$, then $\gamma(\Gamma) \leq \lfloor\frac{n}{2}\rfloor$. Note that if $S_2$ is a dominating set of $\Gamma_2$ such that $|S_2| = \lfloor\frac{n}{2}\rfloor$, then $\gamma(\Gamma_1)(n_2 - 2|S_2|) \geq 0$ and $\partial(\Gamma_1 \times \Gamma_2) \geq \lfloor\frac{n}{2}\rfloor(n_1 - \gamma(\Gamma_1))$. Therefore,

$$\partial(\Gamma_1 \times \Gamma_2) \geq max\{\tfrac{n_1}{2}\lfloor\tfrac{n_2}{2}\rfloor, \tfrac{n_2}{2}\lfloor\tfrac{n_1}{2}\rfloor\}$$

It is easy to see that $\partial(\Gamma_1 \times \Gamma_2) \leq \partial(K_{n_1} \times K_{n_2})$. Let $\{u_1, u_2, \ldots, u_{n_1}\}$ be the vertices of the graph $K_{n_1}$ and let $\{v_1, v_2, \ldots, v_{n_2}\}$ be the vertices of the graph $K_{n_2}$, and suppose that $n_1 < n_2$.

The degree of $(u_1, v_1)$ in $K_{n_1} \times K_{n_2}$ is $\delta(u_1, v_1) = n_2 - 1 + n_1 - 1$. Taking $(u_2, v_2)$, we have that the number of vertices adjacent to $(u_2, v_2)$ in $V(K_{n_1} \times K_{n_2}) \setminus N[(u_1, v_1)]$ is $n_2 - 2 + n_1 - 2$. Following this line of reasoning on the vertices $(u_i, v_i)$ of $K_{n_1} \times K_{n_2}$ for $3 \leq i \leq n_1$, we have that the maximum differential is obtained when $S = \{(u_1, v_1), \ldots, (u_{n_1}, v_{n_1})\}$. Thus,

$$\begin{aligned}
\partial(S) &= (n_2 - 1 + n_1 - 1 + n_2 - 2 + n_1 - 2 + \cdots + n_2 - n_1) - n_1 \\
&= \sum_{i=1}^{n_1}(n_1 + n_2 - 2i) - n_1 = n_1(n_1 + n_2) - n_1 - 2\sum_{i=1}^{n_1} i \\
&= n_1(n_1 + n_2) - n_1 - n_1(n_1 + 1) = n_1 n_2 - 2n_1 = n_1(n_2 - 2).
\end{aligned}$$

If $n_1 = n_2$ then the maximum differential is obtained when $S = \{(u_1, v_1), \ldots, (u_{n_1-1}, v_{n_1-1})\}$,

$$\begin{aligned}
\partial(S) &= (n_1 - 1 + n_1 - 1 + n_1 - 2 + n_1 - 2 + \cdots + n_1 - n_1) - (n_1 - 1) \\
&= \sum_{i=1}^{n_1-1} (2n_1 - 2i) - (n_1 - 1) = (n_1 - 1)(2n_1) - (n_1 - 1) - 2\sum_{i=1}^{n_1-1} i \\
&= 2n_1(n_1 - 1) - (n_1 - 1) - (n_1 - 1)n_1 = (n_1 - 1)^2.
\end{aligned}$$

□

The lower bound is sharp for the graphs $\Gamma = P_3 \times P_2$ and $\Gamma = C_3 \times P_2$. The following result is a consequence of the above proof.

**Corollary 12.**

$$\partial(K_{n_1} \times K_{n_2}) = \begin{cases} n_1(n_2 - 2) & if \quad n_1 < n_2 \\ (n_1 - 1)^2 & if \quad n_1 = n_2 \end{cases}$$

**Theorem 13.** *If $\Gamma_0$ is a graph with minimum degree $\delta_0$ such that $\partial(\Gamma_0) > n - 2\gamma(\Gamma_0)$, then for any graph $\Gamma$ with minimum degree $\delta$*

$$\partial(\Gamma_0 \times \Gamma) \geq \partial(\Gamma_0) + \partial(\Gamma) + \delta_0 + \delta$$

*Proof.* Notice that $\partial(\Gamma_0) = n - 2\gamma(\Gamma_0)$ if and only if, any minimum $\partial$-set is also a dominating set of $\Gamma_0$. Hence, if $\partial(\Gamma_0) > n - 2\gamma(\Gamma_0)$, then there exists a minimum $\partial$-set which is not a dominating set. By Proposition 2, the result follows.

□

**Corollary 14.** *If* $\partial(\Gamma_1 \times \Gamma_2) < \partial(\Gamma_1) + \partial(\Gamma_2) + \delta_1 + \delta_2$*, then* $\partial(\Gamma_1) = n - 2\gamma(\Gamma_1)$ *and* $\partial(\Gamma_2) = n - 2\gamma(\Gamma_2)$.

**Theorem 15.** *Let* $S_i$ *be a minimum* $\partial$*-set of the graph* $\Gamma_i$ *de orden* $n_i$ *for* $i = 1, 2$. *If* $S_1 \times S_2$ *is a minimum* $\partial$*-set of the graph* $\Gamma_1 \times \Gamma_2$*, then*

(a) $S_1$ *is a dominating set of* $\Gamma_1$,

(b) $S_2$ *is a dominating set of* $\Gamma_2$,

(c) $|S_i| = \frac{n_i}{3}$ *for* $i = 1, 2$,

(d) $\partial(\Gamma_1 \times \Gamma_2) = \frac{n_1 n_2}{3}$.

*Proof.* If $S_1 \times S_2$ is a minimum $\partial$-set of $\Gamma_1 \times \Gamma_2$, then we have

$$\partial(\Gamma_1 \times \Gamma_2) = \partial(S_1 \times S_2) = |B(S_1)||S_2| + |S_1||B(S_2)| - |S_1||S_2|.$$

Since $|S_1| \leq \partial(\Gamma_1)$, by Lemma 6, we conclude that $\partial(\Gamma_1 \times \Gamma_2) \leq (|S_2| + |B(S_2)|)\partial(\Gamma_1)$. Using now that $|B(S_2)| \leq n_2 - |S_2|$, we obtain $\partial(\Gamma_1 \times \Gamma_2) \leq n_2\partial(\Gamma_1)$. Following the same line of reasoning, we get the upper bound $\partial(\Gamma_1 \times \Gamma_2) \leq n_1\partial(\Gamma_2)$. By Theorem 9, we have

$$max\{n_1\partial(\Gamma_2), n_2\partial(\Gamma_1)\} \leq \partial(\Gamma_1 \times \Gamma_2) \leq min\{n_1\partial(\Gamma_2), n_2\partial(\Gamma_1)\}.$$

Consequently, all the previous inequalities are equalities. Thus, $|S_i| = \partial(\Gamma_i)$ and $|B(S_i)| = n_i - |S_i|$. Note that $|B(S_i)| = n_i - |S_i|$ if and only if $S_i$ is a dominating set. By Lemma 6 and $|S_i| = \partial(\Gamma_i)$, we have that $|S_i| = \frac{n_i}{3}$ for $i = 1, 2$. Using the previous results we can conclude that $\partial(\Gamma_1 \times \Gamma_2) = \frac{n_1 n_2}{3}$.

□

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